Proportional Taxation: Nonexistence of Stable Structures in an Economy with a Public Good*

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Guesnerie and Oddou [*J. Econom. Theory* 25 (1981), 67–91] raised the open question whether an economy, in which the production of a public good is financed via proportional taxation, has a stable structure. By means of the first example a negative answer to this question is provided. The second example shows that a stable structure may fail to exist even if all the individuals have the same initial endowments in private good. *Journal of Economic Literature* Classification Number: 022. © 1985 Academic Press, Inc.

INTRODUCTION

In two papers Guesnerie and Oddou (G. O. hereafter, [2, 3]) considered an economy with a single private good and a single local public good. The economy "starts" with n individuals, each of which has an initial endowment of private good only. The private good then serves to produce the public good. Each coalition of individuals may form and agree upon a "tax rate" t $(0 \leqslant t \leqslant 1)$ with the implication that each individual in the coalition contribute t fraction of his initial endowment. All contributions serve then to produce public good which is enjoyable by the members of the coalition *only*.

A core outcome of such an economy is thus a tax rate t such that if all individuals form one coalition and pay according to t, no coalition of players can increase the utility of all its members by forming their own community

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and adopting a different tax rate. The existence of core outcomes is guaranteed only under a rather strong condition on the preferences (see [1, 2]). As the core is typically empty, G.O. introduced then the less demanding and much more appealing concept in this context, that is the concept of *stable structure*. A stable structure is a partition of the players into coalitions, together with a system of tax rates, one for each coalition, with the property that no collection of individuals can *all* benefit from splitting from their respective coalitions and forming a new community.

Do stable structures exist in general? Under the standard assumptions on preferences, G.O. proved that the answer is positive for the following special cases:

(i)   For economies with only three individuals,

(ii)   for economies with four individuals, having all the same initial endowments.

They thus raised the question of existence of stable structures in the general case.

In this note we show that the answer to this question is negative: In general, stable structures may not exist. In our first example we exhibit an economy with four individuals and no stable structure. This example is then used to construct an economy with no stable structure even though all individuals have equal initial endowments.

THE MODEL

The economy $\mathscr{E}$ consists of a set $N = \{1, 2, \ldots, n\}$ individuals, a single private good, x, and a single public good, y. Individual $i \in N$ is endowed with $w_i \in R_+$ units of the private good (and with no public good). His preferences are represented by the continuous and quasi-concave utility function $u^i(x, y): R_+^2 \to R$, which is also nondecreasing in the public good. The public good is produced according to the identity production function $y = x$; i.e., one unit of private good is transformed into one unit of public good. (This assumption can be weakened so that production function is any continuous concave nondecreasing function $y = f(x)$. See [1].) It is assumed that if a group of individuals (a coalition) decides to produce the public good jointly, each member contributes for this end the same percentage of his initial endowment. We can therefore define the indirect utility function of individual i, who belongs to a coalition S, in which the tax rate is t, $0 \leqslant t \leqslant 1$, to be

$$\varphi^i(t, S) \equiv u^i((1 - t)w_i, tw(S)), \qquad \text{where } w(S) \equiv \sum_{j \in S} w_j$$

In order to describe our examples, we need now some definitions.

Definitions

A vector $z \in R^n_+$ is *an outcome* of economy $\mathscr{E}$ if there exists a coalition structure $\pi = \{S_j\}_{j=1,\ldots,J}$ (i.e., partition of N into pairwise disjoint coalitions $S_1,\ldots, S_J$) and tax rates t_j, $0 \leqslant t_j \leqslant 1$, $j = 1,\ldots, J$, such that for all i: $z^i = \varphi^i(t_j, S_j)$, where $i \in S_j \in \pi$. The vector z is said to be an outcome *associated with the structure* π.

A coalition $S \subset N$ *blocks* an outcome z if there is $0 \leqslant t \leqslant 1$ such that $z^i < \varphi^i(t, S)$ for each $i \in S$.

An outcome z associated with the structure π is a *C-stable solution* if there exists no coalition which blocks z. In such a case π is called *a stable coalition structure*.

We now provide two examples which show nonexistence of C-stable solution, firstly for 4-individuals economy, and, secondly, for economy where all individuals have the same initial endowments. All the relevant computations are presented in the Appendix.

EXAMPLE 1. Consider the economy $\mathscr{E}$, which consists of a set $N = \{1, 2, 3, 4\}$ of individuals with the utility functions $u^i(x, y) = \min(A_i x + y, x + B_i y)$ where the parameters A_i, B_i and the initial endowments w_i are given by

$$w^1 = w^2 = w^4 = 1, \quad w^3 = 10, \quad A_1 = 17, \quad A_2 = 50, \quad A_3 = 91, \quad A_4 = 121,$$

$$B_1 = 51, \qquad B_2 = 21, \qquad B_3 = 11, \qquad B_4 = 134, 7.$$

Remark. By an arbitrarily small change the utility functions can be made strictly concave and differentiable without changing the special features of our example.

EXAMPLE 2. Consider the economy $\mathscr{E}'$, which consists of a set $N' = \{1, 2, a_1, a_2,\ldots, a_{10}, 4\} - 13$ individuals, where the individuals $\{1\}$, $\{2\}$, and $\{4\}$ are the same as in economy $\mathscr{E}$, and individual $\{3\}$ is replaced by ten identical individuals with the initial endowment of one unit of private good ($w_{a_k} = 1$, $k = 1, 2,\ldots, 10$), and utility function $u^{a_k}(x, y) \equiv u^3(10x, y) = \min(910\, x + y, 10x + 11y)$.

Remark. Note that $\varphi^{a_k}(t, S) \equiv \varphi^0(t, S) = \varphi^3(t, T)$, where $S \subset N'$, $T \subset N$, and $w(S) = w(T)$. Therefore if $\{3\}$ joins a certain coalition $S \subseteq N\backslash\{3\}$ and pays a tax rate t, he achieves (in $\mathscr{E}$) the same utility as each of the players $\{a_k\}$ if they would all join the same coalition (in $\mathscr{E}'$) and pay the same tax rate.

PROPOSITION 1. *The economy $\mathscr{E}$ has no stable structure.*

PROPOSITION 2. *The economy $\mathscr{E}'$ has no stable structure.*

Proof of Proposition 1

For each $i \in N$ let $y_i^0 = \max_t \varphi^i(t, \{i\})$, the individual rationality level (i.r.l. hereafter) of player i. For each $S \subset N$ containing i consider $\{t \mid \varphi^i(t, S) \geqslant y_i^0\}$ the set of the tax rates which are individually rational for i when S forms. In our example it is easily seen that this set is a closed interval which we shall denote by $[\alpha_i^S, \beta_i^S]$ where

$$\alpha_i^S = \frac{y_i^0 - w_i}{B_i w(S) - w_i}; \qquad \beta_i^S = \frac{A_i w_i - y_i^0}{A_i w_i - w(S)}.$$

(Sometimes we shall write $\alpha_i^{i_1, \dots, i_k}$ instead of $\alpha_i^{\{i_1, \dots, i_k\}}$, and similarly for β_i^S.)

Since the dependence of α_i^S and β_i^S on S is only through $w(S)$—the total initial endowment of S—it follows that, for instance, $\alpha_1^{12} = \alpha_1^{14}$, $\alpha_2^{12} = \alpha_2^{24}$, $\alpha_4^{14} = \alpha_4^{24}$, and similarly for β_i^S. Let us make now the following observations.

(i) α_i^S decreases in $w(S)$ and β_i^S increases in $w(S)$. When calculating α_i^S, β_i^S (see values in the Appendix) we find that

(ii) $\alpha_3^N > \beta_4^N$. Combined with (i) this means that no coalition containing both $\{3\}$ and $\{4\}$ can ensure to them their i.r.l., y_3^0 and y_4^0. Therefore no coalition structure with $\{3\}$ and $\{4\}$ in a same coalition can be stable.

(iii) $\alpha_1^{12} < \alpha_4^{24} < \beta_1^{14} < \alpha_2^{12} < \beta_4^{24} < \beta_2^{12}$. Note that the coalitions $\{14\}$ and $\{24\}$ with the tax rates which belong to $(\alpha_4^{14}, \beta_1^{14})$ and $(\alpha_2^{24}, \beta_4^{24})$, respectively, guarantee to their members utility levels which are strictly higher than their i.r.l. Therefore, no coalition structure, in which $\{1\}$ and $\{4\}$, or $\{2\}$ and $\{4\}$, are singletons, can be stable. Moreover, the coalition $\{12\}$ is unable to ensure the i.r.l. for both $\{1\}$ and $\{2\}$; therefore, no coalition structure containing $\{12\}$ can be stable.

(iv) $\alpha_1^{13} < \alpha_2^{23} < \beta_1^{13} < \alpha_3^{13} < \beta_3^{23} < \beta_2^{23}$. Using the same arguments as in (iii) we conclude that any stable structure cannot contain $\{13\}$ as an element and cannot contain both $\{2\}$ and $\{3\}$ as singletons.

In view of observations (i)–(iv) the only remaining candidates for stable structures are

(1) $\{\{124\}\{3\}\}$,   (2) $\{\{123\}\{4\}\}$,   (3) $\{\{1\}\{3\}\{24\}\}$,   (4) $\{\{14\}\{23\}\}$,

and the proof of Proposition 1 will be completed by showing that none of these structures is stable.

By straightforward computations we find that for $i \in S$

$$t_i^S = \frac{(A_i - 1)w_i}{(A_i - 1)w_i + (B_i - 1)w(S)}$$

is the tax rate which yields for player i the maximum utility level he can achieve in S, and $\varphi^i(t, S)$ can be written as

$$\varphi^i(t, S) = w_i + t(B_i w(S) - w_i) \qquad \text{if} \quad t \leqslant t_i^S$$
$$= A_i w_i - t(A_i w_i - w(S)) \qquad \text{if} \quad t \geqslant t_i^S.$$

Consider now the above-mentioned structures one by one:

(1) $\{\{124\}\{3\}\}$. In order to give to $\{1\}$ at least his i.r.l. the tax rate in coalition $\{124\}$ cannot exceed β_1^{124}. But, since $t_2^{124} > \beta_1^{124}$, individual $\{2\}$ cannot achieve more than $\varphi^2(\beta_1^{124}, \{124\})$. Note that $\varphi^2(\alpha_3^{23}, \{23\}) > \varphi^2(\beta_1^{124}, \{124\})$. Therefore there exists $\varepsilon > 0$ small enough, such that

$$\varphi^3(\alpha_3^{23} + \varepsilon, \{23\}) > \varphi^3(\alpha_3^{23}, \{23\}) = y_3^0$$
$$\varphi^2(\alpha_3^{23} + \varepsilon, \{23\}) > \varphi^2(\beta_1^{124}, \{124\}).$$

Thus, the structure $(\{124\}\{3\})$ is unstable.

(2) $\{\{123\}\{4\}\}$. Individual $\{3\}$ achieves his i.r.l. in coalition $\{123\}$ only if the tax rate t in this coalition satisfies $t \geqslant \alpha_3^{123}$. $t_1^{123} < \alpha_3^{123}$ implies that in this case individual $\{1\}$ does not achieve more than $\varphi^1(\alpha_3^{123}, \{123\})$. Note that since $\varphi^1(\alpha_4^{14}, \{14\}) > \varphi^1(\alpha_3^{123}, \{123\})$, there exists $\varepsilon > 0$ small enough such that

$$\varphi^1(\alpha_4^{14} + \varepsilon, \{14\}) > \varphi^1(\alpha_3^{123}, \{123\})$$
$$\varphi^4(\alpha_4^{14} + \varepsilon, \{14\}) > y_4^0.$$

Therefore any outcome associated with coalition structure $\{\{123\}\{4\}\}$ which ensures to $\{3\}$ his i.r.l. is blocked by coalition $\{14\}$ and thus this structure is not stable.

(3) $\{\{1\}\{3\}\{24\}\}$. In order to ensure to individual $\{4\}$ his i.r.l. in coalition $\{24\}$, the tax rate t in this coalition should satisfy $t \leqslant \beta_4^{24}$. But since $t_2^{24} > \beta_4^{24}$, the maximal utility level that $\{2\}$ can achieve is $\varphi^2(\beta_4^{24}, \{24\})$. On the other hand, $\alpha_1^{123} < \alpha_3^{123} < \beta_1^{124} < \beta_3^{123}$; therefore any t with $\alpha_3^{123} < t < \beta_1^{123}$ in coalition $\{123\}$ guarantees to $\{1\}$ and $\{3\}$ their i.r.l. Recalling that $\varphi^2(\alpha_3^{123}, \{123\}) > \varphi^2(\beta_4^{24}, \{24\})$ we conclude, since all the vectors associated with the structure $\{\{1\}\{3\}\{24\}\}$ and ensuring i.r.l. to $\{2\}$ and $\{3\}$ are blocked by $\{123\}$, so this structure is not stable.

(4) $\{\{14\}\{23\}\}$. In order to ensure to individual $\{1\}$ his i.r.l. with a tax t in coalition $\{14\}$, t satisfies $t \leqslant \beta_1^{14}$. But, since $t_4^{14} > \beta_1^{14}$, $\text{Max}_{t \leqslant \beta_1^{14}} \varphi^4(t, \{14\}) = \varphi^4(\beta_1^{14}, \{14\})$. The similar argument shows that any tax rate enabling individual $\{3\}$ to achieve his i.r.l. in coalition $\{23\}$ cannot exceed α_2^{23}. Again $t_2^{23} < \alpha_3^{23}$ implies that individual $\{2\}$ does not

achieve more than $\varphi^2(\alpha_3^{23}, \{23\})$. Now observe that for $\hat{t} = 0.424$ the following inequalities hold:

$$\varphi^4(\hat{t}, \{24\}) > \varphi^4(\beta_1^{14}, \{14\})$$

and

$$\varphi^2(\hat{t}, \{24\}) > \varphi^2(\alpha_3^{23}, \{23\}).$$

Therefore, each vector associated with the coalition structure $\{\{14\}\{23\}\}$ which ensures i.r.l. to individuals $\{1\}$ and $\{3\}$ is blocked by coalition $\{24\}$, i.e., $\{\{14\}\{23\}\}$ is not stable structure, which completes the proof of Proposition 1.

Proof of Proposition 2

Suppose the contrary; i.e., there is a stable structure $\pi = \{S_j\}_{j=1,\ldots,J}$ of economy $\mathscr{E}'$ with corresponding tax rates $\{t_j\}_{j=1,\ldots,J}$, which give rise to the C-stable solution $\mathbf{v} \in R_+^{13}$.

CLAIM.   *There is $S \in \pi$, such that $S_0 \subset S$, where $S_0 = \bigcup_{k=1}^{10} \{a_k\}$.*

Proof.   Assume, by negation, that there are $\{a_k\}$ and $\{a_l\}$ which belong to two different coalitions in π; i.e., without loss of generality $\{a_k\} \in S_1$, $\{a_l\} \in S_2$, and let $v^{a_k} = \varphi^3(t_1, S_1) \geqslant v^{a_l} = \varphi^3(t_2, S_2)$. Then the coalition $S_1 \cup \{a_l\}$ with the tax rate t_1 blocks $\mathbf{v}$, since all members of S_1 become better off with joining $\{a_l\}$ and keeping the tax rate t_1, and also $\varphi^0(t_1, S_1 \cup \{a_l\}) > \varphi^3(t_1, S_1) \geqslant \varphi^3(t_2, S_2) = v^{a_l}$. This contradicts the stability of $\mathbf{v}$.   Q.E.D.

Now consider a structure $\tilde{\pi}$ of economy $\mathscr{E}$ which is obtained from $\pi = \{S_j\}_{j=1,\ldots,J}$ by replacing S_0 by individual $\{3\}$, and an outcome $\mathbf{z} \in R_+^4$ associated with $\tilde{\pi}$ via the same tax rates as in π. Since $w(S_0) = w^3$, $v^i = z^i$ if $i = 1, 2, 4$. Moreover, $v^{a_k} = \varphi^3(t_j, S_j)$, where $S_0 \subset S_j \in \pi$, t_j is a corresponding tax rate, and $z^3 = \varphi^3(t_j, (\{1, 2, 4\} \cap S_j) \cup \{3\})$. Recalling that $\varphi^3(t, S) = \varphi^3(t, T)$ if $w(S) = w(T)$, we conclude that $v^{a_k} = z^3$ for all $k = 1, \ldots, 10$. Now, by Proposition 1, z is not a C-stable solution. Therefore there exists $\tilde{T}$ with tax rate $\tilde{t}$ which blocks z. If $\{3\} \notin \tilde{T}$, then $\tilde{T}$ also blocks $\mathbf{v}$. To complete the proof of Proposition 2, it suffices to show that if $\{3\} \in \tilde{T}$ then $\tilde{S} = (\tilde{T} \backslash \{3\}) \cup S_0$ with the tax rate $\tilde{t}$ blocks $\mathbf{v}$. But clearly

$$\varphi^i(\tilde{t}, \tilde{S}) = \varphi^i(\tilde{t}, \tilde{T}) > z^i = v^i \qquad \text{for } i \in \{1, 2, 4\} \cap \tilde{S}$$

$$\varphi^{a_k}(\tilde{t}, \tilde{S}) = \varphi^3(\tilde{t}, \tilde{T}) > z^3 = v^{a_k} \qquad \text{for all } k = 1, 2, \ldots, 10. \qquad \text{Q.E.D.}$$

Remark.   Facing the negative conclusion that stable structures may not exist even when all individuals have equal initial endowments, one possible direction to proceed is to impose further assumptions on the preferences and

identify situations in which stable structures do exist. In this line, we can prove, for instance, this statement in the case of linear preferences and also for a certain class of utility functions containing Cobb–Douglas preferences.

APPENDIX: The List of the Evaluations Used in the
Proof of Proposition 1

(1) y_i^0, the individually rational level of individual i, is

$$\frac{A_i B_i - 1}{A_i + B_i - 2} w_i.$$

Therefore,

$$y_1^0 = 13{,}121; \qquad y_2^0 = 15{,}203; \qquad y_3^0 = 100; \qquad y_4^0 = 64{,}240.$$

(2) α_i^S and β_i^S, minimal and, respectively, maximal tax rate which guarantees the individually rational level for individual i in coalition S, are

$$\alpha_i^S = \frac{y_i^0 - w_i}{B_i w(S) - w_i}, \qquad \beta_i^S = \frac{A_i w_i - y_i^0}{A_i w_i - w(S)}.$$

Then,

$$\alpha_1^{12} = \alpha_1^{14} = 0.120; \qquad \alpha_1^{13} = 0.022; \qquad \alpha_1^{123} = 0.020;$$

$$\beta_1^{12} = 0.277; \qquad \beta_1^{14} = \beta_1^{12} = 0.259; \qquad \beta_1^{123} = 0.776;$$

$$\alpha_2^{12} = \alpha_2^{24} = 0.346; \qquad \alpha_2^{23} = 0.062; \qquad \beta_2^{12} = \beta_2^{24} = 0.725; \qquad \beta_2^{23} = 0.892;$$

$$\alpha_3^{13} = \alpha_3^{23} = 0.811; \qquad \alpha_3^{123} = 0.738; \qquad \alpha_3^{N} = 0.677;$$

$$\beta_3^{13} = \beta_3^{23} = 0.901; \qquad \beta_3^{123} = 0.902;$$

$$\alpha_4^{14} = \alpha_4^{24} = 0.236; \qquad \beta_4^{14} = \beta_4^{24} = 0.477; \qquad \beta_4^{N} = 0.526.$$

(3) t_i^S, the tax rate which maximizes the utility of individual i in coalition S, is

$$t_i^S = \frac{(A_i - 1) w_i}{(A_i - 1) w_i + (B_i - 1) w(S)}.$$

(Clearly $\alpha_i^S < t_i^S < \beta_i^S$.) Then

$$t_1^{14} = 0.138; \qquad t_1^{123} = 0.026;$$

$$t_1^{14} = 0.170; \qquad t_2^{23} = 0.182; \qquad t_2^{124} = 0.450; \qquad t_2^{24} = 0.551;$$

$$t_4^{14} = t_4^{24} = 0.310.$$

(4)   The indirect utility functions $\varphi^i(t, S)$ are defined by

$$(1 - t\,A_i)w_i + tw(S) \equiv \bar{\varphi}^i(t, S) \qquad \text{if} \quad t \geqslant t_i^S$$
$$(1 - t)w_i + tB_i w(S) \equiv \underline{\varphi}^i(t, S) \qquad \text{if} \quad t \leqslant t_i^S.$$

The values that we need are

$$\varphi^1(\alpha_4^{14}, \{14\}) = \bar{\varphi}^1(0.236, \{14\}) = 13{,}460,$$
$$\varphi^1(\alpha_3^{123}, \{1, 2, 3\}) = 13{,}310,$$
$$\varphi^1(\alpha_3^{13}, \{123\}) = \bar{\varphi}^1(0.811, \{123\} = 12{,}945,$$
$$\varphi^2(\alpha_3^{23}, \{3\}) = \bar{\varphi}(0.811, \{23\}) = 18{,}371,$$
$$\varphi^2(\beta_1^{124}, \{124\}) = \underline{\varphi}^2(0.277, \{124\}) = 18{,}174,$$
$$\varphi^2(\alpha_3^{123}, \{123\}) = \bar{\varphi}^2(0.738, \{123\}) = 21{,}956,$$
$$\varphi^2(\beta_4^{24}, \{24\}) = \underline{\varphi}^2(0.477, \{24\}) = 20{,}557,$$
$$\varphi^2(\hat{t}, \{24\}) = \underline{\varphi}^2(0.424, \{14\}) = 18{,}384,$$
$$\varphi^4(\hat{t}, \{24\}) = \bar{\varphi}^4(0.424, \{24\}) = 70{,}544,$$
$$\varphi^4(\beta_1^{14}, \{14\}) = \underline{\varphi}^4(0.259, \{14\}) = 70{,}510.$$

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