Chapter 6

Radon Measures

6.1 Locally-compact Hausdorff spaces

Definition 6.1 *Let $(\mathbb{X}, \tau)$ be a topological space. It is called* ***hausdorff*** *if every two points $x, y \in \mathbb{X}$ have disjoint open neighborhoods. That is, there exist $U, V \in \tau$ such that*

$$x \in U \quad y \in V \qquad \text{and} \qquad U \cap V = \varnothing.$$

Definition 6.2 *Let $(\mathbb{X}, \tau)$ be a topological space. It is called* ***locally-compact*** *(קומפקטי מקומית) if every point $x \in \mathbb{X}$ has an open neighborhood whose closure is compact.*

Examples:

(a) Every compact space is locally-compact.

(b) $\mathbb{R}^n$ is a locally-compact hausdorff space (and of course, any space homeomorphic to it).

(c) $\{(0,0)\} \cup \{(x,y) \,:\, x > 0, y \in \mathbb{R}\} \subset \mathbb{R}^2$ is hausdorff but not locally-compact, since the origin does not have an open neighborhood whose closure is compact.

This section is concerned with topological spaces that are hausdorff and locally-compact (LCH) . Throughout this chapter, the σ-algebra will be the Borel sets $\mathscr{B}(\mathbb{X})$.

*Definition 6.3 Let $\mathbb{X}$ be a topological space and let $f : \mathbb{X} \to \mathbb{R}$. The **support** (תומך) of f is*

$$\operatorname{Supp}(f) = \overline{\{x \in \mathbb{X} : f(x) \neq 0\}}.$$

Definition 6.4 We denote by $C_c(\mathbb{X})$ the set of continuous functions $\mathbb{X} \to \mathbb{R}$ that have compact support (תומך קומפקטי).

For $f \in C_c(\mathbb{X})$, we denote

$$\|f\|_\infty = \sup_{x\in X} |f(x)|.$$

We endow $C_c(\mathbb{X})$ with the **compact convergence topology**. It is defined through converging sequences: a sequence f_n converges to f if it converges to f uniformly on every compact set

Comment: If $\mathbb{X}$ is compact, then $C_c(\mathbb{X}) = C(\mathbb{X})$, which is a complete normed space.

Comment: When $\mathbb{X}$ is not compact, functions in $C_c(\mathbb{X})$ are not necessarily bounded.

Example: Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 1 - |x-n| & |x-n| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

This sequence belongs to $C_c(\mathbb{R})$ and converges to zero in the compact convergence topology. Note however that $\|f_n\|_\infty = 1$, so that this is not a convergence in norm.
▲▲▲

There are two facts about LCH spaces, which are normally taught in a topology course, and which will be used in this chapter:

Lemma 6.5 (Urysohn) Let $\mathbb{X}$ be an LCH space. Let $U \subset \mathbb{X}$ be open and let $K \Subset U$. Then, there exists an $f \in C_c(\mathbb{X})$, such that $f|_K = 1$ and $\operatorname{Supp}(f) \Subset U$. In particular, taking $U = \mathbb{X}$, there exists a function $f \in C_c(\mathbb{X})$, such that $f|_K = 1$.

Definition 6.6 For open sets $U \subset \mathbb{X}$ and functions $f \in C_c(\mathbb{X})$, we denote by

$$f \prec U$$

the fact that $0 \le f \le 1$ *and* $\operatorname{Supp}(f) \subset U$. *(Note that this condition is stronger than* $0 \le f \le \chi_U$ *since* $\operatorname{Supp}(\chi_U) = \bar{U}$.*)*

In other words, Urysohn's lemma states that for $K \Subset U$ there exists an $\chi_K \le f \prec U$.

Theorem 6.7 (Partition of unity (פיצול יחידה)) Let $\mathbb{X}$ be an LCH space. Let $U_1, \dots, U_n$ be a finite open cover of $K \Subset \mathbb{X}$. Then, there exist functions $\psi_j \prec U_j$, such that

$$\sum_{j=1}^{n} \psi_j = 1 \qquad \text{on } K.$$

6.2 Radon measures

Radon measures are a subset of Borel measures, satisfying regularity conditions. Their importance will be seen in the next section.

Definition 6.8 Let μ be a Borel measure on $\mathbb{X}$ and let $A \in \mathscr{B}(\mathbb{X})$. The measure μ is said to be ***outer-regular*** *(רגולרית חיצונית) on A if*

$$\mu(A) = \inf\{\mu(U) \,:\, A \subset U \in \tau_{\mathbb{X}}\}.$$

It is said to be ***inner-regular*** *(רגולרית פנימית) on A if*

$$\mu(A) = \sup\{\mu(K) \,:\, K \Subset A\}.$$

Example: We proved that locally-finite Borel measures on $\mathbb{R}$ are both outer- and inner-regular. ▲▲▲

Definition 6.9 A ***Radon measure*** *(מידת רדון) on $\mathbb{X}$ is a Borel measure that is*

(a) Finite on all compact sets.

(b) Outer-regular on all Borel sets.

(c) Inner-regular on all open *sets.*

✎ *Exercise 6.1* Show that a Dirac measure on any topological space is a Radon measure.

In view of the fact that every Borel measure on $\mathbb{R}$ (and hence also $\mathbb{R}^n$) which is finite on every compact set is regular, one may wonder whether there are examples of Borel measures satisfying that property which are not regular, and even not Radon measure. We state the following theorem, without a proof:

Theorem 6.10 Let $\mathbb{X}$ be a LCH space in which every open set is σ-compact (i.e., a countable union of compact sets). Then every Borel measure on $\mathbb{X}$ that is finite on compact sets is regular and hence a Radon measure.

Proposition 6.11 In particular, if $\mathbb{X}$ is second-countable, then it satisfies this condition.

6.3 Positive functionals on $C_c(\mathbb{X})$

$C_c(\mathbb{X})$ is a **topological vector space** (מרחב וקטורי טופולוגי) (which means that it is a vector space endowed with a topology, such that vector addition and scalar multiplication are continuous). Such spaces are often studied via spaces of linear functionals that operate on them.

Definition 6.12 A linear functional $I \in \operatorname{Hom}(C_c(\mathbb{X}); \mathbb{R})$ is called ***positive*** *if $I(f) \geq 0$ for every $f \geq 0$.*

Note that positivity does not mention any notion of continuity, however, the following proposition establishes a connection between the two:

Proposition 6.13 Let I be a positive linear functional on $C_c(\mathbb{X})$. Then, for each $K \Subset \mathbb{X}$ there exists a constant C_K, such that

$$|I(f)| \leq C_K \|f\|_\infty$$

for all $f \in C(K)$, where

$$C(K) = \{f \in C_c(\mathbb{X}) \ : \ \operatorname{Supp} f \Subset K\}.$$

Comment: In particular, if $\mathbb{X}$ compact, then positive functionals are bounded.

Proof: Let $K \Subset \mathbb{X}$ be given. Let $\phi \in C_c(\mathbb{X})$ be non-negative and satisfy $\phi|_K = 1$ (here we use Urysohn's lemma). For every $f \in C(K)$,

$$|f| \le \|f\|_\infty \phi.$$

Then,

$$\|f\|_\infty \phi - f \ge 0 \qquad \text{and} \qquad \|f\|_\infty \phi + f \ge 0,$$

and by the linearity and positivity of I,

$$\|f\|_\infty I(\phi) - I(f) \ge 0 \qquad \text{and} \qquad \|f\|_\infty I(\phi) + I(f) \ge 0,$$

from which we get that

$$|I(f)| \le I(\phi)\, \|f\|_\infty,$$

Setting $C_K = I(\phi)$ we obtain the desired result. ■

Lemma 6.14 Let μ be a Borel measure on $\mathbb{X}$. If $\mu(K) < \infty$ for every $K \Subset \mathbb{X}$, then $C_c(\mathbb{X}) \subset L^1(\mu)$.

Proof: Let $f \in C_c(\mathbb{X})$ have support in $K \Subset \mathbb{X}$. Then,

$$\int_{\mathbb{X}} |f|\, d\mu = \int_K |f|\, d\mu \le \|f\|_\infty\, \mu(K) < \infty.$$

■

Corollary 6.15 Let μ be a Borel measure on $\mathbb{X}$, such that $\mu(K) < \infty$ for every $K \Subset \mathbb{X}$. The map

$$f \mapsto \int_{\mathbb{X}} f\, d\mu$$

is a positive linear functional on $C_c(\mathbb{X})$.

Proof: By the previous lemma, the right-hand side is finite. Positivity is immediate. ■ —32h(2018)—

Theorem 6.16 (Riesz representation) For every positive linear functional I on $C_c(\mathbb{X})$ there exists a unique Radon measure μ on X, such that

$$I(f) = \int_{\mathbb{X}} f\,d\mu.$$

Moreover, for all open U,

$$\mu(U) = \sup\{I(f) \;:\; f \prec U\} \tag{6.1}$$

and for all compact K,

$$\mu(K) = \inf\{I(f) \;:\; \chi_K \le f\}. \tag{6.2}$$

Comment: Frigyes Riesz proved it in 1909 for continuous functions on $\mathbb{R}$. Andrei Markov extended it in 1938 for some non-compact spaces. Shizuo Kakutani
—38h(2017)— finally extended it in 1941 for locally-compact hausdorff spaces.

Proof: **Step 1: Uniqueness**: Suppose that μ is a Radon measure satisfying

$$I(f) = \int_{\mathbb{X}} f\,d\mu.$$

We will see that I determines μ uniquely.

Let U be open. Then,

$$\begin{aligned}
\sup\{\mu(K) \;:\; K \Subset U\} &= \sup\left\{\int_{\mathbb{X}} \chi_K\,d\mu \;:\; K \Subset U\right\} \\
&\le \sup\{I(f) \;:\; \chi_K \le f \prec U,\;\; K \Subset U\} \\
&\le \sup\{I(f) \;:\; f \prec U\} \\
&\le \int_{\mathbb{X}} \chi_U\,d\mu \\
&= \mu(U),
\end{aligned}$$

Since μ is inner-regular on U, the left-hand side and the right-hand side are equal, from which we deduce that

$$\mu(U) = \sup\{I(f) \;:\; f \prec U\}.$$

Thus, I determines μ uniquely on all open sets. Since μ is outer-regular, I determines μ on all Borel sets.

Step 2: Define a set function μ on open sets: Let I be given. Define a set function $\mu : \tau_{\mathbb{X}} \to \bar{\mathbb{R}}$ by

$$\mu(U) = \sup\{I(f) \,:\, f \prec U\}.$$

Note that if $U \subset V$, then $f \prec U$ implies that $f \prec V$, hence $\mu(U) \le \mu(V)$; that is, μ is monotone.

Likewise, define a set function $\mu^* : \mathscr{P}(\mathbb{X}) \to \bar{\mathbb{R}}$,

$$\mu^*(Y) = \inf\{\mu(U) \,:\, Y \subset U \in \tau_{\mathbb{X}}\}.$$

Note that $\mu^*(U) = \mu(U)$ for open sets; it follows that μ^* is monotone as well.

Step 3: Prove that μ is sub-additive: Let (U_n) be a sequence of open sets and let $U = \bigcup_{n=1}^\infty U_n$. Let $f \prec U$ and let $K = \text{Supp}(f)$. Since K is compact and (U_n) is an open cover of K, there exists a finite set $n_1, \dots, n_r$, such that

$$K \subset \bigcup_{j=1}^r U_{n_j}.$$

Let $\psi_j \prec U_{n_j}$ be a partition of unity for K, namely,

$$\sum_{j=1}^n \psi_j = 1 \qquad \text{on } K.$$

Now,

$$f = \sum_{j=1}^n \psi_j f \qquad \text{and} \qquad \psi_j f \prec U_{n_j}.$$

Using the linearity of I and the definition of μ for open sets,

$$I(f) = \sum_{j=1}^n I(\psi_j f) \le \sum_{j=1}^n \mu(U_{n_j}) \le \sum_{n=1}^\infty \mu(U_n).$$

Since this holds for every $f \prec U$, taking the supremum over f,

$$\mu(U) \le \sum_{n=1}^\infty \mu(U_n).$$

Step 4: Prove that μ^* is the outer-measure induced by μ: For every $Y \subset \mathbb{X}$, using the sub-additivity of μ,

$$\mu^*(Y) = \inf\{\mu(U) \,:\, Y \subset U \in \tau_{\mathbb{X}}\} \le \inf\left\{\sum_{n=1}^\infty \mu(U_n) \,:\, Y \subset \bigcup_{n=1}^\infty U_n\right\}.$$

This inequality is in fact an equality because we can take $U_1 = U$ and $U_n = \varnothing$ for $n \geq 2$. By Proposition 2.39, μ^* is an outer-measure on $\mathbb{X}$ (the outer-measure induced by μ).

Step 5: Prove that every open set is μ^*-measurable, i.e., $\tau_{\mathbb{X}} \subset \sigma(\mu^*)$: We need to show that for every open set U and $Y \subset \mathbb{X}$,

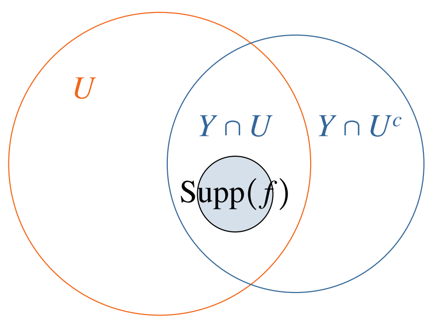
$$\mu^*(Y) = \mu^*(Y \cap U) + \mu^*(Y \cap U^c).$$

It suffices to prove an inequality for sets Y of finite outer-measure. Suppose first that Y is open. Then, $Y \cap U$ is open, hence, given $\varepsilon > 0$, there exists an $f \prec Y \cap U$, such that

$$I(f) > \mu(Y \cap U) - \varepsilon.$$

Moreover, $Y \setminus \operatorname{Supp}(f)$ is open, hence there exists a $g \prec (Y \setminus \operatorname{Supp}(f))$, such that

$$I(g) > \mu(Y \setminus \operatorname{Supp}(f)) - \varepsilon.$$



Putting it together, and since $f + g \prec Y$,

$$\begin{aligned}
\mu(Y) &\geq I(f+g) \\
&= I(f) + I(g) \\
&> \mu(Y \cap U) + \mu(Y \setminus \operatorname{Supp}(f)) - 2\varepsilon \\
&= \mu^*(Y \cap U) + \mu^*(Y \setminus \operatorname{Supp}(f)) - 2\varepsilon \\
&\geq \mu^*(Y \cap U) + \mu^*(Y \cap U^c) - 2\varepsilon.
\end{aligned}$$

Since this holds for every $\varepsilon > 0$, we obtain the desired results, so far, only for open sets.

Let now Y be an arbitrary set. By the definition of μ^*, given $\varepsilon > 0$, there exists an open set $V \supset Y$, such that

$$\mu^*(V) \leq \mu^*(Y) + \varepsilon.$$

Hence,

$$\begin{aligned}\mu^*(Y)+\varepsilon &\geq \mu^*(V)\\ &= \mu^*(V\cap U)+\mu^*(V\cap U^c)\\ &\geq \mu^*(Y\cap U)+\mu^*(Y\cap U^c),\end{aligned}$$

and it remains to take $\varepsilon \to 0$.

Step 6: Prove that every Borel set is μ^*-measurable, i.e., $\mathscr{B}(\mathbb{X}) \subset \sigma(\mu^*)$: Since the collection of μ^*-measurable sets forms a σ-algebra containing the open sets, it contains all the Borel sets.

Step 7: Prove that $\mu^*|_{\mathscr{B}(\mathbb{X})}$ is a Borel measure extending μ: This follows from Carathéodory's theorem; we denote this measure by μ, with no ambiguity.

Step 8: Prove that μ is outer-regular and satisfies (6.1): μ satisfies (6.1) by definition. It is outer-regular by the definition of μ^*. That is, for every Borel set A,

$$\mu(A) = \mu^*(A) = \inf\{\mu(U) \;:\; A \subset U \in \tau_{\mathbb{X}}\}.$$

Step 9: Prove that μ satisfies (6.2): We need to show that for every compact K,

$$\mu(K) = \inf\{I(f) \;:\; \chi_K \leq f \in C_c(\mathbb{X})\}.$$

Let K be compact and $\chi_K \leq f \in C_c(\mathbb{X})$. Let

$$U_\varepsilon = \{x \;:\; f(x) > 1-\varepsilon\}.$$

Note that $K \Subset U_\varepsilon$. Since f is continuous, U_ε is open. Since $f|_{U_\varepsilon} > 1-\varepsilon$, for every $g \prec U_\varepsilon$,

$$\frac{f}{1-\varepsilon} - g \geq 0,$$

and by the positivity of I,

$$\frac{I(f)}{1-\varepsilon} \geq I(g).$$

It follows that

$$\mu(K) \leq \mu(U_\varepsilon) = \sup\{I(g) \;:\; g \prec U_\varepsilon\} \leq \frac{I(f)}{1-\varepsilon}.$$

Since this holds for every $\varepsilon > 0$, $\mu(K) \leq I(f)$, i.e.,

$$\mu(K) \leq \inf\{I(f) \;:\; \chi_K \leq f \in C_c(\mathbb{X})\}.$$

By Urysohn's lemma, there exists for every open U satisfying $K \Subset U$ a function $f \prec U$ satisfying $f \ge \chi_K$, and

$$I(f) \le \mu(U).$$

I.e.,

$$\inf\{I(f) \,:\, \chi_K \le f \in C_c(\mathbb{X})\} \le \mu(U).$$

Since μ is outer-regular,

$$\mu(K) \le \inf\{I(f) \,:\, \chi_K \le f \in C_c(\mathbb{X})\} \le \inf\{\mu(U) \,:\, K \Subset U \in \tau_{\mathbb{X}}\} = \mu(K).$$

Step 10: Prove that μ is finite on compact sets: This is an immediate consequence of (6.2), as $\mu(K) \le I(f)$, for every $\chi_K \le f \in C_c(\mathbb{X})$.

Step 11: Prove that μ is inner-regular on open sets: Let U be open and let $\alpha < \mu(U)$. Choose $f \prec U$, such that $I(f) > \alpha$ (such an f exists by the definition of μ). Let $K = \operatorname{Supp}(f)$ and let $\chi_K \le g \in C_c(\mathbb{X})$. Then, $g - f \ge 0$, hence

$$I(g) > I(f) > \alpha.$$

By (6.2), taking the infimum over all g,

$$\mu(K) > I(f) > \alpha.$$

That is, for every $\alpha < \mu(U)$ there exists a $K \Subset U$, such that $\mu(K) > \alpha$.

Step 12: Prove that $I(f) = \int_{\mathbb{X}} f\,d\mu$: Thus, given I there exists a unique μ satisfying a collection of properties. We still need to show that I is the integral of its argument with respect to μ. Since I is linear, it suffices to prove that

$$I(f) = \int_{\mathbb{X}} f\,d\mu$$

for f whose image is $[0,1]$. Given $N \in \mathbb{N}$, set

$$K_j = \{x \,:\, f(x) \ge j/N\}, \qquad j = 1, \dots, N.$$

These sets are decreasing. Set also $K_0 = \operatorname{Supp}(f)$.

Then define

$$f_j(x) = \begin{cases} 0 & x \notin K_{j-1} \\ f(x) - (j-1)/N & x \in K_{j-1} \setminus K_j \\ 1/N & x \in K_j. \end{cases}$$

One may verify that

$$f = \sum_{j=1}^{N} f_j$$

and

$$\frac{\chi_{K_j}}{N} \le f_j \le \frac{\chi_{K_{j-1}}}{N}.$$

Integrating,

$$\frac{1}{N}\mu(K_j) \le \int_{\mathbb{X}} f_j \, d\mu \le \frac{1}{N}\mu(K_{j-1}).$$

Let U be open with $K_{j-1} \Subset U$. Then, $Nf_j \prec U$ hence $I(f_j) \le \mu(U)/N$. By (6.2) and outer-regularity,

$$\frac{1}{N}\mu(K_j) \le I(f_j) \le \sup\{\mu(U)/N \,:\, K_{j-1} \Subset U\} = \frac{1}{N}\mu(K_{j-1}).$$

Summing over j,

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \le \int_{\mathbb{X}} f \, d\mu \le \frac{1}{N}\sum_{j=1}^{N}\mu(K_{j-1}),$$

and

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \le I(f) \le \frac{1}{N}\sum_{j=1}^{N}\mu(K_{j-1}).$$

It follows that

$$\left| I(f) - \int_{\mathbb{X}} f \, d\mu \right| \le \frac{\mu(K_0) - \mu(K_N)}{N} \le \frac{\mu(\mathrm{Supp}(f))}{N}.$$

It remain to take $N \to \infty$.

■