International Journal of Game Theory, Vol. 16, Issue 2, page 123–143

Remarks on the Reasonable Set of Outcomes in a General Coalition Function Form Game

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Introduction

One basic assumption of game theory is that the players in a game can evaluate every "prospect" that might arise as an outcome of a play. Quoting Shapley (1953) "one would normally expect to be permitted to include, in the class of 'prospects', the prospect of having to play a game. The possibility of evaluating a game is therefore of critical importance". For Shapley this motivates the notion of "value". It is also a basic motivation of the present discussion.

Let us consider a coalition function form game v on a finite set of players $N = \{1, \ldots, i, \ldots n\}$, where v is a real-valued function over the set $\mathcal{P}(N)$ of all possible coalitions (subsets) in N such that $v(\emptyset) = 0$. Our aim is *to look for the largest interval of payoffs that each player can expect getting, before playing the game, when being only informed about the worth of every coalition*. We shall exhibit a correspondence associating to each game v and each player $i \in N$ an interval $R_i(v) = [a_i(v), b_i(v)]$ to be called "the reasonable interval for player $i \in N$ in the game v".

Our theory assigns to a game v a subset, actually an hypercube, in $\mathbb{R}^N$ which are "reasonable outcomes for the players" in that game. Quoting Milnor (1952), "we will take the point of view that it is better to have the set too large rather than too small". Consider indeed the unanimity game u which is such that: $u(N) = 1$, and for every $S \in \mathcal{P}(N)$, $S \neq N$, $u(S) = 0$. In such a game, player $i \in N$ can a priori expect anything in the interval $[0, 1]$, depending upon different parameters: his skill in the negociation, his eagerness, how other players behave, etc. To be sure, he will "reasonably" expect to obtain more than 0 and cannot "reasonably" expect to get more than 1. However, we do not want to assert that all points in the hypercube $[0, 1]^N$ are plausible as an outcome of the game u (most of them are not even feasible), but quoting Milnor (1952) again, "that points outside ... are implausible", whichever play of the game will take place.

The "reasonable interval for a player in a game" provides an a priori evaluation of the outcomes by the player just from looking at v in particular *before knowing which coalition structure (partition of N) will form*. As a matter of fact, the notion of what is a "plausible outcome" for the game should be considered in conjunction with the question of coalition formation.

In our opinion the two basic problems of cooperative game theory, namely what coalition structure emerge and what are the outcomes for the members of each coalition, are intimately related and should be treated jointly. In the present state of research however almost all effort was devoted to the second problem assuming the coalition structure to be given. The study provided in this paper of the set of reasonable outcomes for each coalition structure is a first step towards the joint treatment of the two problems. We do hope that this notion will turn out to be useful in studying the important and difficult problem of coalition formation. Our general idea is that in any dynamic process of coalition formation the "attraction" and the "stability" of coalition structure are largely determined by its reasonable set of outcomes that is all outcomes which are conceivably attainable if that coalition structure forms.

In the present paper we first define a "reasonable payoff correspondence" by what we believe are undisputable axioms. The result, although mathematically simple, catches the intuition of "reasonability" we have in mind. We obtain that for a super-additive game, the reasonable interval for a player is the set of individually rational payoff levels that do not exceed the upperbound introduced by Milnor (1952), namely the player's largest marginal contribution. Since we want to use the set, among other things, to learn about coalition structure, *it is crucial to have the notion also for non super-additive games and for a general coalition structures*. We argue that a natural modification of one of the axioms allows to extend the theory to any game.

In the second part of the paper we go a step further. Our suggested set of "reasonable outcomes" being such that a point outside is nonplausible, it is natural to test the plausibility of alternative solution concepts by verifying whether or not they are included in the set. This will be done for well-known solution concepts such as the Shapley value, Stable sets, the Core, the Bargaining set, the Kernel and the Nucleolus, however extended to non-trivial coalition structures and considered for games which are not necessarily super-additive. Although some of the results were already observed in the literature, at least for the case of the coalition structure $\{N\}$ (Milnor 1952; Shapley 1952; Luce–Raiffa 1957; Wesley 1971; Kikuta 1976; Maschler–Peleg–Shapley 1979) we think that a complete and systematic treatment of such tests is worthwhile to be presented and discussed.

1 Reasonable Outcomes for a Game

Let G_N be the subspace in $\mathbb{R}^{P(N)}$ of all possible coalition functions side-payments games on a finite set N of players, i.e. the set of all real-valued functions v: $P(N) \to \mathbb{R}$ with $v(\emptyset) = 0$. We call *payoff correspondence* a mapping R which associates to each game $v \in G_N$ an hypercube in $\mathbb{R}^N$ denoted:

$$R(v) = \underset{i \in N}{X} R_i(v)$$

For every $i \in N$, $R_i(v) \overset{\text{def}}{=} [a_i(v), b_i(v)]$ is an interval in $\mathbb{R}$ to be thought of as a set of payoff levels which are a priori "reasonable" for the player in the game or rather such that anything outside the set is "unreasonable". We use usual notations for every $v \in G_N$, every $w \in G_N$ and every $i \in N$, $R_i(v) \leqslant R_i(w)$ to mean: $a_i(v) \leqslant a_i(w)$ and $b_i(v) \leqslant b_i(w)$; and $R_i(v) \subseteq R_i(w)$ to mean: if $x \in R_i(v)$ then $x \in R_i(w)$.

Let Σ_N be the set of all permutations of N. In an attempt to select axiomatically a specific payoff correspondence, we consider three axioms.

Symmetry (S): $\forall v \in G_N$, $\forall \sigma \in \Sigma_N$, $\forall i \in N$, $R_i(v) = R_{\sigma(i)}(\sigma_* v)$ where $\sigma_* v \in G_N$ is defined[3] by: $\sigma_* v(S) = v(\sigma^{-1}S)$, for every $S \in P(N)$.

Covariance (C): $\forall v \in G_N$, $\forall \alpha > 0$, $\forall \beta \in \mathbb{R}^N$, if $w \in G_N$ is such that for every $S \in P(N)$ $w(S) = \alpha v(S) + \sum_{i \in S} \beta_i$ then $\forall i \in N$, $R_i(w) = \alpha R_i(v) + \beta_i$.

Monoticity (M): $\forall v \in G_N$, $\forall w \in G_N$, $\forall i \in N$, if $\forall S \in P(N)$, $S \not\ni i$, $v(S \cup \{i\}) - v(S) \geqslant w(S \cup \{i\}) - w(S)$ then $R_i(v) \geqslant R_i(w)$.

A payoff correspondence R which satisfies axioms S, C and M is called a ***reasonable payoff correspondence*** and $R(v)$ are ***reasonable outcomes*** for the game v.

The two first axioms are standard. Axiom S states that reasonable outcomes are "independent of the player's names". Axiom C states that when a game is rescaled by changing the unit and the 0-levels of the players' payoffs, the reasonable sets should also be rescaled accordingly.

The monotonicity axiom M is motivated by the natural feeling that the higher are the marginal contributions of a player in a game, the higher is the payoff the player would expect in that game. A similar attitude was already used by Young (1972) in proving that axioms S and M together with an "efficiency requirement", namely to select in the set of $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = v(N)$, give the Shapley-value for super-additive games. Therefore, in our terminology $a_i(v) = b_i(v) = Sh_i(v)$, the Shapley-value, defines a reasonable payoff correspondence for super-additive games. In fact, this is the unique single-valued reasonable function which is efficient. Without the efficiency requirement one may find many single-valued reasonable functions.

At this point it is interesting to relate the notion of single-valued reasonable function, i.e. a function $f: G_N \to \mathbb{R}^N$ satisfying S, C and M to the notion of semi-value as introduced by Dubey–Neyman–Weber (1981). The relation between the two notions is as follows.

i) A semi-value is a reasonable function

The easiest way to see this is by using the lemma (Dubey–Neyman–Weber 1981, p. 123) according to which any semi-value can be written in the form:

[3] Recall that $v \in G_N$ is *symmetric* iff $\forall \sigma \in \Sigma_N$, $\sigma_* v = v$. Clearly, by Axiom S, if v is symmetric then, for every $i \in N$ and every $j \in N$, $R_i(v) = R_j(v)$.

$$\Psi_i(v) = \sum_{S \subseteq N - \{i\}} P_s^n [v(S \cup \{i\}) - v(S)], \quad i \in N, v \in G_N,$$

where $P_s^n \geqslant 0$ satisfy $\sum_{s=0}^{n-1} \binom{n-1}{s} P_s^n = 1$.

It is easy to check that any such function Ψ is a reasonable function.

ii) A reasonable function may not be a semi-value

Consider *maximal* and *minimal marginal contributions* of player $i \in N$ in game $v \in G_N$ which are the two numbers defined by[4]

$$M_i(v) \overset{\text{def}}{=} \underset{\substack{S \in P(N) \\ S \not\ni i}}{\text{Max}} \; [v(S \cup \{i\}) - v(S))] \quad \text{and}$$

$$m_i(v) \overset{\text{def}}{=} \underset{\substack{S \in P(N) \\ S \not\ni i}}{\text{Min}} \; [v(S \cup \{i\}) - v(S)]$$

Clearly, given $v \in G_N$, if $w \in G_N$ is such that, for every $S \in P(N)$, $w(S) = \alpha v(S) + \sum_{i \in S} \beta_i$, with $\alpha > 0$ and $\beta \in \mathbb{R}^N$, then for every $i \in N, M_i(w) = \alpha M_i(v) + \beta_i$ and $m_i(w) = \alpha m_i(v) + \beta_i$. Also, for every $\sigma \in \Sigma_N$, and every $i \in N, M_i(v) = M_{\sigma(i)}(\sigma_* v)$ and $m_i(v) = m_{\sigma(i)}(\sigma_* v)$.

It is readily seen that the function $f: G_N \to \mathbb{R}^N$ defined by $f_i(v) = M_i(v)$, $i \in N$, $v \in G_N$, satisfies axioms S, C, M. However f is not a semi-value. Take any v such that $m_i(v) = 0$ and $M_i(v) > 0$; then $M_i(-v) = 0$ and hence $f_i(-v) = 0 \neq -f_i(v) = -M_i(v)$. Therefore f does not satisfy the linearity axiom, one of the four axioms defining the semi-value. As a matter of fact, any reasonable function satisfies all other three axioms, and linearity is the only one that may be violated by such a function.

Our interest in the present paper is not in any particular reasonable single-valued function, but rather in the "largest" set-valued function that would "catch" all reasonable single-valued functions. This leads us to the following definition.

A reasonable payoff correspondence R is *maximal* if for every reasonable correspondence $\tilde{R}$ and every game $v \in G_N$, we have $\tilde{R}(v) \subseteq R(v)$, i.e. $\forall i \in N$, $\tilde{R}_i(v) \subseteq R_i(v)$. We now prove:

Theorem 1: The only maximal reasonable payoff correspondence is the payoff correspondence R which is such that

$$\forall v \in G_N, \quad \forall i \in N, \quad R_i(v) = [m_i(v), M_i(v)]$$

[4] The idea of considering the numbers $M_i(v)$ is due to Milnor (1952). The idea of introducing such numbers as $m_i(v)$ appears in Kikuta (1976). However, the author considers the players' marginal contributions to coalitions containing at least two persons. Such "lower bounds" have unsatisfactory properties.

Proof: Define the payoff correspondence $\mathcal{R}$ by $\mathcal{R}_i(v) \stackrel{\text{def}}{=} [m_i(v), M_i(v)]$ for every $v \in G_N$ and every $i \in N$. Since, by definition, $m_i(v) \leqslant M_i(v)$, $\mathcal{R}_i(v) \neq \emptyset$ and $\mathcal{R}$ is well-defined. It is easy to check that $\mathcal{R}$ satisfies axioms $\mathcal{S}$, $\mathcal{C}$ and $\mathcal{M}$. Assume that $\tilde{\mathcal{R}}$ is another payoff correspondence which satisfies $\mathcal{S}$, $\mathcal{C}$ and $\mathcal{M}$ and such that, for every $v \in G_N$ and every $i \in N$, $\tilde{\mathcal{R}}_i(v) = [a_i(v), b_i(v)]$. We want to show that $a_i(v) \geqslant m_i(v)$ and $b_i(v) \leqslant M_i(v)$. Let $o \in G_N$ be the "nul" game which is defined by: $\forall S \in \mathcal{P}(N)$, $o(S) = 0$. Since o is symmetric, by $\mathcal{S}$, $\tilde{\mathcal{R}}_i(o) = \tilde{\mathcal{R}}_j(o)$ for every $i \in N$ and every $j \in N$, $j \neq i$. Since, for every $\alpha > 0$, $\alpha o = o$, by $\mathcal{C}$ we have: $\forall i \in N$, $\tilde{\mathcal{R}}_i(o) = \{0\}$.

Given any game $v \in G_N$, and for a given $i \in N$, consider $\underline{v} \in G_N$ such that, for every $S \in \mathcal{P}(N)$,

$$\underline{v}(S) \stackrel{\text{def}}{=} \begin{cases} v(S) - m_i(v) & \text{if } i \in S \\ v(S) & \text{if } i \notin S \end{cases}$$

For every $S \in \mathcal{P}(N), S \not\ni i$, $\underline{v}(S \cup \{i\}) - \underline{v}(S) = v(S \cup \{i\}) - m_i(v) - v(S) \geqslant 0$.

Thus, using $\mathcal{M}$ with respect to the nul game o, we have $a_i(\underline{v}) \geqslant 0$ and since by $\mathcal{C}$, $a_i(\underline{v}) = a_i(v) - m_i(v)$, we get: $a_i(v) \geqslant m_i(v)$. Similarly for any $i \in N$ consider now $\bar{v} \in G_N$ such that, for every $S \in \mathcal{P}(N)$,

$$\bar{v}(S) \stackrel{\text{def}}{=} \begin{cases} v(S) - M_i(v) & \text{if } i \in S \\ v(S) & \text{if } i \notin S \end{cases}$$

For every $S \in \mathcal{P}(N), S \not\ni i$, $\bar{v}(S \cup \{i\}) - \bar{v}(S) = v(S \cup \{i\}) - M_i(v) - v(S) \leqslant 0$.

Thus, using $\mathcal{M}$ with respect to the nul game, we have $b_i(\bar{v}) \leqslant 0$ and since by $\mathcal{C}$, $b_i(\bar{v}) = b_i(v) - M_i(v)$, we get $b_i(v) \leqslant M_i(v)$. Q.E.D.

Concentrating on super-additive games, we see that Theorem 1 correctly captures the notion of reasonability that we have in mind. Recall that $v \in G_N$ is ***super-additive*** (for short s.a.) iff:

$$\forall S \in \mathcal{P}(N), \quad \forall T \in \mathcal{P}(N), \quad S \cap T \neq \emptyset \Rightarrow v(S) + v(T) \leqslant v(S \cup T).$$

For such games, we have, for every $i \in N$, $m_i(v) = v(\{i\})$. Therefore, the reasonable interval of player $i \in N$ in a super-additive game v given by the maximal reasonable payoff correspondence is the set of individually rational payoffs upper-bounded by his maximal marginal contribution, i.e. $\mathcal{R}_i(v) = [v(\{i\}), M_i(v)]$. This is exactly the notion introduced in Milnor (1952). As a matter of fact, since the class of super-additive games is closed under positive linear transformation, one could formulate Theorem 1 only within this class to obtain:

Corollary: Any payoff correspondence $\tilde{\mathcal{R}}$ which is reasonable for s.a. games satisfies: $\tilde{\mathcal{R}}_i(v) \subseteq [v(i), M_i(v)]$ for any s.a. game v and for all $i \in N$.

However, as mentioned above, we consider as important to obtain a result for *all* games, not only super-additive. We thus first ask whether the reasonable set just introduced is also intuitively acceptable for *non* super-additive games. A negative answer can be seen from the following example.

Example 1: Let $N = \{1, 2, 3\}$ and w be the game defined by: $w(\{i\}) = 0$, for every $i \in N$, $w(\{1,3\}) = w(\{2,3\}) = 0$, $w(\{1,2\}) = -1$, and $w(N) = 0.5$. For player $i = 3$, we have $R_3(w) = [0, 3/2]$. However, this player cannot likely expect a payoff as high as 3/2 (or even 1), since 3/2 is based on his marginal contribution to coalition $\{1, 2\}$ which conceivably will never form.

This simple observation leads to the conclusion that the three axioms which are quite acceptable for super-additive games are actually not as acceptable in non super-additive cases. Since the validity of axioms S and C remains unaffected by the nature of the game, the problem must rely in the validity of the monotonicity axiom M. This is illustrated by the next example.

Example 2: Let w be the game on $N = \{1, 2, 3\}$ defined in Example 1 and u be the simple unanimity game on N, i.e. $u(N) = 1$, and, $\forall S \subsetneq N$, $u(S) = 0$. Looking at marginal contributions of player 3 in the two games we find:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$u(S \cup \{3\}) - u(S)$	0	0	0	1
$w(S \cup \{3\}) - w(S)$	0	0	0	3/2

We see that player 3 has in w higher marginal contributions than in u, and monotonicity would imply that he should expect not less in w than in u. However, intuitively, we feel that he should expect strictly more in u than in w. Indeed, the grand coalition is more efficient in u than in w. In addition player 3 has to consider in game w a coalition which will never form.

We want to modify axiom M to make it more adequate for general games. The key for such modifications is found in Example 2. When $i = 3$ looks at his contribution to coalition $\{1, 2\}$, he must not consider the value $w(\{1,2\}) = -1$ but rather $\hat{w}(\{1,2\}) = 0$, which is the minimal amount that players 1 and 2 will ever accept. In view of this, his effective maximal marginal contribution is 0.5 rather than 3/2.

This indicates that what is relevant for the "expectations of a player in a game" is not his marginal contributions in the game itself, but rather in its super-additive cover. A coalition structure for a coalition $S \in P(N)$ is any partition T of S and Π_S is the set of all possible coalition structures for S. We call *super-additive cover* for $v \in G_N$, the game $\hat{v} \in G_N$ defined by:

$$\hat{v}(S) = \underset{T \in \Pi_S}{\text{Max}} \sum_{T \in T} v(T), \quad S \in P(N).$$

Clearly, $\hat{v}$ is the smallest super-additive game such that $v \leqslant \hat{v}$, and $v = \hat{v}$ if v is super-additive.

The above discussion suggests to modify the monotonicity axiom M by assuming that the player's reasonable intervals are increasing with respect to their marginal contributions in the super-additive cover of the game:

Monotonicity with respect to super-additive cover ($\hat{M}$): $\forall\, i \in N$, $\forall\, v \in G_N$, $\forall\, w \in G_N$, if $\forall\, S \in P(N), S \not\ni i$, $\hat{v}(S \cup \{i\}) - \hat{v}(S) \geqslant \hat{w}(S \cup \{i\}) - \hat{w}(S)$ then $R_i(v) \geqslant R_i(w)$.

We commit a natural abuse of terminology and call from now on *reasonable payoff correspondence*, a payoff correspondence satisfying axioms S, C and $\hat{M}$. For matter of clarification, a payoff correspondence which is reasonable in terms of the axioms S, C and M may be called a *M-reasonable correspondence*. We have the following:

Theorem 2: The only maximal reasonable payoff correspondence is the payoff correspondence $\hat{R}$ such that:

$$\forall\, v \in G_N, \quad \forall\, i \in N, \quad \hat{R}_i(v) = [v(\{i\}), M_i(\hat{v})].$$

Proof: Define $\hat{R}$ by $\hat{R}_i(v) = [v(\{i\}), M_i(\hat{v})]$, for $v \in G_N$ and $i \in N$. We have: $v(\{i\}) = m_i(\hat{v})$ and since $m_i(\hat{v}) \leqslant M_i(\hat{v})$, $\hat{R}_i(v) \neq \emptyset$. It is already checked that $\hat{R}$ satisfies axioms S, C and $\hat{M}$. Assume that $\hat{R}'$ is another payoff correspondence satisfying S, C and $\hat{M}$. By $\hat{M}$ (since $\hat{\hat{v}} = \hat{v}$) we have $R'(v) = R'(\hat{v})$. Since when restricted to s.a. games $\hat{M}$ coincides with M, $\hat{R}'$ satisfied S, C and M for s.a. games. Thus, by the Corollary following Theorem 1 applied to $\hat{v}$ we have:

$$\hat{R}'_i(v) = \hat{R}'_i(\hat{v}) \subseteq [m_i(\hat{v}), M_i(\hat{v})] = [v(\{i\}), M_i(\hat{v})] = \hat{R}_i(v) \qquad \text{Q.E.D.}$$

We observe that the reasonable set (Theorem 2) and the M-reasonable set (Theorem 1) coincide for super-additive games (as do M and $\hat{M}$ in that case).

The discussion following Example 2 also suggests that the maximal bound for players's i reasonable set could be:

$$\underset{\substack{S \in P(N) \\ S \not\ni i}}{\text{Max}} [v(S \cup \{i\}) - \hat{v}(S)] \quad \text{rather than}$$

$$M_i(\hat{v}) = \underset{\substack{S \in P(N) \\ S \not\ni i}}{\text{Max}} [\hat{v}(S \cup \{i\}) - \hat{v}(S)].$$

However, the two notions turn out to coincide as proved by the following:

Proposition 1:

$$\forall v \in G_N, \quad \forall i \in N, \quad M_i(\hat{v}) = \underset{\substack{S \in \mathcal{P}(N) \\ S \not\ni i}}{\mathrm{Max}} [v(S \cup \{i\}) - \hat{v}(S)].$$

Proof: Since $\hat{v} \geqslant v$, we have

$$M_i(\hat{v}) \geqslant \underset{\substack{S \in \mathcal{P}(N) \\ S \not\ni i}}{\mathrm{Max}} [v(S \cup \{i\}) - \hat{v}(S)].$$

On the other hand, let $S^* \in \mathcal{P}(N), S^* \not\ni i$ such that

$$M_i(\hat{v}) = v(S^* \cup \{i\}) - \hat{v}(S^*) \tag{1}$$

We have

$$\hat{v}(S^* \cup \{i\}) = \sum_{T \in \mathcal{T}} v(T), \tag{2}$$

for some $\mathcal{T} \in \Pi_{S^* \cup \{i\}}$. Assume $i \in \hat{T} \in \mathcal{T}$. Since $\hat{v}$ is super-additive:

$$\hat{v}(S^*) \geqslant \hat{v}(\bar{T} - \{i\}) + \sum_{\substack{T \in \mathcal{T} \\ T \neq \bar{T}}} \hat{v}(T) \tag{3}$$

From (1), (2), (3) we derive:

$$M_i(\hat{v}) \leqslant \sum_{T \in \mathcal{T}} v(T) - \hat{v}(\bar{T} - \{i\}) - \sum_{\substack{T \in \mathcal{T} \\ T \neq \bar{T}}} \hat{v}(T) \leqslant v(\hat{T}) - \hat{v}(\bar{T} - \{i\})$$

$$\leqslant \underset{\substack{S \in \mathcal{P}(N) \\ S \not\ni i}}{\mathrm{Max}} [v(S \cup \{i\}) - \hat{v}(S)] \qquad \text{Q.E.D.}$$

Corollary:

$$\forall v \in G_N, \quad \forall i \in N, \quad M_i(\hat{v}) \leqslant M_i(v).$$

2 The Reasonable Payoff Correspondence and Various Solution Concepts

If one agrees with the interpretation of the interval $\hat{R}_i(v)$ as the set of all a priori plausible payoffs for player i viewing the game v, a natural question is whether various solution concepts of v lie in this reasonable set, or, in other words, whether they can be considered as selections in the set:

$$\hat{R}(v) = \{x \in \mathbb{R}^N; \quad \forall\, i \in N, x_i \geqslant v(\{i\}), x \leqslant M(\hat{v})\}.$$

Notice that we consider a set which depends on v only, and not on a coalition structure. This is in accordance with the point of view that the set has to catch any outcome which seems plausible *just from looking at v and before knowing which coalitions will form*.

Given a game $v \in G_N$, denote by $e(S,x) \stackrel{\text{def}}{=} v(S) - x(S)$, with $x(S) \stackrel{\text{def}}{=} \sum_{i \in S} x_i$, *the excess of coalition* $S \in \mathcal{P}(N)$ at $x \in \mathbb{R}^N$ *in v. The pre-imputation space of v for the coalition structure* $\mathcal{B} \in \Pi_N$ on N is defined by $X^*(v, \mathcal{B}) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N; \forall\, B \in \mathcal{B}, e(B,x) = 0\}$, and the *imputation space* is $X(v, \mathcal{B}) \stackrel{\text{def}}{=} \{x \in X^*(v, \mathcal{B}); e(\{i\}, x) \leqslant 0\}$. In the literature, solution concepts of a game v for a coalition structure $\mathcal{B}$ are defined in the space $X(v, \mathcal{B})$ or in the space $X^*(v, \mathcal{B})$. Clearly, for any solution defined in $X(v, \mathcal{B})$ individual rationality is introduced as an a priori condition and the solution satisfies ipso facto the lower bound of the reasonable set $\hat{R}(v)$. From our point of view, a meaningful test for a solution concept must refer to its definition in terms of the pre-imputation space $X^*(v, \mathcal{B})$. In other words, we want to check for a given solution concept whether individual rationality comes out as one of its properties, without being imposed exogenously as a requirement[5].

Since we also want to consider all games, it is convenient to group the various discussions according to two criteria:

1. *Nature of the game:* the game $v \in G_N$ may be general, weakly super-additive (wsa) or super-additive (sa). We say of v that it is *weakly super-additive* if

   $$\forall\, i \in N, \quad \forall\, S \in \mathcal{P}(N), \quad S \not\ni i, \quad v(S \cup \{i\}) - v(S) \geqslant v(\{i\}).$$

   The notion is also known in the literature as 0-monotonicity, which means that the corresponding 0-normalized game v' defined by $v'(S) \stackrel{\text{def}}{=} v(S) - \sum_{i \in S} v(\{i\})$, $S \in \mathcal{P}(N)$, is monotone[6].
   We denote by G_N^{sa} (resp. G_N^{wsa}) the subspace in G_N of all (weakly) super-additive games. It is immediate that: $G_N^{\text{sa}} \subsetneq G_N^{\text{wsa}}$.

2. *Nature of the coalition structure:* whether $\mathcal{B} \neq \{N\}$, or $\mathcal{B} = \{N\}$. In the last case, we use the conventional notations $X^*(v) = X^*(v, \{N\})$ and $X(v) = X(v, \{N\})$ respectively for the pre-imputation and for the imputation spaces[7].

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5 For motivations similar to ours, see Luce–Raiffa (1957), p. 217.

6 Recall that $v \in G_N$ is *monotone* iff $\forall\, S \in \mathcal{P}(N)$, $\forall\, T \in \mathcal{P}(N)$, $S \subseteq T \Rightarrow v(S) \leqslant v(T)$. For such a game, we clearly have: $\forall\, i \in N$, $\forall\, S \in \mathcal{P}(N), S \not\ni i, v(S \cup \{i\}) - v(S) \geqslant 0$.

7 See Aumann–Drèze (1974) for a general presentation of alternative solution concepts for games with non trivial coalition structures.

2.1 The Shapley-Value

The coalition structure $\{N\}$ being fixed, the *Shapley-value* (Shapley 1953) is the function Sh from G_N to $\mathbb{R}^N$ such that:

$$\forall i \in N, \quad \forall v \in G_N, \quad Sh_i(v) = \sum_{\substack{S \in \mathcal{P}(N)\setminus\emptyset \\ S \not\ni i}} \alpha_S [v(S \cup \{i\}) - v(S)]$$

where $\alpha_S \overset{\text{def}}{=} \dfrac{|S|!\,(|N| - |S| - 1)!}{|N|!}$. The extension of the notion to general $\mathcal{B}$ is due to Aumann--Dreze (1974).

We call *value of a game $v \in G_N$ for a coalition structure* $\mathcal{B} \in \Pi_N$ the payoff vector $\varphi(v, \mathcal{B}) \in \mathbb{R}^N$ which is such that:

$$\forall B \in \mathcal{B}, \quad \forall i \in B, \quad \varphi_i(v, \mathcal{B}) = Sh_i(v|_B),$$

where $v|_B$ is the game v restricted[8] to $B \in \mathcal{B}$. Of course, we have $\varphi(v, \{N\}) = Sh(v)$.

It follows readily from the definition of φ as a weighted average of marginal contributions that, for every $v \in G_N$ and $\mathcal{B} \in \Pi_N$,

$$m(v) \leqslant \varphi(v, \mathcal{B}) \leqslant M(v).$$

The left hand side inequality gives individual rationality of the value $\varphi(v, \mathcal{B})$ if the game v is weakly super-additive, a fortiori if it is super-additive. We can easily construct an example of a *non* weakly super-additive game in which the Shapley value is not individually rational[9]. As for the upper-bound, the right hand side inequality gives readily that the value $\varphi(v, \mathcal{B})$ is bounded by $M(\hat{v})$ if the game v is super-additive. The following example shows that relaxing super-additivity would usually yield violation of the upper-bound, even when remaining within weakly super-additive games for $\mathcal{B} = \{N\}$.

Example 3: Let v on $N = \{1, 2, 3, 4, 5\}$ be such that:

$$v(\{i\}) = 0, \quad i \in N, \quad v(\{1, 2\}) = v(\{3, 4\}) = 2, \quad v(N) = 3$$

and the other values of $v(S)$ determined as the least values to render the game monotone (hence weakly super-additive). In particular, $v(\{1, 2, 3, 4\}) = 2$ which yields: $v(N) - v(N - \{5\}) = 1$, and thus $Sh_5(v) > 0$. On the other hand, in $\hat{v}$, player $i = 5$ is a dummy and $M_5(\hat{v}) = 0$.

We thus conclude by a statement which generalizes an observation initially made by Milnor (1952) for the Shapley-value:

$$\forall v \in G_N^{sa}, \quad \forall \mathcal{B} \in \Pi_N, \quad \varphi(v, \mathcal{B}) \in \hat{R}(v).$$

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[8] Given $v \in G_N$ and $\mathcal{B} \in \Pi_N$, for any $B \in \mathcal{B}$, $v|_B \in G_B$ is defined by letting: $\forall S \in \mathcal{P}(N)$, $v|_B(S) = v(B \cap S)$.

[9] Let $N = \{1, \ldots i \ldots n\}$, $v \in G_N$, 0-normalized, for which there is a player $i \in N$ such that, $\forall S \in \mathcal{P}(N)$, $S \not\ni i$, $S \neq N - \{i\}$, $v(S \cup \{i\}) - v(S) = 0$ and $v(N - \{i\}) - v(N) < 0$. Such a game is not monotone and, as a consequence, not weakly super-additive. A straightforward computation gives: $Sh_i(v) < 0 = v(\{i\})$.

2.2 Stable Sets

Consider $v \in G_N$. *A domination* is defined on $\mathbb{R}^N$ by: $\forall x \in \mathbb{R}^N$, $\forall y \in \mathbb{R}^N$, $x \succ y \overset{\text{def}}{\Leftrightarrow} \exists S \in \mathcal{P}(N)/e(S,x) \geqslant 0$ and $\forall i \in S, x_i > y_i$. We also write $x \underset{S}{\succ} y$ for $e(S,x) \geqslant 0$ and $\forall i \in S, x_i > y_i$, and say that "$x$ dominates y via coalition S". Let $Q \subseteq X^*(v, \mathcal{B})$, Q is a *stable set of v for the coalition structure* $\mathcal{B} \in \Pi_N$ if two conditions hold

1) *Internal consistency:* $\nexists (x, x') \in Q^2 / x \succ x'$
2) *External domination:* $\forall x' \in X^*(v, \mathcal{B}),\ x' \notin Q,\ \exists x \in Q/x \succ x'$.

Stability is usually considered for (subsets of) the imputation space. In particular, for the coalition structure $\{N\}$, stable sets of a game v in $X(v)$ are called *Von Neumann Morgenstern solutions of the game* (Von Neumann Morgenstern 1944).

The following result generalizes a property initially stated by Milnor (1952) – and attributed to Gillies – for Von Neumann Morgenstern solutions of super-additive games.

Proposition 2: For every $v \in G_N$ and every $\mathcal{B} \in \Pi_N$, if $Q \subseteq X^*(v, \mathcal{B})$ is a stable set of v for $\mathcal{B}$ then: $\forall x \in Q, x \leqslant M(\hat{v})$.

Proof: Let Q be a stable set of v for $\mathcal{B}$. Take $x \in Q$ and assume:

$$\exists i \in N, \quad i \in B \in \mathcal{B}/\forall S \in \mathcal{P}(N), \quad x_i > \hat{v}(S) - \hat{v}(S - \{i\}). \tag{1}$$

So, let $\delta > 0$ be such that:

$$\forall S \in \mathcal{P}(N), \quad x_i - \delta > \hat{v}(S) - \hat{v}(S - \{i\}). \tag{2}$$

In particular, for $B \in \mathcal{B}$, $B \ni i$, we have, using the definition of $\hat{v}$ and the fact that $x \in X^*(v, \mathcal{B})$:

$$x_i - \delta > \hat{v}(B) - \hat{v}(B - \{i\}) \geqslant v(B) - \hat{v}(B - \{i\}) = x(B) - \hat{v}(B - \{i\}),$$

which gives:

$$\hat{v}(B - \{i\}) - \delta > x(B - \{i\}). \tag{3}$$

Letting $\mathcal{T} \in \Pi_{B - \{i\}}$ be the partition of $B - \{i\}$ such that $\hat{v}(B - \{i\}) = \sum_{T \in \mathcal{T}} v(T)$, (3) gives:

$$\sum_{T \in \mathcal{T}} v(T) - \delta > \sum_{T \in \mathcal{T}} x(T)$$

Thus there is a coalition $T \in \mathcal{T}$, $\mathcal{T} \in \Pi_{B - \{i\}}$, and a number $\eta > 0, \eta \leqslant \delta$, such that

$$v(T) - \eta > x(T). \tag{4}$$

Consider $x' \in X^*(v, \mathcal{B})$ such that:

$$x'_i = x_i - \eta$$
$$x'_j = x_j + \frac{\eta}{|T|} \quad \text{if } j \in T \tag{5}$$
$$x'_j = x_j \quad \text{if } j \notin T, j \neq i$$

We have, using (4), $v(T) > x'(T) = x(T) + \eta$, i.e. $e(T, x') > 0$; and since, $\forall j \in T$, $x'_j > x_j$, (4) and (5) imply $x' \underset{T}{\succ} x$.

Since $x \in Q$, we have by internal consistency: $x' \notin Q$. Applying external domination we find:

$$\exists y \in Q, \quad \exists S \in \mathcal{P}(N)/e(S,y) \geqslant 0 \quad \text{and} \quad \forall j \in S, \quad y_j > x'_j. \tag{6}$$

If $S \not\ni i$, since by (5) $x'_j \geqslant x_j$ for every $j \in S$, we deduce from (6): $y \underset{S}{\succ} x$, contradicting internal consistency. If $S \ni i$, using (2), (4) and (5), we get:

$$y_i > x'_i = x_i - \eta \geqslant x_i - \delta > \hat{v}(S) - \hat{v}(S - \{i\}),$$

i.e. $\hat{v}(S - \{i\}) > \hat{v}(S) - y_i$. Letting $V \in \Pi_{S-\{i\}}$ be the partition of $S - \{i\}$ such that $\hat{v}(S - \{i\}) = \sum_{V \in V} v(V)$, combined with $\hat{v}(S) \geqslant v(S)$ and (6) we obtain:

$$\sum_{V \in V} v(V) = \hat{v}(S - \{i\}) > \hat{v}(S) - y_i \geqslant v(S) - y_i \geqslant y(S) - y_i$$
$$= y(S - \{i\}) = \sum_{V \in V} y(V)$$

Thus: $\exists V \subseteq S - \{i\}/v(V) > y(V)$, i.e. $e(V,y) > 0$, and $\forall j \in V, y_j > x_j$ giving $y \underset{V}{\succ} x$, which again contradicts internal consistency. Q.E.D.

Proposition 2 shows that a stable set is always nicely upper-bounded. It remains to be seen whether or not it is also lower-bounded. We have a negative answer with the next example (inspired by Shapley 1952) of a super-additive game v, where a stable set for $\{N\}$ is not included in the set of individually rational outcomes.

Example 4: Let v be a 4-persons quota game. The quotas are $w_1 = w_2 = w_3 = 0.4$ and $w_4 = -0.2$. The resulting coalition function v on $N = \{1, 2, 3, 4\}$ is

$$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 0.8,$$
$$v(\{1,4\}) = v(\{2,4\}) = v(\{3,4\}) = 0.2,$$
$$v(N) = 1 \quad \text{and} \quad v(\{i,j,k\}) = 0.8, \quad v(\{i\}) = 0 \quad \text{for every } i \in N, j \in N \text{ and } k \in N.$$

We have $v \in G_N^{sa}$, and given $\mathcal{B} = \{N\}$, the pre-imputation and the imputation spaces are:

$$X^*(v) = \left\{ y \in \mathbb{R}^4 ; \sum_{i=1}^{4} y_i = 1 \right\} \quad \text{and} \quad X(v) = \left\{ y \in \mathbb{R}_+^4 ; \sum_{i=1}^{4} y_i = 1 \right\}$$

Define:

$$P = \{(0.4, x, 0.8 - x, -0.2); 0 \leqslant x \leqslant 0.8\}$$
$$R = \{(x, 0.8 - x, 0.4, -0.2); 0 \leqslant x \leqslant 0.4\}$$

and $Q = P \cup R$. We have $Q \subsetneq X^*(v)$ but Q is not included in $X(v)$. We show in the Appendix that Q is a stable set of v for $\{N\}$.

Thus, for any game and any coalition structure, stable sets satisfy the upper-bound defining the reasonable set; however may violate individual rationality, even for super-additive games and the coalition structure $\{N\}$.

2.3 The Core

We call *Core of a game $v \in G_N$ for a coalition structure* $\mathcal{B} \in \Pi_N$ the set

$$C_0(v, \mathcal{B}) \overset{\text{def}}{=} \{x \in X^*(v, \mathcal{B}); \nexists\, x' \in X^*(v, \mathcal{B})/x' \succ x\}$$

Clearly, we have

$$C_0(v, \mathcal{B}) = \{x \in X^*(v, \mathcal{B}); \forall S \in \mathcal{P}(N), e(S, x) \leqslant 0\},$$

and any Core outcome is individually rational. On another hand since as easily checked $C_0(v, \mathcal{B})$ is in the intersection of all stable sets of v for $\mathcal{B}$, by Proposition 2, it is upper-bounded by $M(\hat{v})$. We conclude.

Proposition 3:

$$\forall\, v \in G_N, \quad \forall\, \mathcal{B} \in \Pi_N, \quad C_0(v, \mathcal{B}) \subseteq \hat{\mathcal{R}}(v)$$

Thus, the reasonable set can be viewed as a "Core catcher", a term introduced by Tijs (1981) who looks for super-set of the Core. Unlike the set defined by Tijs, the reasonable set is always non-empty and defined axiomatically.

2.4 Bargaining Set

Consider a game $v \in G_N$ and a coalition structure $\mathcal{B} \in \Pi_N$. Denote, for $i \in N$ and $j \in N$: $T_{ij} \overset{\text{def}}{=} \{S \in \mathcal{P}(N); S \ni i, S \not\ni j\}$. Given $B \in \mathcal{B}$, an objection of *$i \in B$ against $j \in B$*

at $x \in \mathbb{R}^N$ is a pair $(y, S) \in \mathbb{R}^S \times T_{ij}$ such that:

$$e(S,y) \geqslant 0 \quad \text{and} \quad \forall\, l \in S, \quad y_l > x_l$$

A *counter-objection* of $j \in B$ *against* (y, S) is a pair $(z, T) \in \mathbb{R}^T \times T_{ji}$ such that:

$$e(T,z) \geqslant 0 \quad \text{and} \quad \forall\, l \in T - S, \quad z_l \geqslant x_l, \quad \forall\, k \in T \cap S, \quad z_k \geqslant y_k.$$

Player $i \in B$ has a *justified objection against* $j \in B$ *at* $x \in \mathbb{R}^N$, denoted $i \underset{x}{\succ} j$, if there is an objection of i against j at x and no counter-objection of j with respect to it.

The *pre-Bargaining set of* $v \in G_N$ *for* $\mathcal{B} \in \Pi_N$ is the set[10]:

$$M_{(1)}(v, \mathcal{B}) \overset{\text{def}}{=} \{x \in X^*(v, \mathcal{B}); \forall\, B \in \mathcal{B}, \forall\, i \in B, \forall\, j \in B, \text{no } (i \underset{x}{\succ} j)\}$$

It is different from the *"Classical" Bargaining set* which is defined with respect to imputations and is such that: $M^i_{(1)}(v, \mathcal{B}) = M_{(1)} \cap X(v, \mathcal{B})$. In case of the coalition structure $\mathcal{B} = \{N\}$, we use the notation: $M_{(1)}(v) = M_{(1)}(v, \{N\})$. As can been easily seen, we have $C_0(v, \mathcal{B}) \subseteq M_{(1)}(v, \mathcal{B})$.

Unfortunately, we have the following negative assertion[11].

Proposition 4: The pre-Bargaining set $M_{(1)}(v, \mathcal{B})$ may contain points outside the reasonable set $\hat{R}(v)$ and violations of the bounds on both directions are possible even for super-additive games and the coalition structure $\mathcal{B} = \{N\}$. In case of the upper-bound, the violation occurs already for the "Classical" Bargaining set $M^i_{(1)}(v)$.

Although not founded in literature, this phenomenon was already noticed and in fact the following example is a slight simplification of an example provided by B. Peleg for the upper-bound.

Example 5: Let $N = \{1, 2, 3, 4, 5, 6\}$ and let

$$v(S) = \begin{cases} 1 & \text{if } |S - \{6\}| \geqslant 3, \\ 0 & \text{otherwise.} \end{cases}$$

It is a matter of straightforward verification to check that

$$x = \left(\frac{1}{5} - \epsilon, \frac{1}{5} - \epsilon, \frac{1}{5} - \epsilon, \frac{1}{5} - \epsilon, \frac{1}{5} - \epsilon, 5\epsilon\right) \quad \text{and}$$

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[10] The subscript (1) refers, as usual, to the fact that only individual players object and counter-object.

[11] Proposition 4 has to be compared with Theorem 2 in Kikuta (1976), p 204 which proves the validity of the upper-bound $M(v)$ for weak super-additive games and for a variant of the Bargaining set in which only some special coalitions are permitted.

$$y = \left(\frac{1}{5} + \eta, \frac{1}{5} + \eta, \frac{1}{5} + \eta, \frac{1}{5} + \eta, \frac{1}{5} + \eta, -5\eta\right)$$

are both in $M_{(1)}(v)$ for $\epsilon > 0$, $\eta > 0$ sufficiently small, e.g. $\epsilon \leqslant \frac{2}{35}$, and $\eta \leqslant \frac{2}{65}$. However, player 6 is dummy and nevertheless receives in $x : 5\epsilon > M_6(\hat{v}) = 0$, when he receives in $y : -5\eta < v(\{6\}) = 0$.

We conclude that even for the "Classical" Bargaining set where individual rationality is externally imposed, one does not obtain reasonability.

2.5 The Kernel

Given a game $v \in G_N$, let $e_{ij}(x) = \underset{S \in T_{ij}}{\text{Max}}\ e(S,x)$, be the maximal excess with respect to $x \in \mathbb{R}^N$ of a coalition containing $i \in N$ but not $j \in N$.

The *pre-Kernel of v for* $\mathcal{B} \in \Pi_N$ is the subset of $M_{(1)}(v, \mathcal{B})$ defined by:

$$K^*(v, \mathcal{B}) = \{x \in X^*(v, \mathcal{B});\ \forall B \in \mathcal{B}, \forall i \in B, \forall j \in B, e_{ij}(x) = e_{ji}(x)\}.$$

The *Kernel of v* for $\mathcal{B}$ is the subset of $M^i_{(1)}(v, \mathcal{B})$ defined by

$$K(v, \mathcal{B}) = \{x \in X(v, \mathcal{B});\ \forall B \in \mathcal{B}, \forall i \in B, \forall j \in B,\ e_{ij}(x) \leqslant e_{ji}(x) \text{ or } x_j = v(\{j\})\}$$

By definition, we have: $K^*(x, \mathcal{B}) \cap \{x \in \mathbb{R}^N; \forall i \in N, x_i \geqslant v(\{i\})\} \subseteq K(x, \mathcal{B})$. For the coalition structure $\mathcal{B} = \{N\}$, the sets are denoted $K^*(v)$ and $K(v)$.

Proposition 5: For any super-additive game $v \in G_N^{sa}$ and any coalition structure $\mathcal{B} \in \Pi_N$, the pre-Kernel $K^*(v, \mathcal{B})$, and hence the Kernel $K(v, \mathcal{B})$, are upper-bounded by $M(\hat{v}) = M(v)$.

Proof: Let $x \in K^*(v, \mathcal{B})$. Assume that for some $i \in N$, $x_i > M_i(v)$, i.e. $\forall S \in \mathcal{P}(N)$, $S \not\ni i, x_i > v(S \cup \{i\}) - v(S)$. We prove that this implies $x \notin K^*(v, \mathcal{B})$.

Let $B \in \mathcal{B}$ such that $i \in B$, then:

$$x_i > v(B) - v(B - \{i\}) = x(B) - v(B - \{i\}) = x_i - e(B - \{i\}, x)$$

implying $e(B - \{i\}, x) > 0$. Hence $A = \{S \subseteq N; e(S,x) > 0\} \neq \emptyset$.

Let $S^\circ \subseteq N$ such that $e(S^\circ, x) \geqslant e(S,x)$, $\forall S \in A$.

Assertion 1: $i \notin S^\circ$. Since if $i \in S^\circ$, then:

$$e(S^\circ - \{i\}, x) = v(S^\circ - \{i\}) - x(S^\circ - \{i\})$$
$$= e(S^\circ, x) + [x_i - (v(S^\circ) - v(S^\circ - \{i\}))]$$

Since, by assumption $x_i > v(S^\circ) - v(S^\circ - \{i\})$, we have $e(S^\circ - \{i\}, x) > e(S^\circ, x)$, contradicting the definition of S°.

Assertion 2: $S^\circ \cap (B - \{i\}) \neq \emptyset$. Since otherwise, recalling that the game is s.a. and $e(B - \{i\}, x) > 0$, $S^\circ \cap (B - \{i\}) = \emptyset$ would imply:

$$e(S^\circ \cup (B - \{i\}), x) = e(S^\circ, x) + e(B - \{i\}, x) > e(S^\circ, x),$$

contradicting the definition of S°.

To complete the proof, let now $j \in S^\circ \cap (B - \{i\})$. Clearly $e_{ji}(x) = e(S^\circ, x) > e_{ij}(x)$.
Q.E.D.

Weakening the super-additivity assumption may lead to violation of the upper-bound, even for w.s.a. games and $\mathcal{B} = \{N\}$, not only for $K^*(v)$ but also for the smaller set $K(v)$. This is shown by the following example:

Example 6: Consider the following 7 players game in which the set of players $N = \{1, 1', 2, 2', 3, 3', 4\}$ is composed of three couples $C_1 = \{1, 1\}$, $C_2 = \{2, 2'\}$, $C_3 = \{3, 3'\}$ and an exceptional player 4. Let $\epsilon > 0$ and $\epsilon < \frac{1}{3}$ and define v by:

$$v(\{i\}) = 0 \quad \text{for every } i \in N, \quad v(C_1) = v(C_2) = v(C_3) = 2,$$

$$v(C_i \cup C_j \cup \{4\}) = 4 + 4\epsilon, \quad v(N) = 6$$

and for any other S, $v(S)$ is determined by taking the monotone cover of the above values: i.e. $v(S) = \underset{R \subseteq S}{\text{Max}}\ v(B)$. Let

$$x = (1 - \epsilon, 1 - \epsilon, 1 - \epsilon, 1 - \epsilon, 1 - \epsilon, 1 - \epsilon, 6\epsilon);$$

we claim that x is in the Kernel although $x_4 = 6\epsilon > M_4(\hat{v}) = 4\epsilon$. Clearly, any two players in a couple are interchangeable players who receive in x the same amount and therefore $e_{ii'}(x) = e_{i'i}(x)$. Consider a pair $\{i, j\}$, $j \neq i'$; then $e_{ij}(x) = 2\epsilon$ which is achievable by $S = \{i, i'\}$ or $S = \{i, j', k, k', 4\}$, $k \neq j$. Similarly $e_{ji}(x) = 2\epsilon$. Consider finally a pair $\{i, 4\}$. The only coalition with positive excess containing i and not 4 is $\{i, i'\}$ with excess 2ϵ. Therefore $e_{i4}(x) = 2\epsilon$. Now 4 against i has an excess of 2ϵ via the coalition $\{j, j', k, k', 4\}$, and therefore $e_{4i}(x) = 2\epsilon$.

Wesley (1971) has shown that for $v \in G_N^{w\,sa}$ and for $x \in K^*(v)$, $x_i \leq M_i(v)$ for all $i \in N$. For $v \in G_N^{sa}$, this is a special case of Proposition 4 for $\mathcal{B} = \{N\}$. For

$v \in G_N^{wsa} \backslash G_N^{sa}$, the bounds $M_i(v)$ may be strictly higher than the bounds $M_i(\hat{v})$. In Example 6, for instance, $M_4(v) = 2 + 4\epsilon$ (player's 4 contribution to coalition $C_i \cup C_j$) compared to $M_4(\hat{v}) = 4\epsilon$. The example shows that Wesley's result for such v cannot be strengthened to replace $M_i(v)$ by $M_i(\hat{v})$.

As for the lower bound, one has the following statement.

Proposition 6: For $v \in G_N^{wsa}$ and for $\mathcal{B} = \{N\}$, $K^*(v) = K(v)$ and hence they are both individually rational.

This is a result of Maschler–Peleg (1967) (see Maschler–Peleg–Shapley 1972, p. 77). It cannot be extended by relaxing one of the two conditions. Example 7 shows that a payoff in the pre-Kernel may not be individually rational for non weakly[12] super-additive games, even for $\mathcal{B} = \{N\}$. Example 8 shows that for $\mathcal{B} \neq \{N\}$, a payoff in the pre-Kernel may not be individually rational even for super-additive games.

Example 7: Let $N = \{1, 2, 3\}$ and the coalition function v:

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \quad v(\{1, 2\}) = 20, \quad v(\{1, 3\}) = 50,$$

$$v(\{2, 3\}) = 10, \quad v(\{1, 2, 3\}) = 40.$$

The super-additive cover $\hat{v}$ is such that $\hat{v}(\{1, 2, 3\}) = 50$, and $\hat{v}(S) = v(S)$ otherwise. We have: $M_1(\hat{v}) = 50$, $M_2(\hat{v}) = 20$ and $M_3(\hat{v}) = 50$. The pre-Kernel is $K^*(v) = \{(27.5, -5, 17.5)\}$. Thus we have $K^*(v) \leqslant M(\hat{v})$ but $K^*(v)$ is not individually rational for player 2. The Kernel is $K(v) = \{(25, 0, 15)\}$ and thus we have $K(v) \in \hat{\mathcal{R}}(v)$.

Example 8: Let $N = \{1, 2, 3\}$, and v be the 3-person simple game in which $\{1, 2\}$ is the only minimal winning coalition, i.e.:

$$v(\{1, 2\}) = v(N) = 1, \quad v(S) = 0 \quad \text{otherwise}.$$

For $\mathcal{B} = \{\{1\}, \{2, 3\}\}$, we have $x = (0, 1/2, -1/2) \in K^*(v, \mathcal{B})$.

2.6 The Nucleolus

The last solution concept to be considered is the Nucleolus, initially introduced by Schmeidler (1969) to select a point in the Kernel (thus in the Bargaining set).

Given a game on N, let θ be the function from $\mathbb{R}^N$ to $\mathbb{R}^{2^n}$ defined by letting:

$$\forall t = 1 \ldots 2^n, \quad \theta_t(x) = e(S_t, x) \quad \text{with} \quad \theta_s(x) > \theta_t(x) \text{ whenever } t > s.$$

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[12] However, Maschler–Peleg–Shapley (1979), p. 327, Corollary 4–13, extends to certain non-weakly super-additive games the class of games for which $K^*(v) = K(v)$, and thus Proposition 6 holds.

We say that $\theta(y)$ is lexicographically greater than $\theta(x)$, denoted $\theta(y) \underset{L}{>} \theta(x)$ iff:

$$\exists\, i_0 / \forall\, i, i < i_0, \theta_i(y) = \theta_i(x) \quad \text{and} \quad \theta_{i_0}(y) > \theta_{i_0}(x).$$

We write: $\theta(x) \underset{L}{\geqq} \theta(y)$ for $\theta(y) \underset{L}{>} \theta(x)$.

The *pre-Nucleolus of* $v \in G_N$ *for* $\mathcal{B} \in \Pi_N$ is defined by:

$$N_u^*(v, \mathcal{B}) = \{x \in X^*(v, \mathcal{B});\ \forall\, y \in X^*(v, \mathcal{B}), \theta(x) \underset{L}{\leqq} \theta(y)\}.$$

The *Nucleolus of* v *for* $\mathcal{B}$ is:

$$N_u(v, \mathcal{B}) = N_u^*(v, \mathcal{B}) \cap X(v, \mathcal{B}).$$

We have (Schmeidler 1969; Kohlberg 1971) that $N_u(v, \mathcal{B})$ (resp. $N_u^*(v, \mathcal{B})$) selects a unique point in $K(v, \mathcal{B})$ (resp. $K^*(v, \mathcal{B})$).

By Proposition 5, for any super-additive game $v \in G_N^{sa}$ and any coalition structure $\mathcal{B} \in \Pi_N$, the pre-Nucleolus $N_u^*(v, \mathcal{B})$ and hence the Nucleolus $N_u(v, \mathcal{B})$, is super-bounded by $M(\hat{v}) = M(v)$. As in the case of the Kernel, this positive result cannot be strengthened by relaxing either the super-additivity of the game or the condition $\mathcal{B} = \{N\}$. In fact, noticing that x of Example 6 is also the Nucleolus of that game shows that the Nucleolus (a fortiori the pre-Nucleolus) in non super-additive games may not been bounded by $M(\hat{v})$ even for a weakly super-additive game and $\mathcal{B} = \{N\}$. The following example shows that for general coalition structure $\mathcal{B}$, even weakly super-additive games may have their nucleolus unbounded by $M(\hat{v})$.

Example 9: Let $N = \{1, 2, 3, 4, 5, 6\}$ and v be such that:

$$v(\{i\}) = 0 \quad \text{for every } i \in N, \quad v(\{2, 3\}) = 10$$

$$v(\{4, 5\}) = v(\{4, 6\}) = v(\{5, 6\}) = 13, \quad v(\{1, 2, 3\}) = v(\{4, 5, 6\}) = 15$$

$$v(\{1, 4, 5\}) = v(\{1, 4, 6\}) = v(\{1, 5, 6\}) = v(\{1, 4, 5, 6\}) = 19;$$

otherwise $v(S)$ is the least value which makes v monotone, hence wsa. Let $\mathcal{B} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$; we have:

$$x = N_u(v, \mathcal{B}) = (7, 4, 4, 5, 5, 5) \quad \text{and}$$

$$x_1 = 7 > M_1(v) = \underset{S \in P(N)}{\text{Max}} [v(S) - v(S - \{i\})] = 6.$$

Recalling that $M_i(v) \geqslant M_i(\hat{v})$, we obtain the desired conclusion.

As for the lower bound, the situation is exactly as for the pre-Kernel and Kernel respectively. This follows for positive results from the facts that $N_u^*(v, \mathcal{B}) \in K^*(v, \mathcal{B})$ and $N_u(v, \mathcal{B}) \in K(v, \mathcal{B})$. For negative results, notice that in all counter-examples concerning the Kernel, the special points are actually the Nucleolus.

The results of the present section concerning the reasonability of various solution concepts are summarized in Table 1.

Table 1

Criteria	Solution concept	Value $\varphi(v, \mathcal{B})$	Stable sets of v for $\mathcal{B}$ with respect to pre-imputations	Core $C_0(v, \mathcal{B})$	Pre-Bargaining set $\mathcal{M}_{(1)}(v, \mathcal{B})$	pre-Kernel $\mathcal{K}^*_u(v, \mathcal{B})$ and pre-Nucleolus $\mathcal{N}^*_u(v, \mathcal{B})$
$\mathcal{B} = \{N\}$	$M(\hat{v})$	Yes for sa	Yes for all v	Yes for all v	Counter-example (Ex. 5) for sa even for the "classical" Bargaining set $\mathcal{M}^i_{(1)}(v, \mathcal{B})$	Yes for sa
		Counter-example (Ex. 3) for wsa				Counter-example (Ex. 6) for wsa even for $\mathcal{N}_u(v) \in \mathcal{K}(v)$
	Individual Rationality	Yes for wsa	Counter-example (Ex. 4) for sa	Yes for all v	Counter-example (Ex. 5) for sa	Yes for wsa
		Counter-example for non wsa				Counter-example (Ex. 7) for non wsa
$\mathcal{B} \neq \{N\}$	$M(\hat{v})$	Same as for $\mathcal{B} = \{N\}$	Yes for all v	Yes for all v		Yes for sa
						Counter-example (Ex. 9) for wsa even for $\mathcal{N}_u(v, \mathcal{B}) \in \mathcal{K}(v, \mathcal{B})$
	Individual Rationality	Same as for $\mathcal{B} = \{N\}$		Yes for all v		Counter-example (Ex. 8) for sa

wsa: weak super-additivity; sa: super-additivity

Appendix

Proposition: In Example 3, Q is a stable set of v for $\{N\}$.

Proof: 1) Internal consistency. Consider $y \in Q$ and $\tilde{y} \in Q$. No domination can occur if both vectors belong to the same subset of Q, namely P or R. Assume w.l.og, that $y \in P$ and $\tilde{y} \in R$. We can have $y \succ \tilde{y}$ or $\tilde{y} \succ y$ only via coalitions $\{1,2\}$, $\{1,3\}$, $\{2,3\}$. But:

$y \underset{\{1,2\}}{\succ} \tilde{y}$ is impossible since $\tilde{y}_1 < 0.4$ implies $\tilde{y}_2 > 0.4$, thus $y_2 > 0$ and $y_1 + y_2 > 0.8 = v(\{1,2\})$.

$\tilde{y} \underset{\{1,2\}}{\succ} y$ is impossible since one would have $\tilde{y}_1 > 0.4$.

$y \underset{\{1,3\}}{\succ} \tilde{y}$ is impossible since $y_3 > 0.4$ implies $y_1 + y_3 > 0.8 = v(\{1,3\})$.

$\tilde{y} \underset{\{1,3\}}{\succ} y$ is impossible since one would have $\tilde{y}_1 > 0.4$.

$y \underset{\{2,3\}}{\succ} \tilde{y}$ is impossible since one would have $y_3 > 0.4$ implying $y_2 < 0.4$ and thus $\tilde{y}_2 < 0.4$.

$\tilde{y} \underset{\{2,3\}}{\succ} y$ is impossible since one would have $y_3 < 0.4$ implying $y_2 > 0.4$ thus $\tilde{y}_2 > 0.4$ and $\tilde{y}_2 + \tilde{y}_3 > 0.8 = v(\{2,3\})$.

2) External domination. Consider $y \in X^*(v) - Q$. Clearly, if $y_4 < -0.2$, any $\tilde{y} \in Q$ dominates y via $\{4\}$. If $y_4 > -0.2$ let $y_4 = -0.2 + 3\epsilon$, where $\epsilon > 0$, and consider $\tilde{y} = (y_1 + \epsilon, y_2 + \epsilon, y_3 + \epsilon, -0.2)$. Note that $\tilde{y}(\{1,2,3\}) = 1.2$. So $\tilde{y}$ is feasible for at least one of the coalitions $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$; thus $\tilde{y} \succ y$ via that coalition. If $\tilde{y} \in Q$, $\tilde{y}$ is the required outcome. Assume that $\tilde{y} \notin Q$. We show that in this case $\exists z \in Q$ such that $z \succ \tilde{y}$ via $\{1,2\}$ or $\{1,3\}$ or $\{2,3\}$. This concludes the proof since such z dominates y via the same coalition. In fact, $\tilde{y} \in Q$ implies in particular that $\tilde{y}_1 \neq 0.4$ and $\tilde{y}_3 \neq 0.4$: if $\tilde{y}_1 > 0.4$, then $\tilde{y}_2 + \tilde{y}_3 < 0.8$ and $\exists z \in P$ such that $z \underset{\{2,3\}}{\succ} \tilde{y}$; if $\tilde{y}_1 < 0.4$ and $\tilde{y}_3 < 0.4$, then $\exists z \in R$ such that $z \underset{\{1,3\}}{\succ} \tilde{y}$; if $\tilde{y}_1 < 0.4$ and $\tilde{y}_3 > 0.4$, then $\tilde{y}_1 + \tilde{y}_2 < 0.8$ and again $\exists z \in R$ such that $z \underset{\{1,2\}}{\succ} y$. Q.E.D.

Acknowledgements: We are deeply indebted to B. Peleg and to M. Maschler for very useful discussions and for helping us to trace the existing results in this subject. We are also grateful to an anonymous referee for helpful comments, in particular for asking about relation between reasonable functions and semi-values. The usual caveat applies.

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Received June 1984
Revised version October 1985