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The Normal Distribution and Repeated Games

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Abstract: For a reperated zero-sum two-person game with incomplete information discussed by *Zamir*, it is proved here that $\lim_{n\to\infty} \sqrt{n}\, v_n(p) = \phi(p)$ where $\phi(p)$ is the normal density function evaluated at its p-quantile (i.e. $\phi(p) = \frac{1}{\sqrt{2\pi}} e^{-(1/2)x_p^2}$ where $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_p} e^{-(1/2)x^2}\, dx = p$). Here for $0 \leqslant p \leqslant 1$, $(p, 1-p)$ is the a priori probability distribution on two states of nature, the actual state of nature is known to the maximizer but not to the minimizer. $v_n(p)$ is the minimax value of the game with n stages.

1. The Result

$$\text{Let } A_1 = \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

For any p; $0 \leqslant p \leqslant 1$ and any positive integer n define a two-person zero-sum game $\Gamma_n(p)$ played as follows:

Stage – 0: A chance move chooses an element $k \in \{1, 2\}$ with probabilities: $Pr(k = 1) = p$; $Pr(k = 2) = p'$ $(= 1 - p)$. Player I is informed of the value of k but player II is not.

Stage – m: $(m = 1, 2, \ldots, n)$ Players I and II choose $i_m \in \{1, 2\}$ and $j_m \in \{1, 2\}$ respectively. After being chosen, the pair (i_m, j_m) is announced to both players.

At the end of the n-th stage player II pays player I the amount $\frac{1}{n} \sum_{m=1}^{n} a^k_{i_m j_m}$, k being the element chosen by change at stage-0. (a^k_{ij} is the (i, j) element of A_k).

Denote by $v_n(p)$ the (minmax) value of $\Gamma_n(p)$.

Amazingly enough the above described simple looking discrete game turns out to involve in its value none other but the well known normal distribution:

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Theorem 1.1 (main result): *For* $0 \leqslant p \leqslant 1$,

$$\lim_{n \to \infty} \sqrt{n}\; v_n(p) = \phi(p), \tag{1.1}$$

where

$$\phi(p) = \frac{1}{\sqrt{2\pi}}\, e^{-(1/2)x_p^2}\,;\quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_p} e^{-(1/2)x^2}\, dx = p\,.$$

In words; the limit of $\sqrt{n}\, v_n(p)$ is the standard normal density function evaluated at its p-quantile.

2. Background

The idea behind the above described game is the following: We think of k as the *state of nature* about which there is an a priori probability distribution (p, p'). Player I knows the real state of nature but player II does not know. However, player II can learn in general something about the real state of nature k by observing the moves $i_1, i_2, \ldots,$ of player I, since those would usually depend on k.

Such repeated games were first discussed by *Aumann* and *Maschler* [1966] who proved that for general A_1 and A_2 $\lim v_n(p) = \operatorname{Cav} u(p)$, $u(p)$ being the (minmax) value of the matrix game $\Delta(p) = pA_1 + p'A_2$ and $\operatorname{Cav} u(p)$ is the concavification on $[0,1]$ of $u(p)$. For the game under consideration:

$$\Delta(p) = pA_1 + p'A_2 = \begin{pmatrix} 3p + 2p' & -(p + 2p') \\ -(3p + 2p') & p + 2p' \end{pmatrix}$$

and it is easily checked that $u(p) =$ value of $\Delta(p) = 0$ for $0 \leqslant p \leqslant 1$. Hence $\lim_{n \to \infty} v_n(p) = \operatorname{Cav} u(p) = 0; 0 \leqslant p \leqslant 1$. Noticing that also 0 is the value of the game in which *both players* know the actual state of nature, we may interpret this result as saying that in the long run, player I's advantage in information is washed out and eventually he will have to release all his information (or else not to use it).

What is the optimal speed for releasing the information? Mathematically speaking, what is the speed of convergence of $v_n(p)$ to 0? *Aumann* and *Maschler* [1966] have shown that in such games the speed of convergence is bounded in order of magnitude by $1/\sqrt{n}$. For the game under conderation this means:

$$v_n(p) \leqslant \frac{c(p)}{\sqrt{n}}\,;\; 0 \leqslant p \leqslant 1;\; n = 1, 2, \ldots, \tag{2.1}$$

where $c(p)$ is a constant depending on p.

Zamir [1971–1972] has constructed this game just to show that it satisfies:

$$v_n(p) \geqslant \frac{c^*(p)}{\sqrt{n}};\ 0 \leqslant p \leqslant 1;\ n = 1, 2, \ldots, \tag{2.2}$$

for some other p-dependent constant $c^*(p)$. In other words $1/\sqrt{n}$ is not only an upper bound but also *the least upper bound* for the speed of convergence. Combination of (2.1) yields:

$$c^*(p) \leqslant \sqrt{n}\ v_n(p) \leqslant c(p);\ \ 0 \leqslant p \leqslant 1;\ \ n = 1, 2, \ldots . \tag{2.3}$$

Does $\sqrt{n}\ v_n(p)$ converge? If so what is then the limit as a function of the variable p, which measures in a way the initial advantage in information of player I over player II? Theorem 1.1 answers these questions making use of a result on the variation of bounded martingales obtained recently by the authors. The role of bounded martingales in repeated games of incomplete information is so important and so essential that it is worth taking a short while to explain it:

For a given strategy σ of player I (say a behavioral strategy) and a given *history* of moves $\lambda_{m-1} = (i_1, i_2, \ldots, i_{m-1})$ define the conditional probability

$$p_m = Pr(k = 1 \mid \sigma, \lambda_{m-1}).$$

We also write $p_m = Pr(k = 1 \mid \sigma, \cdot)$ and think of p_m as a random variable. It is easily seen then that $\{p_m\}_0^n$, with $p_0 \equiv p$, is a martingale bounded in [0,1].

It turns out that the variation $E(|p_{m+1} - p_m||p_m)$ plays the most important role in the analysis: On one hand it is proportional to the *extra profit* made by player I in stage-m by using his information about k (compared to what he could get if he would play independently of k). On the other hand $E(|p_{m+1} - p_m||p_m)$ measures in a certain way the *amount of information concerning k released by player* I *at stage-m.* For example if player I's strategy at stage-m is to play T (Top row) if $k = 1$ and B (Bottom row) if $k = 2$ then p_{m+1} would be either 1 (if T was chosen) or 0 (if B was chosen). This may be called a *completely revealing strategy* since once it is played, player II knows k with probability 1. For such a strategy $E(|p_{m+1} - p_m||p_m) = 2p_m p'_m$. On the other extreme player I may use the same probability distribution on (T, B) say (s, s'), independently of the value of k. In doing so he is completely ignoring his information at that stage and there is certainly no way for player II to gain any information from such a behavior. In fact in this case $p_{m+1} \equiv p_m$ and $E(|p_{m+1} - p_m||p_m) = 0$.

With this observation it is quite understandable that the accumulated gain of player I due to his extra information will turn out to be measured by $\sum_{m=0}^{\infty} E(|p_{m+1} - p_m|)$ which is nothing else but the variation of the martingale of conditional probabilities $\{p_m\}_0^\infty$.

Let us now turn to the proof of Theorem 1.1.

3. Proof of Theorem 1.1

For any p, $0 \leqslant p \leqslant 1$ let

$$\Delta(p) = pA_1 + p'A_2 = \begin{pmatrix} 3p + 2p' & -(p + 2p') \\ -(3p + 2p') & p + 2p' \end{pmatrix}$$

and observe that the value of $\Delta(p)$ is 0 for $0 \leqslant p \leqslant 1$. To find an upper bound for $\sqrt{n}\ v_n(p)$ we consider the Maxmin of the payoffs. Let $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n)$ be a strategy of player I, where $\sigma^m = (s_1^m, s_2^m)$ is his strategy at stage-m, meaning to play (T, B) with probabilities $(s_1^m, s_1^{m'})$ if $k = 1$ and with probabilities $(s_2^m, s_2^{m'})$ if $k = 2$. ($0 \leqslant s_i^m \leqslant 1$ and $s_i^{m'} = 1 - s_i^m$ for $i = 1, 2$). The strategy σ defines the martingale of conditionals $\{p_m\}_0^n$ by: $p_0 \equiv p$ and for $m = 1, 2, \ldots, n-1$:

$$Pr(p_{m+1} = p_{m+1}(T) \mid p_m) = \overline{s^m}; Pr(p_{m+1} = p_{m+1}(B) \mid p_m) = 1 - \overline{s^m}$$

$$\text{where: } \overline{s^m} = p_m s_1^m + p'_m s_2^m \tag{3.1}$$

$$p_{m+1}(T) = p_m \frac{s_1^m}{\overline{s^m}} \ ; \ p_{m+1}(B) = p_m \frac{s_1^{m'}}{1 - \overline{s^m}} \, .$$

Against σ consider the following response of player II: $\tau = (\tau^1, \tau^2, \ldots, \tau^n)$ where τ^m is an optimal strategy (of player II) in $\Delta(p_m)$. Denote by $H_m(\sigma, \tau)$ the expected payoff at stage-m given σ, τ and p_m, then:

$$H_m(\sigma, \tau) = p_m \sigma_1^m A_1 \tau^m + p'_m \sigma_2^m A_2 \tau^m$$

where

$$\sigma_i^m = (s_i^m, s_i^{m'}) \ ; \ i = 1, 2 \ \text{ and } \ \tau^m = \begin{pmatrix} t^m \\ t^{m'} \end{pmatrix} .$$

Using (3.1) we write

$$H_m(\sigma, \tau) = p_m \bar{\sigma} A_1 \tau^m + [p_m(\sigma_1^m - \bar{\sigma})A_1 + p'_m(\sigma_2^m - \bar{\sigma})A_2]\tau^m + p'_m \bar{\sigma} A_2 \tau^m$$

where (3.2)

$$\bar{\sigma} = (\overline{s^m}, 1 - \overline{s^m}).$$

Now by (3.1):

$$\sigma_1^m - \bar{\sigma} = p'_m(\sigma_1^m - \sigma_2^m)$$

$$\sigma_2^m - \bar{\sigma} = p_m(\sigma_2^m - \sigma_1^m),$$

also since τ^m is optimal in $\Delta(p_m)$:

$$p_m \bar{\sigma} A_1 \tau^m + p'_m \bar{\sigma} A_2 \tau^m = \bar{\sigma} \Delta(p_m) \tau^m \leqslant \text{value of } \Delta(p_m) = 0.$$

From (3.2) we thus obtain:

$$H_m(\sigma, \tau) \leqslant p_m p'_m (\sigma_2^m - \sigma_1^m)(A_2 - A_1)\tau^m. \tag{3.3}$$

But

$$(A_2 - A_1)\tau^m = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t^m \\ t^{m'} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{Hence}$$

$$H_m(\sigma, \tau) \leqslant 2 p_m p'_m (s_1^m - s_2^m) \tag{3.4}$$

(we may assume $s_1^m \geqslant s_2^m$ since otherwise $H_m(\sigma, \tau)$ could be increased by interchanging s_1^m and s_2^m).
By (3.1) we have also (since $s_1^m \geqslant s_2^m$)

$$E(|p_{m+1} - p_m \| p_m) = \overline{s^m}\left(p_m \frac{s_1^m}{\overline{s^m}} - p_m\right) - (1 - \overline{s^m})\left(p_m \frac{1 - s_1^m}{1 - \overline{s^m}} - p_m\right)$$

$$= 2 p_m p'_m (s_1^m - s_2^m).$$

Thus

$$H_m(\sigma, \tau) \leqslant E(|p_{m+1} - p_m \| p_m) \tag{3.5}$$

Taking expectation over all p_m and summing on m from 1 to n we obtain:

$$f_n(\sigma, \tau) \leqslant \sum_{m=1}^{n} E(|p_{m+1} - p_m|) \tag{3.6}$$

where $f_n(\sigma, \tau)$ is the expected accumulated payoff in the n stages of $\Gamma_n(p)$ (i.e. n times the payoff) when σ and τ are played. So:

$$v_n(p) \leqslant \max_\sigma \min_\tau \frac{1}{n} f_n(\sigma, \tau) \tag{3.7}$$

$$\leqslant \max_\sigma \frac{1}{n} \sum_{m=1}^{n} E(|p_{m+1} - p_m|).$$

We now recall the following result of ours (*Mertens* and *Zamir* [1975] Theorem 2.4):

$$\lim_{n\to\infty} \sup_{X_0^n(p)} \frac{1}{\sqrt{n}} V(X_0^n(p)) = \phi(p), \tag{3.8}$$

where

$X_0^n(p) = \{X_m\}_0^n$ is an n-martingale bounded in $[0,1]$ with $E(X_0) = p$ and the sup is taken over all such martingales. V denotes the variation i.e.

$$V(X_0^n(p)) = \sum_{i=0}^{n} E(|X_{i+1} - X_i|).$$

If we denote by $P_0^n = \{p_m\}_0^n$ the martingale of conditional probabilities we have by (3.7)

$$v_n(p) \leq \frac{1}{\sqrt{n}} \max_{\sigma} \frac{1}{\sqrt{n}} V(P_0^n) \leq \frac{1}{\sqrt{n}} \operatorname{Sup}_{X_0^n(p)} \frac{1}{\sqrt{n}} V(X_0^n(p)) \tag{3.9}$$

and by (3.8);

$$\operatorname*{Lim\,Sup}_{n\to\infty} \sqrt{n}\, v_n(p) \leq \phi(p). \tag{3.10}$$

To get a lower bound for $\sqrt{n}\, v_n(p)$ we notice that if in $\Gamma_{n+1}(p)$ player I plays in the first stage: (s, s') if $k = 1$ and (t, t') if $k = 2$, he guarantees for the first stage:

$$\operatorname{Min}(p(3s - 3s') + p'(2t - 2t');\ p(-s + s') + p'(-2t + 2t')). \tag{3.11}$$

Depending on whether he plays T or B in the first stage we have:

$$p_1(T) = Pr(k = 1|T) = \frac{ps}{Pr(T)};\ p_1(B) = Pr(k = 1|B) = \frac{ps'}{Pr(B)}, \tag{3.12}$$

where $Pr(T) = 1 - Pr(B) = ps + p't$.
After the first move player I may play optimally in $\Gamma_n(p_1(T))$ or in $\Gamma_n(p_1(B))$ according to whether he played T or B in the first stage. It follows that:

$$\begin{aligned} v_{n+1}(p) \geq \frac{1}{n+1} \operatorname*{Max}_{0\leq s,t\leq 1} \Big[&\operatorname{Min}(p(3s - 3s') + p'(2t - 2t'); p(-s + s') \\ &+ p'(-2t + 2t')) + n\Big(Pr(T)\, v_n\Big(\frac{ps}{Pr(T)}\Big) + Pr(B)\, v_n\Big(\frac{ps'}{Pr(B)}\Big)\Big)\Big] \end{aligned} \tag{3.13}$$

Actually (3.13) holds as an equality (see *Zamir* [1971–1972] p. 184) but we do not need this stronger statement here.

Restrict now the domain of maximization in (3.13) by; $t = \frac{1}{2} - \frac{p(s - s')}{2p'}$. Then the expression (3.11) becomes $p(s - s')$ and also $Pr(T) = Pr(B) = \frac{1}{2}; p_1(B) = 2ps'$. Denoting $\omega_n(p) = \sqrt{n}\, v_n(p)$ we get from (3.10):

$$\omega_{n+1}(p) \geqslant \frac{1}{\sqrt{n+1}} \underset{0 \leqslant s \leqslant 1}{\operatorname{Max}} \left[p(s-s') + \frac{\sqrt{n}}{2}(\omega_n(2ps) + \omega_n(2ps'))\right] \tag{3.14}$$

$$0 \leqslant \tfrac{1}{2} - \frac{p(s-s')}{2p'} \leqslant 1.$$

We may assume that $s \geqslant \frac{1}{2}$ since otherwise replacing s by s' would increase the expression to be maximized in (3.14).

Define now x by: $2ps = p + x$ which impliy $2ps' = p - x$ and $x = p(s - s')$. Also $0 \leqslant \frac{1}{2} - \frac{p(s-s')}{2p'} \leqslant 1$ implies $0 \leqslant x \leqslant p'$ while $x = p(s-s')$ and $\frac{1}{2} \leqslant s \leqslant 1$ implies $0 \leqslant x \leqslant p$. In terms of the variable x we then rewrite (3.14) as:

$$\omega_{n+1}(p) \geqslant \frac{1}{\sqrt{n+1}} \underset{0 \leqslant x \leqslant p \wedge p'}{\operatorname{Max}} \left[x + \frac{\sqrt{n}}{2}(\omega_n(p+x) + \omega_n(p-x))\right]. \tag{3.15}$$

Define now a sequence $\{U_n(p)\}_0^\infty$ by $U_0(p) \equiv 0$ and

$$U_{n+1}(p) = \frac{1}{\sqrt{n+1}} \underset{0 \leqslant x \leqslant p \wedge p'}{\operatorname{Max}} \left[x + \frac{\sqrt{n}}{2}(U_n(p+x) + U_n(p-x))\right] \tag{3.16}$$

for $n = 0, 1, 2, \ldots$.

Clearly:

$$U_n(p) \leqslant \omega_n(p) \quad \text{for} \quad 0 \leqslant p \leqslant 1 \quad \text{and} \quad n = 1, 2, \ldots . \tag{3.17}$$

Proposition:

$$\lim_{n \to \infty} \operatorname{Inf} U_n(p) \geqslant \phi(p). \tag{3.18}$$

Proof: The proof of this proposition is actually identical to the proof of *Mertens* and *Zamir* [1975] Lemma 3.8 (for a slightly different sequence of functions):

For any n we claim that:

$$U_{n+k}(p) \geqslant \phi(p) - \frac{\sqrt{n}}{2\sqrt{n+k}} - \sum_{i=n}^{n+k} \frac{c}{i^2}, \tag{3.19}$$

for $k = 0, 1, 2, \ldots$ where c is a constant satisfying:

$$\frac{1}{\sqrt{n+1}} \underset{0 \leqslant x \leqslant p \wedge p'}{\operatorname{Max}} \left[x + \frac{\sqrt{n}}{2}(\phi(p+x) + \phi(p-x))\right] \geqslant \phi(p) - \frac{c}{n^2} \tag{3.20}$$

(See *Mertens* and *Zamir* [1975] Lemma 3.5).

(3.19) is proved by induction on k: For $k = 0$ clearly:

$U_n(p) \geqslant 0 \geqslant \phi(p) - \frac{1}{2}$, Assuming (3.19) for k and using (3.16) we have:

$$U_{n+k+1}(p)=\frac{1}{\sqrt{n+k+1}}\underset{0\leqslant x\leqslant p\wedge p'}{\operatorname{Max}}\left\{x+\frac{\sqrt{n+k}}{2}\left(U_{n+k}(p+x)+U_{n+k}(p-x)\right)\right\}$$

$$\geqslant\frac{1}{\sqrt{n+k+1}}\underset{0\leqslant x\leqslant p\wedge p'}{\operatorname{Max}}\left\{x+\frac{\sqrt{n+k}}{2}\left[\phi(p+x)-\frac{\sqrt{n}}{2\sqrt{n+k}}-\sum_{i=n}^{n+k}\frac{c}{i^2}+\right.\right.$$

$$\left.\left.\phi(p-x)-\frac{\sqrt{n}}{2\sqrt{n+k}}-\sum_{i=n}^{n+k}\frac{c}{i^2}\right]\right\}$$

$$\geqslant\frac{1}{\sqrt{n+k+1}}\underset{0\leqslant x\leqslant p\wedge p'}{\operatorname{Max}}\left\{x+\frac{\sqrt{n+k}}{2}\left(\phi(p+x)+\phi(p-x)\right)\right\}-$$

$$\frac{\sqrt{n}}{2\sqrt{n+k+1}}-\sum_{i=n}^{n+k}\frac{c}{i^2},$$

so by (3.20)

$$U_{n+k+1}(p)\geqslant\phi(p)-\frac{c}{(n+k+1)^2}-\frac{\sqrt{n}}{2\sqrt{n+k+1}}-\sum_{i=n}^{n+k}\frac{c}{i^2}$$

$$=\phi(p)-\frac{\sqrt{n}}{2\sqrt{n+k+1}}-\sum_{i=n}^{n+k+1}\frac{c}{i^2}$$

which establishes (3.19) from which we get:

$$\lim_{n\to\infty}\operatorname{Inf}U_n(p)=\lim_{k\to\infty}\operatorname{Inf}U_{n+k}(p)\geqslant\phi(p)-\sum_{i=n}^{\infty}\frac{c}{i^2}.$$

Since this must hold for every n and since $\sum_{i=n}^{\infty}\frac{c}{i^2}<\infty$, the proof of our proposition is completed.

Combining (3.17) and (3.18) and (3.10) we have:

$$\lim_{n\to\infty}\operatorname{Inf}\sqrt{n}\,v_n(p)\leqslant\lim_{n\to\infty}\operatorname{Inf}U_n(p)\geqslant\phi(p)\geqslant\lim_{n\to\infty}\operatorname{Sup}\sqrt{n}\,v_n(p).$$

Thus, finally we conclude that:

$$\lim_{n\to\infty}\sqrt{n}\,v_n(p)=\phi(p)$$

which is what we claimed in Theorem 1.1.

4. On the Heuristics of the Result

In view of the fact that our result is not very intuitive and its proof is based on a result on the maximal variation of a bounded martingale, which is by itself surprising and requires a rather lengthy proof, we feel it would be helpful to outline the heuristic arguments that lead to the result. We hope that this argument will make the result less puzzling.

Our departure point is the rather intuitive recursive formula for $v_n(p)$ proved in *Zamir* [1971–1972, p. 184]. This is just (3.13) written as equality instead of inequality.

1. We assume that, as is usually the case in such a maximization problem, the Max $0 < s, t < 1$ is achieved for s and t that equalizes the two functions under the Min sign. i.e.

$$p(3s - 3s') + p'(2t - 2t') = p(-s + s') + p'(2t + 2t').$$

This leads to the restriction $t = \frac{1}{2} - \frac{p(s-s')}{2p'}$ and hence to the recursion formula (3.16) in which $x = p(s - s')$ and $U_n(p) = \sqrt{n}\, v_n(p)$.

2. Assuming convergence of the functions U_n in (3.16) to some function φ one gets, letting $x = \frac{\alpha_n}{\sqrt{n}}$:

$$\sqrt{1 + \frac{1}{n}}\,\varphi(p) \cong \operatorname*{Max}_{\alpha_n}\left[\frac{\alpha_n}{n} + \frac{1}{2}\left(\varphi\left(p + \frac{\alpha_n}{\sqrt{n}}\right) + \varphi\left(p - \frac{\alpha_n}{\sqrt{n}}\right)\right)\right]$$

$$\cong \operatorname*{Max}_{\alpha_n}\left[\frac{\alpha_n}{n} + \varphi(p) + \frac{\alpha_n^2}{2n}\varphi''(p)\right]$$

$$= \varphi + \frac{1}{n}\operatorname*{Max}_{a_n}\left[\alpha_n + \frac{\alpha_n^2}{2}\varphi''\right] = \varphi - \frac{1}{2n\varphi''}.$$

On the other hand:

$$\sqrt{1 + \frac{1}{n}}\,\varphi \cong \varphi + \frac{1}{2n}\varphi.$$

Thus $\varphi = \frac{-1}{\varphi''}$. In other words, assuming that $\sqrt{n}\, v_n(p)$ converges, the limit is a solution of the differential equation $\varphi\varphi'' + 1 = 0$.

3. From the differential equation $\varphi\varphi'' + 1 = 0$ we have $-\varphi'(p) = \int_{1/2}^{p} \frac{1}{\varphi}\,dp$. We have chosen $\frac{1}{2}$ as a bound of the integration so as to have $\varphi'\ \frac{1}{2} = 0$ which is implied by the symmetry of $\varphi(p)$ about $p = \frac{1}{2}$.

Letting $z(p) = -\varphi'(p) = \int_{1/2}^{p} \frac{1}{\varphi}\,dp$ we have $z'(p) = \frac{1}{\varphi}$ and thus $\varphi = \frac{dp}{dz}$.

4. Now replace in our differential equation the variable p by the variable z:

$$\varphi'_z = \varphi'_p \frac{dp}{dz} = \varphi'_p \varphi = -z\varphi$$

and thus

$$\ln \varphi = K - \frac{1}{2} z^2$$

or

$$\varphi = A \frac{1}{\sqrt{2\pi}} e^{-1/2z^2} \tag{4.1}$$

Since $\varphi = \dfrac{dp}{dz}$ we get:

$$p = c + \int_{-\infty}^{z(p)} A \frac{1}{\sqrt{2\pi}} e^{-1/2x^2}\, dx \tag{4.2}$$

Denoting by $F(x)$ the cumulative standard normal distribution we have therefore:

$$\varphi(z) = AF'(z) \tag{4.3}$$

$$p = c + AF(z) \tag{4.4}$$

Now $\varphi \geqslant 0$ and $\varphi \not\equiv 0$ implies $A > 0$ from which follows by (4.2) that $z(p)$ is monotonously increasing with p. Since $\varphi(0) = \varphi(1) = 0$ we have by (4.1) : $z(0) = -\infty$, $z(1) = +\infty$

From (4.2) we thus have:

$$1 = c + A. \tag{4.4}$$

From $\varphi'\left(\frac{1}{2}\right) = 0$ we have $z\frac{1}{2} = \varphi'\frac{1}{2} = 0$, hence from (4.2):

$$\frac{1}{2} = c + \frac{1}{2} A. \tag{4.5}$$

We conclude from (4.4) and (4.5) that $c = 0$ and $A = 1$, thus finally:
$\varphi(p) = F'(z)$; $p = F(z)$ i.e. the limit $\varphi(p)$ is the standard normal density evaluated at its p-quantile.

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