The Annals of Statistics
1973, Vol. 1, No. 4, 791-796

ON THE NOTION OF VALUE FOR GAMES WITH INFINITELY MANY STAGES[1]

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The value of a zero-sum two-person game with infinite number of stages can be defined either directly or as the limit of the values v_n of the truncated games with n stages. It is shown that these two concepts are not equivalent. There are games in which $\lim v_n$ exists but which do not have values as infinite stage games.

Introduction. We consider a relatively new class of multistage games called *repeated games of incomplete information*: A chance move chooses, (according to a known distribution of finite support) the payoffs of a zero-sum two-person game Γ given by a finite matrix. Each player is given partial information (possibly no information) about Γ which is then played repeatedly between the same two players. After each stage, each player receives some information which may depend only on the actual game chosen by chance and on both players' moves in that stage. We are interested in the *average payoff per stage* in such a multistage game. Denote by Γ_n the game with n stages, and by Γ_∞ the game with infinitely many stages. Clearly Γ_n has a (minimax) value. As for the notion of value for Γ_∞, two alternative approaches seem quite natural:

(i) *The asymptotic approach* according to which the game with infinite number of stages is a "limit" of the n-stage games Γ_n as $n \to \infty$. The suggested value would then be $\lim_{n\to\infty} v_n$ where v_n is the value of Γ_n, provided that this limit exists.

(ii) *The direct approach* according to which an infinite stage game Γ_∞ is defined directly with infinite dimensional strategy spaces. In spite of some technical difficulties Γ_∞ and its value v_∞ can be defined in this way. Clearly v_∞ may or may not exist.

Besides the mathematical difference, the two approaches reflect a difference in the situation modeled by the infinite stage game: In the asymptotic approach the infinite stage game is an approximation for a game with a very large—but known—number of stages. In the second approach Γ_∞ is a model for a game with a very large number of stages in which the players do not know exactly the number of stages that are going to be played.

The main mathematical question in this context is: Are the two notions of value, $\lim v_n$ and v_∞, equivalent? In particular, does the existence of one of them imply the existence of the other?

In most of the games treated in the literature, including stochastic games (see

Received October 1972; revised December 1972.

[1] This research was supported in part by NSF Grant No. GP-33431X.

for example [3] and [5]), v_∞ and $\lim v_n$ were equivalent and the computation of $\lim v_n$ was considered a "method" to find the value v_∞.

It is shown in this paper that $\lim v_n$ *and* v_∞ *are not equivalent*. There are games in which $\lim v_n$ exists while v_∞ does not exist.

In Section 1 we define Γ_n, Γ_∞ and v_∞. In Section 2 we discuss an auxiliary game which will be used in Section 3 to construct a game in which v_∞ does not exist while $\lim v_n$ exists.

1. The notion of value. We denote by L and M the finite sets of pure strategies of player I and II respectively in Γ. The sets of pure strategies in Γ_n will be $L^n = L \times \cdots \times L$ and $M^n = M \times \cdots \times M$. In Γ_∞ the sets will be $L^\infty = L \times L \times \cdots$ and $M^\infty = M \times M \times \cdots$. Let $\lambda_k{}^{\mathrm{I}}$ and $\lambda_k{}^{\mathrm{II}}$ denote the kth-stage information state of player I and II respectively, i.e. $\lambda_k{}^{\mathrm{I}}(\lambda_k{}^{\mathrm{II}})$ is all the relevant information known to player I (player II) after the $(k-1)$th stage as he comes to play the kth stage. ($\lambda_k{}^{\mathrm{I}}$ may contain among other things any available information about previous moves of the players or chance.) Denote by $\Lambda_k{}^{\mathrm{I}}$ and $\Lambda_k{}^{\mathrm{II}}$ the set of all possible kth-stage information states for player I and player II respectively. (In this paper $\Lambda_k{}^{\mathrm{I}}$ and $\Lambda_k{}^{\mathrm{II}}$ will be finite for all k.)

Let Ω be the measure space resulting from imposing the Lebesgue measure on the unit interval $[0, 1]$. *A mixed strategy* of player I is a sequence $\sigma = (\sigma_1, \sigma_2, \cdots)$ (n-tuple in the case of Γ_n and an infinite sequence in the case of Γ_∞) of measurable functions $\sigma_k : \Omega \times \Lambda_k{}^{\mathrm{I}} \to L$. A mixed strategy for player II is defined similarly. (This is the definition suggested by Aumann in [1] (page 638).)

In comparing Γ_n and Γ_m for $n \neq m$, the relevant notion of payoff is that of "payoff per stage." Therefore we define the payoff in Γ_n to be $n^{-1} \sum_{j=1}^n h_j$ where h_j is the payoff in Γ at stage j. With this convention Γ_n is well defined and all Γ_n, $n = 1, 2, 3, \cdots$, are comparable both in payoffs and in values.

In Γ_∞ the natural definition of payoff would be $\lim_{n\to\infty} n^{-1} \sum_{j=1}^n h_j$, but as pointed out by Aumann and Maschler in [2], this limit may fail to exist for a given pair of strategies. Nevertheless, the value v_∞ can be defined without defining the payoff: For any pair of mixed strategies σ and τ in Γ_∞ let $\rho_n(\sigma, \tau) = E(n^{-1} \sum_{j=1}^n h_j(\sigma, \tau))$ be the expected average payoff in the first n stages.

DEFINITION. The infinite stage game Γ_∞ is said to have a value v_∞ if for each $\varepsilon > 0$ there exist a pair of strategies; σ_ε for player I and τ_ε for player II such that:

$$\limsup \rho_n(\sigma, \tau_\varepsilon) \leqq v_\infty + \varepsilon\,, \qquad \text{for all } \sigma\,, \tag{1}$$

$$\liminf \rho_n(\sigma_\varepsilon, \tau) \geqq v_\infty - \varepsilon\,, \qquad \text{for all } \tau\,. \tag{2}$$

2. An auxiliary game. We describe our auxiliary example schematically by:

$$\begin{array}{lcc} & \tfrac{1}{2} & \tfrac{1}{2} \\ \text{payoff matrices} & G_1 = \begin{pmatrix} 0 & 8 \\ 0 & 8 \end{pmatrix} & G_2 = \begin{pmatrix} 8 & 0 \\ 8 & 0 \end{pmatrix} \\ \text{information matrices} & H_1 = \begin{pmatrix} a & a \\ b & c \end{pmatrix} & H_2 = \begin{pmatrix} a & a \\ d & c \end{pmatrix}. \end{array}$$

The game Γ_n (or Γ_∞) is played as follows: Chance chooses one of the two 0-sum 2-person games G_1 and G_2 each with probability $\frac{1}{2}$. In addition to the payoff we are interested in the stage of information of the players after a play of the game which is described by the "information matrices" H_1 and H_2 respectively. This may be interpreted to mean that after each play of the game a referee announces one of four letters, a, b, c, or d according to which strategies were chosen by the players and which game was chosen by chance. (Thus if a is announced, player II learns that player I played the top strategy; if c is announced, both players learn which strategy was chosen by the other player, and if b or d is announced, both players find out which strategy was chosen by the other player *and* which game was chosen by chance.)

After chance has chosen one of the games, *neither* player is informed of the choice. The chosen game is then played repeatedly: n times in the case of Γ_n, infinitely many times in the case of Γ_∞. After each play of the chosen game, the players are credited (or debited) with the appropriate payoff, and the referee makes an announcement in accordance with the information matrix. The players are *not* informed directly of the payoff. (It may be thought of as being held for them until the end of the supergame.) They *do* remember, in addition to the information announced for previous stages, the strategies they themselves chose at the previous stages.

The main feature of the information matrices in our example is that no information about the choice of chance is gained by either player unless I plays the bottom strategy and II plays the left strategy, in which case the choice of chance is completely revealed to both players, and the average payoff from that stage on, is 0 (up to an error term of the order of $1/n$).

LEMMA 2.1. *Both v_∞ and $\lim_{n\to\infty} v_n$ exist and*

$$v_\infty = \lim_{n\to\infty} v_n = 4 .$$

PROOF. We denote the (pure) strategies of player I by T (top) and B (bottom) and those of player II by L (left) and R (right).

Let σ_0 be the strategy of player I to play T in all stages. Clearly, for any strategy τ of player II and any n,

$$\rho_n(\sigma_0, \tau) = 4 .$$

Let τ_0 be the strategy of player II to play L until (if ever) d is announced and then to play R repeatedly. Then for any strategy σ of player I and any n we have $\rho_n(\sigma, \tau_0) \leqq 4$. Hence $v_n = 4$ which yields $\lim_{n\to\infty} v_n = 4$. Evidently we have also: $\liminf \rho_n(\sigma_0, \tau) = \limsup \rho_n(\sigma, \tau_0) = 4$ for all σ and τ, which means $v_\infty = 4$.

3. The main example. The game in our main example is of the same type as that of the auxiliary game. The only difference is that here chance chooses one of *three* 0-sum 2-person games. The probabilities, the payoff matrices and the information matrices are:

$$\begin{array}{llll} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \text{payoff matrices} & G_1 = \begin{pmatrix} 0 & 8 \\ 0 & 8 \end{pmatrix} & G_2 = \begin{pmatrix} 8 & 0 \\ 8 & 0 \end{pmatrix} & G_3 = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \\ \text{information matrices} & H_1 = \begin{pmatrix} a & a \\ b & c \end{pmatrix} & H_2 = \begin{pmatrix} a & a \\ d & c \end{pmatrix} & H_3 = \begin{pmatrix} a & a \\ e & f \end{pmatrix}. \end{array}$$

Again we denote the (pure) strategies of the players by T, B and L, R. The supergames Γ_n and Γ_∞ are played as described in the auxiliary game.

Let us consider first the situation resulting in the supergame when a certain pair of (pure) strategies is played at some stage:

(T, L) No information about the choice of chance is gained (by any player) and the expected payoff is 2 (for the stage under consideration).

(T, R) No information about the choice of chance is gained and the expected payoff is 0.

(B, L) The choice of chance is completely revealed to both players and the (average) expected payoff in the supergame is determined to be 0 (up to an error term) as the players will play optimally in the chosen game.

(B, R) The choice of chance is partly revealed, namely: with probability $\frac{1}{2}$, f will be announced which implies that chance has chosen G_3 and therefore the payoff in the supergame is determined to be 0. With probability $\frac{1}{2}$, c will be announced which brings the players to the situation described in the auxiliary game. If in that situation the players play optimally, the payoff in the supergame is determined to be 4. We conclude that when (B, R) is once played, the expected payoff to the supergame is actually determined to be $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 4 = 2$.

THEOREM 3.1.

$$1 \leqq v_n \leqq 1 + \frac{1}{n}, \qquad n = 1, 2, \cdots.$$

PROOF. We will outline the proof. It is straightforward to fill up the formal setup and details. It can be found in [7].

Let τ_0 be the following strategy of player II: Play $(\frac{1}{2}, \frac{1}{2})$ as long as a is being announced and switch to the corresponding optimal strategy when b, c, d, e, or f is once announced. It is easily seen that $\rho_n(\sigma, \tau_0) \leqq 1 + (1/n)$ for any strategy σ of player I.

In constructing an optimal strategy for player I in Γ_n we make use of the minimax theorem and assume that player I knows player II's strategy τ, from which he can calculate at each stage i the (conditional) probability t_i of choosing L. Now let σ_0 be the following strategy: At stage i play the corresponding optimal strategy if b, c, d, e or f has already been announced; otherwise play T if $t_i \geqq \frac{1}{2}$ and B if $t_i < \frac{1}{2}$. Since clearly $\rho_n(\sigma_0, \tau) \geqq 1$ for any strategy τ of player II, this concludes the proof.

COROLLARIES.

(i) $\lim_{n\to\infty} v_n = 1$.

(ii) *Player* II *can guarantee* 1 *in* Γ_∞. *In fact for the above described strategy* τ_0 *regarded as a strategy in* Γ_∞ *we have*:

$$\limsup \rho_n(\sigma, \tau_0) \leqq 1 \qquad \textit{for all } \sigma .$$

(iii) *For any strategy* τ *of player* II *in* Γ_∞, *player* I *has a strategy—namely the above described* σ_0*—such that*:

$$\liminf \rho_n(\sigma_0, \tau) \geqq 1 .$$

THEOREM 3.2. *For any strategy* σ *of player* I *in* Γ_∞ *and for any* $\varepsilon > 0$, *there is a strategy* τ *of player* II *such that* $\limsup \rho_n(\sigma, \tau) < \varepsilon$.

PROOF. Given σ let:

$$s_n = \text{Prob}\,\{\text{I plays } T \text{ in stage } n \mid \text{only } a \text{ is announced in the first } n-1 \text{ stages}\} .$$

Let $\alpha = \prod_{n=1}^{\infty} s_n$. ($\alpha$ surely exists and satisfies $0 \leqq \alpha \leqq 1$) and consider the two cases:

(i) $\alpha = 0$. Then the required "answer" τ is: Play L until something other than a is announced and then switch to the corresponding optimal strategy. When σ and τ are played, the choice of chance will be completely revealed (i.e. b, d, or e announced) with probability $1 - \prod_{n=1}^{\infty} s_n = 1 - \alpha = 1$, which means that with probability 1 the payoff will be 0 from some stage on. This implies easily that $\limsup \rho_n(\sigma, \tau) = 0$.

(ii) $\alpha > 0$. Let M and N be positive integers such that $(1 - \prod_{M+1}^{\infty} s_n) < \varepsilon/4$ and $N > 4M/\varepsilon$. The answer τ to σ in this case is: Play L repeatedly. If at any stage $i \leqq M$ an information other than a is announced play the corresponding optimal strategy from that stage on; otherwise play R from stage $M+1$ on. Up to stage M, any information other than a reduces the payoff to 0 from that stage on. The probability of first announcing anything other than a *after* the Mth stage is $\prod_1^M s_n(1 - \prod_{M+1}^{\infty} s_n) < \varepsilon/4$. Since in such an event the expected payoff is 2 (and this is the only event that involves any expected payoff greater than 0 after the Mth stage) we have that when σ and τ are played:

$$n > N \Rightarrow \rho_n(\sigma, \tau) < \frac{2M}{n} + 2 \cdot \frac{\varepsilon}{4} < \varepsilon ,$$

which completes the proof of the theorem.

CONCLUSION. Theorem 3.2 and Corollary (iii) of Theorem 3.1 imply that in the game under consideration v_∞ *does not exist*, while $\lim_{n\to\infty} v_n$ *exists* by Corollary (i) of Theorem 3.1.

REMARKS. Examples of games with existing $\lim v_n$ and nonexisting v_∞ can also be found in [4] but there the proof of this fact requires three rather complicated works, namely [2], [4], and [6]. The advantage of our example is its simplicity and its self-containment. It shows clearly the essential difference between Γ_∞ and $\lim \Gamma_n$: In Γ_∞ a player can afford to suffer losses during a finite number of stages however big this number may be. These losses will "wash out" because of the infinite horizon of the game. In the games with a finite horizon, Γ_n, such losses may be of great importance. To put it in different words: A truncation of an optimal Bayesian strategy in Γ_∞ (e.g. the one described for player II in the proof of Theorem 3.2), may not be close to an optimal Bayesian strategy in Γ_n.

It may be also remarked that the example here preceded and motivated the results in [4].

Acknowledgment. I am indebted to Professor R. J. Aumann for many important discussions on this subject.

REFERENCES

[1] AUMANN, R. J. (1964). Mixed and behavior strategies in an infinite extensive game. *Ann. Math. Studies*, No. 52, Princeton Univ. Press. 627–650.

[2] AUMANN, R. J. and MASCHLER, M. (1968). Repeated games with incomplete information. The zero-sum extensive case. Report to the U.S. Arms Control and Disarmament Agency, Washington, D.C.; final report on contract ACDA/ST-143, prepared by MATHEMATICA, Princeton. 25–108.

[3] BLACKWELL, D. and FERGUSON, T. S. (1968). The big match. *Ann. Math. Statist.* **39** 159–163.

[4] MERTENS, J. F. and ZAMIR, S. (1971). The value of two-person zero-sum repeated games with lack of information on both sides. *Internat. J. Game Theory* **1** 39–64.

[5] SHAPLEY, L. S. (1953). Stochastic games. *Proc. Nat. Acad. Sci. U.S.A.* **39** 1095–1100.

[6] STEARNS, R. E. (1967). A formal information concept for games with incomplete information. Report to the U.S. Arms Control and Disarmament Agency, Washington, D.C.; final report on contract ACDA/AT-116, prepared by MATHEMATICA, Princeton. New Jersey, 405–433.

[7] ZAMIR, S. (1970). The inequivalence between the two approaches to repeated games with incomplete information. *Research Program in Game Theory and Mathematical Economics*, Department of Mathematics, The Hebrew University of Jerusalem.

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