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TRIPLE POINTS: FROM NON-BROWNIAN FILTRATIONS TO HARMONIC MEASURES

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Introduction

A smooth boundary is two-sided: $\partial U_1 \cap \partial U_2 \cap \partial U_3 = \emptyset$ for any smoothly bounded domains (connected open sets) U_1, U_2, U_3 of $\mathbb{R}^d$ or any manifold. For piecewise smooth boundaries, $\dim(\partial U_1 \cap \partial U_2 \cap \partial U_3) \le d-2$. Triple points are rare. For irregular domains the situation differs (the phenomenon of Brouwer, see [L]). There is an infinite sequence of pairwise nonoverlapping domains $U_n \subset \mathbb{R}^3$ with equal boundaries: $\partial U_1 = \partial U_2 = \dots$ (in addition, ∂U_n may be of positive three-dimensional Lebesgue measure). Each U_n reminds one of the blood circulation system, branching from artery to capillary vessels, while ∂U_n is the tissue supplied with blood. Each boundary point is accessible from each domain by a continuous path of finite length.

A boundary is conjectured to be two-sided for arbitrary domains under the right definition for sides, pointed out by Bishop [Bi, Sect. 6] in terms of the Martin boundary: its natural projection to the topological boundary should be at most 2 to 1 almost everywhere w.r.t. the natural measure on the Martin boundary.

An equivalent formulation without Martin boundaries can be given (Bishop [Bi], Eremenko, Fuglede, Sodin [ErFuSo2]) in terms of harmonic measures. Consider the idea for the "blood vessel-type" domains. A Brownian particle starting from an interior point of such a domain U exits from U through the wall of a vessel (but probably not through the tissue), which is essential for boundary-value problems in U. The probability distribution of the exit point is the well-known harmonic measure for U. The measure depends on the starting point, but its type does not (the type of a measure means the class of all equivalent measures, where equivalence is mutual absolute continuity). Let us define the harmonic boundary of U as the above measure type on $\overline{U} \setminus U$. The intersection of several harmonic boundaries may be defined as another measure type such that a measure is absolutely continuous w.r.t. the intersection if and only if it is absolutely continuous w.r.t. each of the given harmonic boundaries. The intersection is said to be empty if the only such measure is zero.

Considerable progress has been made recently, and is furthered in this paper, to understand harmonic boundaries. In contrast to the topological boundary $\overline{U} \setminus U$, the harmonic boundary of a three-dimensional domain is of Hausdorff dimension strictly less than 3 [Bo] but sometimes greater than 2 [Wo]. For each d there is a finite N_d such that the intersection of $N_d + 1$ harmonic boundaries is always empty; see [Bi], where the following estimates are given: $N_d \leq 10$ for $d \geq 4$, and $N_3 \leq 4$. See also [FriH], [O1,2] for related estimates. The equality $N_d = 2$ is conjectured for all d ([Bi, Sect. 6] and [ErFuSo2]) but proved only for $d = 2$ ([Bi], [ErFuSo1]): two-dimensional topology excludes the "blood vessels" phenomenon (not to be confused with the "Wada lakes" mentioned in [Bi, p. 20]; these fail to access their boundary points by continuous paths).

One of the two main results of the present paper (Theorem 7.4) states that $N_d = 2$ for all d. Thus, "Problem a" of [ErFuSo2] is solved, and the conjecture of Bishop [Bi] is proved. The intersection of three harmonic boundaries is always empty. In this respect there is no distinction between smoothly bounded domains and irregular domains! Anyway, triple points are rare; a boundary is two-sided.

Probabilistic arguments are usual when dealing with harmonic measures. The phrase "We think about harmonic measure in terms of hitting probability of Brownian motion" [Bo, p. 478] is equally applicable here, but the following phrase is not: "It seems clear that all — or almost all — the arguments involving Brownian motion in this paper can be translated into the language of classical potential theory" [O1, p. 180]. The result $N_d = 2$ will be proved by using Brownian motion not only as a suitable language. Stochastic analysis is involved far beyond the customary strong Markov property. This is a challenge: can the result $N_d = 2$ be achieved by non-stochastic arguments?

The motif of triple points brings together the topic of classical analysis, discussed above, and the following topic of stochastic analysis: diffusion processes on graphs. A graph is treated here not as a discrete scheme but as a one-dimensional topological space with branching points. In such a space a harmonic boundary need not be two-sided, see [ErFuSo1, Sect. 7, "An example from axiomatic potential theory"]. A diffusion process on a graph is a simple, natural, and useful idea, arising when considering the movement of nutrients in the root system of a plant [FrDu], small random perturbations of Hamiltonian dynamical systems [Fre], large-scale geometry of discrete groups [V], and some other topics (see [BPY]). A canonical case, well-known as Walsh's Brownian motion (see [W], [BPY]) can be described as a complex-valued continuous martingale $Z(t)$ such that $|Z(t)|^2 - t$ is also a martingale, and $Z^3(t) \in [0, +\infty)$ for all $t \in [0, +\infty)$. Its phase space $\{z \in \mathbb{C} : z^3 \in [0, +\infty)\} = \{re^{2\pi ki/3} : r \in [0, +\infty), k = 0, 1, 2\}$ consists of three rays connected at the triple point 0. The two-ray counterpart of $Z(t)$, a continuous real-valued martingale $B(t)$ such that $B^2(t) - t$ is a martingale, is just the usual one-dimensional Brownian motion. Processes $|Z(t)|$ and $|B(t)|$ are identically distributed; each is a so-called reflecting Brownian motion. In fact, $B(t)$ can be obtained from the reflecting Brownian motion by assigning independent equiprobable random signs ± 1 to its excursions, and $Z(t)$ can be obtained similarly by assigning independent equiprobable random phases 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$. The simple description suggests that the distinction between processes $Z(t)$ and $B(t)$ should not be deeper than that between $B(t)$ and $|B(t)|$, which is misleading: we cannot assign the phases (or signs) in real time. (I apologise for the non-standard terminology, formally introduced only after Def. 1.1 but, hopefully, it is self-explanatory on the intuitive level.) It is well-known (see sect. 1) that a real-time deterministic machine can produce a Brownian motion from a reflecting Brownian motion (not by assigning signs, of course). Can it produce Walsh's Brownian motion? This was an open problem [BPY, Problem 2]. One of the two main results of this paper (Theorem 4.14) solves the problem: $Z(t)$ cannot be produced in real time from a Brownian motion, nor from a finite or countable collection of independent Brownian motions.

Consider the three rays as domains U_1, U_2, U_3 in the one-dimensional space $\{z \in \mathbb{C} : z^3 \in [0, +\infty)\}$. The harmonic measure of U_k is concentrated at 0, thus the intersection of three harmonic boundaries is nonempty, which is impossible in $\mathbb{R}^d$. Accordingly, the Brownian motion on $\mathbb{R}^d$ cannot produce Walsh's Brownian motion. The triple point is an essential singularity, while an endpoint is not!

The result pertaining to stochastic analysis will be proved first. The other result, pertaining to classical analysis, will follow. A brighter light is shed on their relation by a recent result of M. Barlow, M. Emery, F. Knight, S. Song and M. Yor [BEKSoY]: in some sense (see also the end of sect. 4), a boundary is two-sided in the infinite-dimensional space of Brownian sample paths, which implies the result for $\mathbb{R}^d$.

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1 From Stochastic Calculus to Stochastic Topology

For time-challenged non-probabilists, I suggest a short presentation of stochastic calculus [Me2] (containing no proofs) and Chapter 1 of [E] to introduce the subject. Books for further reading are recommended therein (see also Bass [Ba]). The book by Revuz and Yor [RY] will be referred to frequently.

A filtration is an increasing family $\mathcal{F} = (\mathcal{F}(t))_{t\in[0,\infty)}$ of sub-σ-fields $\mathcal{F}(t) \subset \mathcal{F}(\infty)$ on a probability space $(\Omega, \mathcal{F}(\infty), P)$. (Traditionally, a probability space is denoted by $(\Omega, \mathcal{F}, P)$ and a filtration by $(\mathcal{F}_t)_{t\in[0,\infty)}$, which becomes inconvenient when dealing with several filtrations $\mathcal{F}_1, \dots, \mathcal{F}_n$.) The triple $(\Omega, \mathcal{F}, P) = (\Omega, (\mathcal{F}(t))_{t\in[0,\infty)}, P)$ is called a filtered probability space (or a stochastic basis). It is assumed that $(\Omega, \mathcal{F}(\infty), P)$ is standard, that is, isomorphic mod 0 to $[0,1]$ with Lebesgue measure, or a smaller (maybe empty) interval plus atoms; $\mathcal{F}(0)$ contains all sets of probability 0; $\mathcal{F}(t) = \cap_{\varepsilon>0}\mathcal{F}(t+\varepsilon)$; and $\mathcal{F}(\infty)$ is the least σ-field containing all $\mathcal{F}(t)$. A random process is a map $X : [0,\infty) \times \Omega \to \mathbb{R}$ whose restriction to each $[0,t]\times\Omega$ is jointly measurable w.r.t. the Borel σ-field on $[0,t]$ and the σ-field $\mathcal{F}_t$ on Ω. (That is, only progressively measurable processes are considered, see [RY, I.4.7].) The assumptions of this paragraph are implicit throughout the paper. The reader may restrict himself to filtrations satisfying the "absolute continuity condition" (see the following two paragraphs). The restriction can be relaxed, as noted after Def. 2.3.

Each filtration $\mathcal{F}$ determines the corresponding set $\mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ of all local martingales (assumed right-continuous, starting from 0) and is uniquely determined by $\mathcal{M}_{\mathrm{loc}}(\mathcal{F})$; in fact, $\mathcal{F}$ is generated by a countable subset of $\mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ (just take a basis (X_k) of $L_2(\Omega, \mathcal{F}(\infty), P)$ and let $M_k(t) = E(X_k \big| \mathcal{F}(t))$). It can be shown by some tricks (see [Sk1, Example 2 on p. 168] for one of them) that any filtration is generated by a single martingale, but we do not need it. Usually defined via stopping (see [Me2, p. 138]; [E, 1.5]; [RY, IV.1.5 and in addition, 1.7, 4.1, and V.1.24, 2.13]), $\mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ may be defined equivalently as the closure of the separable Hilbert space of L_2-bounded martingales in a weaker topology, so-called ucp-topology, metrizable (see [E, 1.3]) but not normed, corresponding to the convergence in probability, uniformly on finite time intervals. (See also [E, 4.43].) We restrict ourselves to filtrations $\mathcal{F}$ satisfying the following *"continuity condition":* each $M \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ is continuous (almost surely). The filtration generated by a Brownian motion (the so-called Brownian filtration) satisfies the continuity condition, as well as the filtration generated by a continuous Feller process (that is, a time-homogeneous sample-continuous Markov process whose semigroup sends continuous functions into continuous functions). Walsh's Brownian motion is another example. We borrow convenient notation from [E]: for any continuous processes X and Y,

$$X \overset{\mathrm{m}}{=} Y \quad \text{means} \quad X - Y \in \mathcal{M}_{\mathrm{loc}}\,.$$

Any $M \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ such that $M^2(t) - t$ is also a local martingale, necessarily is a Brownian motion (P. Lévy's characterization theorem [RY, IV.3.6]). Any $M \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ determines its "quadratic variation" — an increasing process $\langle M, M\rangle$ such that $M^2 - \langle M, M\rangle \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F})$. (Informally, $\langle M, M\rangle(t) = \int_0^t (dM(s))^2$.) The process $\langle M, M\rangle$ may be treated as another time, then M turns into a Brownian motion (see [RY, V.1]). We restrict ourselves to filtrations $\mathcal{F}$ satisfying the following *"absolute continuity condition":* the above continuity condition is satisfied, and for any $M \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ the process $\langle M, M\rangle$ is absolutely continuous (that is, almost all sample paths of $\langle M, M\rangle$ are absolutely continuous functions on $[0, \infty)$). The condition is satisfied by a filtration generated by a Brownian motion or a Walsh Brownian motion.

Such notions as "Brownian motion" or "Walsh's Brownian motion" are subordinate to a given filtration. If a process is a Brownian motion then its sample paths are distributed according to the Wiener measure, but the converse does not hold. Let sample paths of a process X be distributed according to the Wiener measure. Then, indeed, X is a Brownian motion w.r.t. the filtration $\mathcal{F}_X$ generated by X; $\mathcal{F}_X(t) \subset \mathcal{F}(t)$. Future increments $X(t + \Delta t) - X(t)$ do not depend on the past $\mathcal{F}_X(t)$ of the process X, but still may depend on the whole past $\mathcal{F}(t)$, which is forbidden for a Brownian motion.

An isomorphism between two filtered probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ or (abusing the language) between two filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$ is, by definition, a $\operatorname{mod} 0$ isomorphism between probability spaces $(\Omega_1, \mathcal{F}_1(\infty), P_1)$ and $(\Omega_2, \mathcal{F}_2(\infty), P_2)$ sending $\mathcal{F}_1(t)$ to $\mathcal{F}_2(t)$ for each t. If $\mathcal{F}_1$ is generated by a random process X and $\mathcal{F}_2$ is generated by a random process Y, then an isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_2$ is what we call a reversible real-time transformation of X into Y. A famous example (Lévy, Skorokhod)is a reversible real-time transformation of a Brownian motion $X(t)$ into a reflecting Brownian motion $Y(t)$, given by the formula

$$Y(t) = X(t) - \inf_{s \in [0,t]} X(s)\,.$$

The inverse transformation is far from being evident:

$$X(t) = Y(t) - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(Y(s)<\varepsilon)} \, ds \, .$$

For a detailed discussion see [RY, VI.2]. The evident non-invertible transformation $Y(t) = |X(t)|$ is an example of an irreversible real-time transformation, as defined below. (Some interesting transformations of a Brownian motion into itself are described in [Y2, Sect. 17.3].) The definition is chosen such that the formula $Y(t) = X(t-1)$ for $t \ge 1$, $Y(t) = 0$ for $0 \le t \le 1$ is *not* a real-time transformation of a Brownian motion X into another process Y. The reason is that Y is a martingale w.r.t. its own filtration $\mathcal{F}_Y$, but not w.r.t. $\mathcal{F}_X$: knowing $X(t)$ we can predict $Y(t+1)$ with certainty. The following definition stipulates that, given the past of Y, knowledge of the past of X gives no additional information about the future of Y. The notion defined below (though not in the same words) may be found in [GS, Sect. 7], [IWa, Chap.2, Def.7.1], [DFSmT, Def.6.1].

1.1 Definition. A morphism from a filtered probability space $(\Omega_1, \mathcal{F}_1, P_1)$ to a filtered probability space $(\Omega_2, \mathcal{F}_2, P_2)$, or (abusing the language) from a filtration $\mathcal{F}_1$ to a filtration $\mathcal{F}_2$, is a measure preserving map $\pi : \Omega_1 \to \Omega_2$ satisfying two conditions:

(1) π is measurable from $(\Omega_1, \mathcal{F}_1(t), P_1)$ to $(\Omega_2, \mathcal{F}_2(t), P_2)$ for any t,
(2) for any t and any $A \in \mathcal{F}_2$

$$\mathbb{P}\left(A \middle| \mathcal{F}_2(t)\right)(\pi\omega) = \mathbb{P}\left(T^{-1}(A) \middle| \mathcal{F}_1(t)\right)(\omega)$$

for almost all $\omega \in \Omega_1$.

Here $\mathbb{P}(\cdot|\cdot)$ means the conditional probability; we suppress probability measures (P, P_n, Q and others) whenever they are uniquely determined by context; this time, $\mathbb{P}$ means $\mathbb{P}_{P_2}$ on the left-hand side but $\mathbb{P}_{P_1}$ on the right-hand side.

Conditions (1), (2) are equivalent to the following: π induces an embedding of $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}_2)$ into $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}_1)$. That is, for any local martingale $M_2 \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F}_2)$ the following process M_1 belongs to $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}_1)$: $M_1(t,\omega) = M_2(t, \pi\omega)$.

So, the phrase "the process X can be transformed into the process Y in real time" means that there is a morphism from $\mathcal{F}_X$ to $\mathcal{F}_Y$.

It is well-known that a one-dimensional Brownian motion cannot be transformed in real time into a two-dimensional Brownian motion. For a Brownian filtration, $\mathcal{F}_B$, the space $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}_B)$ reminds one, in some sense, of a one-dimensional manifold. Of course, $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}_B)$ is an infinite-dimensional linear space. However, each $M \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F}_B)$ can be written as the stochastic integral $M(t) = \int_0^t H(s)\,dB(s)$ of some $\mathcal{F}_B$-adapted process H (see [RY, V.3.4]). Informally, $dM(t) = H(t)\,dB(t)$ means that, for fixed t and ω the differential dM belongs to the one-dimensional linear space $\{hdB : h \in \mathbb{R}\}$ as if it were the tangent space.

It may seem that stochastic analysis is inherently non-smooth (has no natural smooth structure) since the very Brownian motion is not smooth. However, smoothness in time is not the point. A Brownian motion is rather an infinite collection of independent variables, and $M(t) = \int_0^t H(s)\,dB(s)$ is a function of the variables. Any bounded $\mathcal{F}_B(\infty)$-measurable function f is of the form $f = M(\infty)$ for some $M \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F}_B)$, therefore $f = \int_0^\infty H(t)\,dB(t)$ for some H, the differential $H(t)\,dB(t)$ being well-defined almost everywhere. In this sense stochastic analysis is inherently smooth: measurability implies differentiability. Why? Because $f(\omega) = M(\infty, \omega)$ is a kind of boundary value of $M(t, \omega)$. For a deeper discussion, see the Malliavin calculus [M], [N].

The filtration $\mathcal{F}_{B^n}$ generated by an n-dimensional Brownian motion $B^n(t)$ is, in some sense, n-dimensional: each $M \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F}_{B^n})$ can be written as the stochastic integral $M(t) = \int_0^t H^n(s)\,dB^n(s)$ of some n-dimensional $\mathcal{F}_{B^n}$-adapted process H^n. The space $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}_{B^n})$ reminds one of an n-dimensional manifold. Accordingly, $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}_{B^{n+1}})$ cannot be embedded into $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}_{B^n})$, that is, $B^{n+1}(t)$ cannot be produced in real time from $B^n(t)$. See [J, Chap. 4] for a definition and properties of the so-called instant dimension of $\mathcal{M}_{\mathrm{loc}}(\mathcal{F})$. Instant dimension is defined there for so-called stable subspaces of $\mathcal{M}_{\mathrm{loc}}(\mathcal{F})$, but we need it for the whole $\mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ only. Note that a single martingale can generate a filtration of instant dimension greater than 1; an example: $M(t) = \int_0^t B_1(s)\,dB_2(s)$ for independent Brownian motions B_1, B_2, see [RY, V.4.13]. See also [Sk1, Example 2 on p. 168] for infinite instant dimension (Skorokhod calls it "rank").

Stochastic analysis investigates quantitative properties of random processes. Their qualitative properties, insensitive to reversible real-time transformations, are properties of filtrations, considered up to isomorphisms. A theory of filtrations could be called stochastic topology! Its starting point is the instant dimension, the only[1] known invariant whose meaning is evident (though its invariance is not so evident, which is similar to classical topology).

A Brownian motion in a topological group G can be defined as a path-

[1] As far as discontinuous martingales are excluded.

continuous time-homogeneous process X with independent increments on the left, starting from the unit element of the group [Bax]. For $G = \mathbb{R}$ (implying the additive group) these are of the form $X(t) = aB(t) + bt$ where B is the standard Brownian motion and a, b are constants. For a finite-dimensional Lie group G any such X generates a filtration isomorphic to the Brownian filtration $\mathcal{F}_{B^n}$ for some $n \in \{0, 1, \dots, \dim G\}$. (See [Bax] and references therein; an infinite-dimensional case is treated in [Bax].) In fact, a naturally constructed invertible real-time transformation produces X from a Brownian motion Y in the tangent space of G at its unit (the linear space of the Lie algebra). The global topology of G is irrelevant, since a sample path is continuous and insensitive to self-intersections. The dimension n of the least linear subspace embracing Y exhausts the classification.

Waiving the group structure and the independence of increments we turn to diffusion processes in manifolds, or in $\mathbb{R}^n$, which is the same for stochastic topology due to its local nature. Two main approaches to diffusion processes are used: martingale problems and stochastic differential (or integral) equations. The latter is a technique for connecting a diffusion process and a Brownian motion by an invertible real-time transformation.

Invertibility of a real-time transformation may seem to be a simple matter. In terms of stochastic differentials, the transformation has its differential, defined almost everywhere. The differential is a linear measure-preserving map from one finite-dimensional linear space, equipped with a Gaussian measure, into another. Clearly, such a map is invertible (mod 0) if and only if the two Gaussian measures are of the same dimension. Remembering that stochastic topology is smooth and local, we may expect that a real-time transformation is invertible if and only if it does not reduce the instant dimension. Strangely enough, it does not hold. Invertibility can be violated by a delicate combination of two phenomena. The first, an initial value problem for an ordinary differential equation[2] can have more than one solution. The second, a partition of a probability space into (a continuum of) measurable sets can be an immeasurable partition. As a consequence, the theory of stochastic differential equations is forced to distinguish a strong solution (also called solution-process) from a weak solution (also called solution-measure). The latter is a morphism of $\mathcal{F}_X$ into $\mathcal{F}_{B^n}$ (never reducing the instant dimension), the former is an isomorphism

[2] I mean a classical (not stochastic) differential equation whose right-hand side is continuous but need not satisfy Lipschitz condition.

(some examples may be found in [RY, Chap. IX, 1.19, 3.6, 3.17, 3.18]) provided, however, that the stochastic differential equation is non-degenerate. Otherwise, a strong solution is a (non-invertible) morphism $\mathcal{F}_{B^n} \to \mathcal{F}_X$, while a weak solution is not a morphism at all.

Irreversibility of a given transformation does not mean that the two filtrations are nonisomorphic: another transformation may be invertible. The first example of a filtration of instant dimension 1 (identically), nonisomorphic to Brownian filtration, is given by [DFSmT]. It is of the form $(\Omega, \mathcal{F}_B, \lambda P)$, where $(\Omega, \mathcal{F}_B, P)$ is the Brownian filtered probability space and λ is a density (that can be chosen such that both λ and $1/\lambda$ are bounded [FT]). Unexpectedly, equivalent measures can lead to nonisomorphic filtered probability spaces. The density λ depends on the remote past in a complicated way, thus the measure change $P \mapsto \lambda P$ turns the Brownian motion B into a highly non-Markovian process.

It is natural to ask about an invariant distinguishing the filtration of [DFSmT] from Brownian filtration. In fact, [DFSmT] deals mostly with discrete time filtrations; a sequence $t_n \downarrow 0$ is considered rather than $t \in [0, \infty)$, and Vershik's theory of decreasing sequences of measurable partitions [Ver] is used. The relevant invariant takes on only two values ("standard" and "non-standard"). Richer invariants exist [F] but are somewhat bizarre.

Walsh's Brownian motion was conjectured to give a natural example of a non-Brownian filtration of instant dimension 1 (see [RY, the text after Question 6 at the end of Chapter V]), in addition to the artificial example of [DFSmT]. We will prove the conjecture by means of a new invariant of filtrations (that is, of filtered probability spaces), taking on two values ("cozy" and "not cozy"). The invariant is defined only for continuous-time filtrations. See also [BEKSoY] for a new integer-valued invariant: splitting multiplicity.

2 Joining Two Copies of a Filtration

A joining of two probability spaces $(\Omega_k, \mathcal{F}_k(\infty), P_k)$, $k = 1, 2$, is usually defined as a probability distribution Q on $(\Omega_1 \times \Omega_2, \mathcal{F}_1(\infty) \otimes \mathcal{F}_2(\infty))$ whose marginals are P_1 and P_2, but we may also define it as consisting of another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}(\infty), \tilde{P})$ and two measure preserving maps $\pi_k : \tilde{\Omega} \to \Omega_k$. Any such $J = ((\tilde{\Omega}, \tilde{\mathcal{F}}(\infty), \tilde{P}), \pi_1, \pi_2)$ determines Q_J,

$$Q_J(A) = \tilde{P}\big(\{\tilde{\omega} \in \tilde{\Omega} : (\pi_1(\tilde{\omega}), \pi_2(\tilde{\omega})) \in A\}\big)$$

for $A \subset \Omega_1 \times \Omega_2$, $A \in \mathcal{F}_1(\infty) \otimes \mathcal{F}_2(\infty)$. Any Q with given marginals is Q_J for some J. Though, $Q_{J_1} = Q_{J_2}$ does not mean that J_1 and J_2 are isomorphic, but it does not matter. We need only the case $(\Omega_1, \mathcal{F}_1(\infty), P_1) = (\Omega_2, \mathcal{F}_2(\infty), P_2)$.

2.1 DEFINITION. (a) A self-joining over a probability space $(\Omega, \mathcal{F}(\infty), P)$ is a triple $J = ((\tilde{\Omega}, \tilde{\mathcal{F}}(\infty), \tilde{P}), \pi_1, \pi_2)$ consisting of a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}(\infty), \tilde{P})$ and two measure preserving maps $\pi_k : \tilde{\Omega} \to \Omega$.

(b) A self-joining over a filtered probability space $(\Omega, \mathcal{F}, P)$ or (abusing the language) over a filtration $\mathcal{F}$ is a triple $J = ((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \pi_1, \pi_2)$ consisting of a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and two morphisms π_k from $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ to $(\Omega, \mathcal{F}, P)$.

The following simple example is especially important. Consider filtrations $\mathcal{F} = \mathcal{F}_B$ and $\tilde{\mathcal{F}} = \mathcal{F}_{B^2}$ generated by one- and two-dimensional Brownian motions B and $B^2 = (B_1, B_2)$, respectively. For a given number $\rho \in [-1, +1]$ define π_1, π_2 by

$$B(t) \circ \pi_1 = B_1(t) \,, \qquad B(t) \circ \pi_2 = \rho B_1(t) + \sqrt{1-\rho^2} B_2(t) \,,$$

then $B \circ \pi_1$ and $B \circ \pi_2$ are ρ-correlated copies of B. The self-joining will be denoted by $J(\rho)$. Below, "Var" means "variance" (the squared distance from the one-dimensional space of constants in L_2) and "Cov" means "covariance" (the bilinear form corresponding to the quadratic form "variance").

2.2 LEMMA. *For the self-joining $J(\rho)$, for any $f, g \in L_2(\mathcal{F}_B(\infty))$*

(a) $|\mathrm{Cov}(f \circ \pi_1, g \circ \pi_2)| \le |\rho| \sqrt{\mathrm{Var}(f)} \sqrt{\mathrm{Var}(g)}$ *for* $\rho \in [-1, +1]$;

(b) $\mathrm{Cov}(f \circ \pi_1, f \circ \pi_2) \to \mathrm{Var}(f)$ *for* $\rho \to +1$.

Proof. Let f_n, g_n be the orthogonal projections of f and g on the n-th Wiener chaos, then $f = f_0 + f_1 + \dots$, $\mathrm{Var}(f) = \|f_1\|^2 + \|f_2\|^2 + \dots$, and $\mathbb{E}(f \circ \pi_2 \mid \pi_1^{-1} \mathcal{F}_B(\infty)) = (f_0 + \rho f_1 + \rho^2 f_2 + \dots) \circ \pi_1$, that is, $\mathbb{E}(f(\rho B_1(\cdot) + \sqrt{1-\rho^2} B_2(\cdot)) \mid B_1(\cdot)) = (f_0 + \rho f_1 + \rho^2 f_2 + \dots)(B_1(\cdot))$, see [Ma, Chapter 1, Sect. 6] or[N, §1.4.1]. It follows that $\mathrm{Cov}(f \circ \pi_1, g \circ \pi_2) = \mathrm{Cov}((f_0 + f_1 + \dots) \circ \pi_1, (g_0 + \rho g_1 + \rho^2 g_2 + \dots) \circ \pi_1) = \mathrm{Cov}(f_0 + f_1 + \dots, g_0 + \rho g_1 + \rho^2 g_2 + \dots) = \rho \mathbb{E}(f_1 g_1) + \rho^2 \mathbb{E}(f_2 g_2) + \dots$ which makes (a) and (b) evident. □

The correlation coefficient ρ can be randomized by a simple generalization of the self-joining $J(\rho)$: multiplying $\tilde{\Omega}$ by $[-1, +1]$ we construct a larger $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ supporting both a two-dimensional Brownian motion B^2 and an $\tilde{\mathcal{F}}(0)$-measurable random variable ρ independent of B^2. Any probability measure on $[-1, +1]$ can be chosen as the distribution of ρ. The above formulas for π_1, π_2 still work, but ρ is now randomized. The simplest interesting case is this: ρ takes on two values 0 and $+1$ only, $P(\rho = 1) = p \in (0,1)$, $P(\rho = 0) = 1 - p$. Clearly, $\mathrm{Cov}(f \circ \pi_1, g \circ \pi_2) = p\mathrm{Cov}(f,g)$ for any $f, g \in L_2(\mathcal{F}_B(\infty))$. This means a positive minimal angle between the subspaces $L_2^0(\pi_1^{-1}\mathcal{F}(\infty)) = L_2^0(\tilde\Omega, \pi_1^{-1}\mathcal{F}(\infty), \tilde P)$ and $L_2^0(\pi_2^{-1}\mathcal{F}(\infty))$ of the Hilbert space $L_2(\tilde{\mathcal{F}}(\infty)) = L_2(\tilde\Omega, \tilde{\mathcal{F}}(\infty), \tilde P)$; here L_2^0 means the subspace of all zero-mean functions of L_2. The angle averages the correlation, since the L_2-norm is of a global nature, in contrast with the quadratic variation used in the following definition.

2.3 Definition. Let $J = ((\tilde\Omega, \tilde{\mathcal{F}}, \tilde P), \pi_1, \pi_2)$ be a self-joining over a filtered probability space $(\Omega, \mathcal{F}, P)$. The maximal correlation $\rho_{\max}(J)$ of the self-joining J is defined as the least number of $[0,1]$ such that for any $X, Y \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ the following process on $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde P)$ is an increasing process:

$$\langle X_1 - Y_2, X_1 - Y_2 \rangle - (1 - \rho_{\max})\big(\langle X_1, X_1\rangle + \langle Y_2, Y_2\rangle\big) ;$$

here $X_1 = X \circ \pi_1$, $Y_2 = Y \circ \pi_2$.

The filtration $\mathcal{F}$ is assumed to satisfy the continuity condition of section 1, but $\tilde{\mathcal{F}}$ is not. Throughout the paper, the condition is assumed for a filtration if and only if its notation does not contain a tilde (˜), unless the reader prefers to assume the absolute continuity condition to hold everywhere, which can be done with no essential harm.

For any $f \in L_2(\mathcal{F}(\infty))$ the process $X(t) = \mathbb{E}(f|\mathcal{F}_t)$ ($\mathbb{E}(\cdot|\cdot)$ means the conditional expectation) is a local martingale (in fact, an L_2-bounded martingale), $f = X(\infty)$, and[3] $\mathrm{Var}(f) = E\langle X, X\rangle(\infty)$. Applying (2.3) to X, Y obtained this way from $f, g \in L_2(\mathcal{F}(\infty))$ we get

$$\begin{aligned}
&X_1(t) - Y_2(t) = \mathbb{E}(f_1 - g_2 \mid \tilde{\mathcal{F}}_t) ;\\
&\mathrm{Var}(f_1 - g_2) \ge (1 - \rho_{\max})\big(\mathrm{Var}(f_1) + \mathrm{Var}(g_2)\big) ;\\
&2\mathrm{Cov}(f_1, g_2) \le \rho_{\max}\big(\mathrm{Var}(f_1) + \mathrm{Var}(g_2)\big) ;\\
&\mathrm{Cov}(f \circ \pi_1, g \circ \pi_2) \le \rho_{\max}\sqrt{\mathrm{Var}(f)}\sqrt{\mathrm{Var}(g)} .
\end{aligned}$$

The cosine of the minimal angle does not exceed $\rho_{\max}(J)$, and can be strictly smaller. In fact, $\rho_{\max}(J)$ is the cosine of the minimal angle between so-called stable subspaces generated by $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}) \circ \pi_1$ and $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}) \circ \pi_2$; a definition can be found in [J, Chap. 4] or [RY, IV.5.11], but we do not need it. The notion of the quadratic covariation,

$$\langle X, Y\rangle = \tfrac12\langle X, X\rangle + \tfrac12\langle Y, Y\rangle - \tfrac12\langle X - Y, X - Y\rangle ,$$

[3] It is assumed that $\mathcal{F}(0)$ is trivial; otherwise $L_2^0(\mathcal{F}(\infty))$ should be replaced with $L_2(\mathcal{F}(\infty)) \ominus L_2(\mathcal{F}(0))$.

allows us to write the process $\langle X_1 - Y_2, X_1 - Y_2\rangle - (1-\rho_{\max})(\langle X_1, X_1\rangle + \langle Y_2, Y_2\rangle)$ in the form

$$\rho_{\max}\big(\langle X_1, X_1\rangle + \langle Y_2, Y_2\rangle\big) - 2\langle X_1, Y_2\rangle\,.$$

If $(\Omega, \mathcal{F}, P)$ satisfies the absolute continuity condition of sect. 1, then $\langle X_1, X_1\rangle$ and $\langle Y_2, Y_2\rangle$ are absolutely continuous, which implies absolute continuity of $\langle X_1, Y_2\rangle$, since $\langle X_1, X_1\rangle + \langle Y_2, Y_2\rangle \pm 2\langle X_1, Y_2\rangle = \langle X_1 \pm Y_2, X_1 \pm Y_2\rangle$ increases. Thus, $\rho_{\max}$ is the least number satisfying

$$\rho_{\max}\tfrac{d}{dt}\big(\langle X_1, X_1\rangle + \langle Y_2, Y_2\rangle\big) \ge 2\tfrac{d}{dt}\langle X_1, Y_2\rangle$$

almost everywhere. Multiplying X_1 by c, Y_2 by $1/c$, and minimizing in c, we get

$$\big|\tfrac{d}{dt}\langle X_1, Y_2\rangle\big| \le \rho_{\max}\sqrt{\tfrac{d}{dt}\langle X_1, X_1\rangle}\sqrt{\tfrac{d}{dt}\langle Y_2, Y_2\rangle}\,.$$

In general, the ratio

$$\frac{\frac{d}{dt}\langle X, Y\rangle}{\sqrt{\frac{d}{dt}\langle X, X\rangle}\sqrt{\frac{d}{dt}\langle Y, Y\rangle}}$$

is the instant correlation between X, Y. So, $\rho_{\max}(J)$ is the supremum over all $X, Y \in \mathcal{M}_{\mathrm{loc}}(\mathcal{F})$ of the essential supremum in t and ω of the instant correlation between $X \circ \pi_1$ and $Y \circ \pi_2$ (with an evident caveat about zero denominator).[4]

For a filtration of instant dimension 1, the instant correlation ρ does not depend on the choice of X, Y up to a sign, and $\rho_{\max}(J)$ is the essential supremum of $|\rho|$ over $\Omega \times [0, \infty)$. (For a larger instant dimension, an instant maximal correlation may be defined such that $\rho_{\max}(J)$ is its essential supremum.) In particular, $\rho_{\max}(J) = |\rho|$ for the self-joining $J(\rho)$ (with a constant ρ), and $\rho_{\max}(J) = 1$ for the example with randomized $\rho \in \{0, 1\}$.

2.4 Definition. A filtered probability space $(\Omega, \mathcal{F}, P)$, or (abusing the language) a filtration $\mathcal{F}$, is called cozy, if there is a sequence (J_n) of self-joinings $J_n = ((\tilde\Omega_n, \tilde{\mathcal{F}}_n, \tilde P_n), \pi_1^{(n)}, \pi_2^{(n)})$ over $\mathcal{F}$ such that

(a) $\rho_{\max}(J_n) < 1$ for each n,

(b) $\mathrm{Cov}(f \circ \pi_1^{(n)}, f \circ \pi_2^{(n)}) \to \mathrm{Var}(f)$ when $n \to \infty$ for any $f \in L_2(\mathcal{F}(\infty))$.

2.5 Lemma. *A filtration generated by a finite or countable collection of independent Brownian motions is cozy.*

[4] All that has been said can be generalized easily for a filtration satisfying the continuity condition rather than the absolute continuity condition.

Proof. For a single Brownian motion, it follows from Lemma 2.2 that we may choose $\rho_n < 1$, $\rho_n \to 1$ and let $J_n = J(\rho_n)$. A generalization for many dimensions is straightforward. □

2.6 LEMMA. *There is no morphism from a cozy filtration to a non-cozy filtration.*

Proof. Let $\mathcal{F}_1$ be a cozy filtration and π a morphism from $\mathcal{F}_1$ to another filtration $\mathcal{F}_2$; we have to prove that $\mathcal{F}_2$ is also cozy. Any self-joining $J = ((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \pi_1, \pi_2)$ over $(\Omega_1, \mathcal{F}_1, P_1)$ induces a self-joining $\pi J = ((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \pi \circ \pi_1, \pi \circ \pi_2)$ over $(\Omega_2, \mathcal{F}_2, P_2)$. We have $\rho_{\max}(\pi J) \le \rho_{\max}(J)$ since $\mathcal{M}_{\mathrm{loc}}(\mathcal{F}_1) \supset \mathcal{M}_{\mathrm{loc}}(\mathcal{F}_2) \circ \pi$. On the other hand, $\mathrm{Cov}(f \circ (\pi \circ \pi_1), f \circ (\pi \circ \pi_2)) = \mathrm{Cov}((f \circ \pi) \circ \pi_1, (f \circ \pi) \circ \pi_2))$ for any $f \in L_2(\mathcal{F}_2(\infty))$. Therefore, a sequence (J_n) of self-joinings over $\mathcal{F}_1$, satisfying conditions (a), (b) of Definition 2.4, induces the sequence (πJ_n) of self-joinings over $\mathcal{F}_2$, satisfying the conditions. □

Remembering that a strong solution of a stochastic differential equation may be thought of as a morphism $\mathcal{F}_{B^n} \to \mathcal{F}_X$ we conclude that the strong solution generates a cozy filtration. It is especially interesting for degenerate equations, that is, non-invertible morphisms.

The main result of the section follows immediately from Lemmas 2.5, 2.6.

2.7 Theorem. *There is no morphism from a filtration generated by a finite or countable collection of independent Brownian motions to a non-cozy filtration.*

It will be shown that Walsh's Brownian motion generates a non-cozy filtration, and therefore it cannot be produced in real time from a Brownian motion of any dimension, finite or infinite!

3 Joining Two Copies of Walsh's Brownian Motion

A continuous semimartingale may be defined as the sum of a continuous local martingale and a continuous process of (locally) finite variation: $X(t) - X(0) = M_X(t) + A_X(t)$, $M_X(0) = 0$, $A_X(0) = 0$. The martingale part M_X and the compensator A_X are uniquely determined by X. Quadratic variation for X may be introduced by $\langle X, X \rangle = \langle M_X, M_X \rangle$. (See [RY, IV.1.17–18].) If A_X increases, X is called a local submartingale; if A_X decreases, X is called a local supermartingale. The set of all continuous semimartingales is a linear subset of the linear topological space of all continuous processes (equipped with the ucp-topology, as in sect. 1), but the linear subset is not closed. In fact, it is dense (since continuous processes of finite variation are dense). The maps $X \mapsto M_X$ and $X \mapsto A_X$ are linear but not continuous. The set (convex, not linear) of all continuous local submartingales is closed. In fact, it is the closure of the set of all continuous submartingales. Being restricted to the closed set, the maps $X \mapsto M_X$, $X \mapsto A_X$ are continuous. The same applies for supermartingales. However, a process can be both a supermartingale and a local martingale without being a martingale, even if it is positive and bounded in L_2! See [RY, V.2.13].

Walsh's Brownian motion $Z(t)$ (see the Introduction) can be described alternatively by three processes $X_1(t), X_2(t), X_3(t)$ such that

$$Z(t) = X_1(t) + e^{2\pi i/3} X_2(t) + e^{4\pi i/3} X_3(t)$$

and X_1, X_2, X_3 do not overlap in the following sense: no more than one of them differs from zero at any given t (and ω). Each X_k belongs to the following class (see also [RY, VI.4.4]); another example is $|Z(t)|$ (the same as $|B(t)|$ for Brownian B).

3.1 Definition. A process of class Σ_+ is a continuous semimartingale X of the form $M + V$ where M is a local martingale and V is an increasing process such that $X(0) = 0$, $X(t) \ge 0$, and $\int_0^t 1_{(X(s)>0)}\, dV(s) = 0$ for all $t \in [0, \infty)$.

For any $X \in \Sigma_+$ the increasing process V is half of the local time of X at 0 (see [RY, VI.1]),

$$V(t) = \tfrac{1}{2} L_t(X) = \tfrac{1}{2} \lim_{\varepsilon \downarrow 0} \tfrac{1}{\varepsilon} \int_0^t 1_{(X(s)<\varepsilon)}\, d\langle X, X\rangle(s) = \int_0^t 1_{(X(s)=0)}\, dX(s)\,.$$

That is, $X \overset{\mathrm{m}}{=} \frac{1}{2} L(X)$ for $X \in \Sigma_+$. It follows that $X - \frac{1}{2} L(X)$ is a time-changed Brownian motion, and (by Skorokhod's lemma [RY, VI.2.1]) any $X \in \Sigma_+$ is a time-changed reflecting Brownian motion.

For any $X_1, X_2, X_3 \in \Sigma_+$ the process $Z(t) = X_1(t) + e^{2\pi i/3} X_2(t) + e^{4\pi i/3} X_3(t)$ is a local martingale if and only if $L_t(X_1) = L_t(X_2) = L_t(X_3)$ for all t. The reason is simple: $Z \overset{\mathrm{m}}{=} \frac{1}{2}(L(X_1) + e^{2\pi i/3} L(X_2) + e^{4\pi i/3} L(X_3))$, so, $Z \in \mathcal{M}_{\mathrm{loc}}$ if and only if $L(X_1) + e^{2\pi i/3} L(X_2) + e^{4\pi i/3} L(X_3) = 0$, that is, $L(X_1) = L(X_2) = L(X_3)$.

Non-overlapping processes $X_1, X_2, X_3 \in \Sigma_+$ such that $L(X_1) = L(X_2) = L(X_3)$ describe Walsh's Brownian motion if and only if

$\langle X_1, X_1\rangle(t) + \langle X_2, X_2\rangle(t) + \langle X_3, X_3\rangle(t) = t$ for all t. Otherwise they describe a time-changed Walsh's Brownian motion or, equivalently, a complex-valued continuous local martingale Z such that $Z^3(t) \in [0, \infty)$ for all t (a special case of so-called spider-martingales [Y2, Def. 17.2], [BEKSoY, Sect. 2]; for the change of time see [Y2, Prop. 17.6]).

So, let a filtration $\mathcal{F}_Z$ be generated by Walsh's Brownian motion $Z = X_1 + e^{2\pi i/3}X_2 + e^{4\pi i/3}X_3$. Striving to prove that $\mathcal{F}_Z$ is not cozy (which will be done in next sections) we consider a sequence (J_n) of self-joinings over $\mathcal{F}_Z$ satisfying condition (b) of Definition 2.4: $\operatorname{Cov}(f\circ\pi_1^{(n)}, f\circ\pi_2^{(n)}) \to \operatorname{Var}(f)$ when $n \to \infty$ for any $f \in L_2(\mathcal{F}(\infty))$. The following result shows that for large n processes $Z\circ\pi_1^{(n)}$ and $Z\circ\pi_2^{(n)}$ must have many common zeros, which hints at a singular nature of the triple point 0 of the space $\{z \in \mathbb{C} : z^3 \in [0, +\infty)\}$.

3.2 Lemma. *Let (J_n) be as above, and processes $R_1^{(n)}, R_2^{(n)}$ be defined by $R_k^{(n)}(t) = |Z(t)|\circ\pi_k^{(n)}$. Then for any $t \in [0, \infty)$*

$$\mathbb{E}\int_0^t 1_{(R_1^{(n)}\neq 0)}\, dL(R_2^{(n)}) \to 0 \quad \textit{for } n\to\infty\,.$$

The lemma follows from the next lemma, see below. Of course, $\mathbb{E} = \mathbb{E}_{\tilde{P}_n}$ means integration w.r.t. $\tilde{\mathbb{P}}_n$.

The natural measure on the set of zeros of the second copy $Z\circ\pi_2^{(n)}$ of Z is $dL(R_2^{(n)})$. Most of these zeros are also zeros of the first copy $Z\circ\pi_1^{(n)}$ (for large n, with a high probability), and most zeros of the first copy are also zeros of the second copy. That is strange! Usually, if two functions are close, we may expect at the utmost that their zeros are close, but need not coincide. Abundance of zeros is not an explanation, since Lemma 3.2 has no counterpart for the two-ray case, the Brownian motion $B(t)$, nor for the one-ray case, $|B(t)|$. Remember the self-joining $J(\rho)$ over $\mathcal{F}_B$ with a constant correlation coefficient ρ. It is easy to see that $B\circ\pi_1$ and $B\circ\pi_2$ (as well as $|B|\circ\pi_1$ and $|B|\circ\pi_2$) have no common zeros for $\rho \neq \pm 1$. *We observe emergence of the distinction mentioned in the Introduction: a triple point is an essential singularity, while an endpoint is not.*

The proof involves the following "geodesic" metric on the space $\{z \in \mathbb{C} : z^3 \in [0, +\infty)\}$:

$$\operatorname{dist}\big(r_1 e^{2\pi k_1 i/3}, r_2 e^{2\pi k_2 i/3}\big) = \begin{cases} |r_1 - r_2|\,, & \text{when } k_1 = k_2\,,\\ |r_1| + |r_2|\,, & \text{otherwise}\,. \end{cases}$$

Given a self-joining J over $\mathcal{F}_Z$, we introduce the process D_J,

$$D_J(t) = \operatorname{dist}\big(Z(t)\circ\pi_1, Z(t)\circ\pi_2\big)\,.$$

If a sequence (J_n) of self-joinings satisfies condition (b) of Definition 2.4, then $\mathbb{E}(D_{J_n}(t))^2 \to 0$ for $n\to\infty$. Indeed, $\operatorname{Cov}(f\circ\pi_1^{(n)}, f\circ\pi_2^{(n)}) \to \operatorname{Var}(f)$ implies $\mathbb{E}|f\circ\pi_1^{(n)} - f\circ\pi_2^{(n)}|^2 \to 0$ for any $f\in L_2(\mathcal{F}_Z(\infty))$. Combining $f=\operatorname{Re} Z(t)$ and $f=\operatorname{Im} Z(t)$ we have $\mathbb{E}|Z(t)\circ\pi_1^{(n)} - Z(t)\circ\pi_2^{(n)}|^2\to 0$. It remains to note that $\operatorname{dist}(z_1,z_2)\le \mathrm{const}\cdot|z_1-z_2|$ for all $z_1,z_2\in\{z\in\mathbb{C}: z^3\in[0,+\infty)\}$.

Lemma 3.2 follows immediately from the next lemma showing that $\mathbb{E}\int_0^t 1_{(R_1\ne 0)}\,dL(R_2)\le 6\mathbb{E}D_J(t)$. Here $R_k(t)=|Z(t)|\circ\pi_k$.

3.3 Lemma. *For any self-joining J over $\mathcal{F}_Z$, the following process is a submartingale:*

$$X(t) = D_J(t) - \frac{1}{6}\int_0^t 1_{(R_1\ne0)}\,dL(R_2) - \frac{1}{6}\int_0^t 1_{(R_2\ne0)}\,dL(R_1)\,.$$

Here is an intuitive explanation. For any t and ω one of the following simple cases takes place within a small neighborhood of t:

Case 1: $R_1(t)>0$ and $R_2(t)>0$. Then $X=D_J+\mathrm{const}$. Take $k,l\in\{1,2,3\}$ such that $R_1(t)=X_k(t)\circ\pi_1$ and $R_2(t)=X_l(t)\circ\pi_2$. If $k=l$ then $D_J=|X_k\circ\pi_1 - X_l\circ\pi_2|$ is a submartingale, since $X_k\circ\pi_1$ and $X_l\circ\pi_2$ are martingales. If $k\ne l$ then $D_J = X_k\circ\pi_1 + X_l\circ\pi_2$ is a martingale.

Case 2: $R_1(t)>0$ but $R_2(t)=0$. Then $X=D_J-(1/6)L(R_2)+\mathrm{const}$. Take k such that $R_1(t)=X_k(t)\circ\pi_1$. Assume that $k=1$ (other cases are similar), then $D_J = X_1\circ\pi_1 - (X_1-X_2-X_3)\circ\pi_2 = (\text{a martingale}) - \frac12\big(L(X_1)-L(X_2)-L(X_3)\big)\circ\pi_2 = (\text{a martingale}) + \frac16 L(R_2)$.

Case 3: $R_1(t)=0$ and $R_2(t)=0$. Infinitesimally, D_J can only increase, and $X=D_J+\mathrm{const}+(\text{negligible term})$.

The idea is quite clear for cases 1, 2 but vague for 3. Also, assembling the local descriptions needs justification. Instead, the proof given below uses stochastic integration.

For any continuous function f of finite variation, $\int_0^t 1_{(f(s)=0)}\,df(s)=0$. Stochastic integration is more subtle: for a continuous semimartingale X, the integral $\int_0^t 1_{(X(s)=0)}\,dX(s)$ need not vanish. If $X(t)\ge0$ for all t, then [RY, VI.1.7]

$$\int_0^t 1_{(X(s)=0)}\,dX(s) = \frac12 L_t(X)\ge 0\,,$$

therefore [RY, IV.2.10]

$$\int_0^t H(s)1_{(X(s)=0)}\,dX(s) = \frac{1}{2}\int_0^t H(s)\,dL_s(X)$$

for any bounded predictable process H. The following general lemma will be used in the proof of Lemma 3.3.

3.4 LEMMA. *Let $X_1,\dots,X_n$ be continuous semimartingales and $H_1,\dots,H_n$ be predictable processes such that for all t*

$$H_k(t) \ge 0 \quad \textit{for all } k\,; \qquad \sum_k H_k(t) = 1\,;$$

$$\sum_k H_k(t)X_k(t) = \max_k X_k(t)\,.$$

Then the following process is increasing:

$$A(t) = \max_k X_k(t) - \sum_k \int_0^t H_k(s)\,dX_k(s)\,.$$

Proof. Let $Y_k(t) = -X_k(t) + \max_i X_i(t)$, then $Y_k(t) \ge 0$, $\sum_k H_k(t)Y_k(t) = 0$, and $A(t) = \max_k X_k(0) + \sum_k \int_0^t H_k(s)\,dY_k(s)$. However, $H_k(t)Y_k(t) = 0$, therefore (see the formula before the lemma)

$$\int_0^t H_k(s)\,dY_k(s) = \int_0^t H_k(s)1_{(Y_k(s)=0)}\,dY_k(s) = \frac{1}{2}\int_0^t H_k(s)\,dL_s(Y_k)\,,$$

which evidently increases. □

Proof of Lemma 3.3. Define processes Y_1,Y_2,Y_3: $Y_k = (2X_k - X_1 - X_2 - X_3)\circ\pi_1 - (2X_k - X_1 - X_2 - X_3)\circ\pi_2$, then $D_J(t) = \max_{k=1,2,3}|Y_k(t)| = \max_{k=1,\dots,6} Y_k(t)$, where $Y_4 = -Y_1$, $Y_5 = -Y_2$, $Y_6 = -Y_3$. (In fact, $D_J(t) = \max_{k=1,2,3} Y_k(t) = \max_{k=1,2,3}(-Y_k(t))$, but all the six terms are needed for the proof.) Introduce processes $H_1,\dots,H_6$ such that $H_k(t) \ge 0$, $\sum_{k=1,\dots,6} H_k(t) = 1$, and $\sum_{k=1,\dots,6} H_kY_k = D_J$ as follows:

if $R_2 < R_1 = X_k\circ\pi_1$ then $H_k = 1\,;$

if $R_1 < R_2 = X_l\circ\pi_2$ then $H_{3+l} = 1\,;$

if $R_1 = X_k\circ\pi_1 = R_2 = X_l\circ\pi_2 > 0$ then $H_k = 1/2$ and $H_{3+l} = 1/2\,;$

if $R_1 = R_2 = 0$ then $H_1 = \dots = H_6 = 1/6\,.$

Lemma 3.4 states that $A = D_J - \sum_{k=1,\dots,6}\int H_k\,dY_k$ is an increasing process. We have

$$\begin{aligned} Y_k &\overset{\mathrm{m}}{=} \tfrac12\big(2L(X_k) - L(X_1) - L(X_2) - L(X_3)\big)\circ\pi_1 \\ &\quad - \tfrac12\big(2L(X_k) - L(X_1) - L(X_2) - L(X_3)\big)\circ\pi_2 \end{aligned}$$

$$= -\tfrac{1}{6}L(R_1) + \tfrac{1}{6}L(R_2) \quad \text{for } k = 1,2,3\,;$$

$$\begin{aligned}\mathcal{M}_{\mathrm{loc}} \ni \sum_{k=1}^{3} \int (H_k - H_{3+k})\, d\bigl(Y_k + \tfrac{1}{6}L(R_1) - \tfrac{1}{6}L(R_2)\bigr) \\ = D_J - A + \frac{1}{6}\int (H_1 + H_2 + H_3 - H_4 - H_5 - H_6)\, d\bigl(L(R_1) - L(R_2)\bigr) \\ = D_J - A + \frac{1}{6}\int \operatorname{sgn}(R_1 - R_2)\, d\bigl(L(R_1) - L(R_2)\bigr) \\ = D_J - A - \frac{1}{6}\int 1_{(R_1\neq 0)}\, dL(R_2) - \frac{1}{6}\int 1_{(R_2 \neq 0)}\, dL(R_1) = X - A\,.\end{aligned}$$

It remains to prove that the above process is a martingale (not only a local martingale). The integrand is bounded: $|H_k - H_{3+k}| \le 1$. The integrator is the sum of two (correlated) Brownian motions: $\langle 2X_k - X_1 - X_2 - X_3, 2X_k - X_1 - X_2 - X_3\rangle(t) = \langle X_1, X_1\rangle(t) + \langle X_2, X_2\rangle(t) + \langle X_3, X_3\rangle(t) = t$, since $\langle X_k, X_l\rangle = 0$ for $k \ne l$. □

In fact, $2X_k - X_1 - X_2 - X_3$ is a so-called skew Brownian motion with parameter $\alpha = 1/3$, see [W], [HSh], and [RY, X.2.24, XII.2.16].

The main result of this section, Lemma 3.2, follows from Lemma 3.3 proved above. So, common zeros cannot be rare for two copies of Walsh's Brownian motion Z, if the sequence (J_n) of self-joinings of $\mathcal{F}_Z$ satisfies condition (b) of Definition 2.4. This is a half of the way toward non-coziness of $\mathcal{F}_Z$. The second half is this: common zeros are rare for two copies of Z, if the self-joining J of $\mathcal{F}_Z$ satisfies condition (a) of Definition 2.4: $\rho_{\max}(J) < 1$. This statement will be proved in the next section for reflecting Brownian motion $|B|$, which is enough due to the following argument.

If $Z(t)$ is Walsh's Brownian motion, then $|Z(t)|$ is (another model of) reflecting Brownian motion, and there is a natural morphism π from $\mathcal{F}_Z$ to $\mathcal{F}_{|Z|}$. Any self-joining J of $\mathcal{F}_Z$ induces a self-joining πJ of $\mathcal{F}_{|Z|}$ (see the proof of Lemma 2.6), and $\rho_{\max}(\pi J) \le \rho_{\max}(J)$.

4 Joining Two Copies of Reflecting Brownian Motion

An interesting geometric property of two-dimensional Brownian trajectories, well-known since 1985 [Bu], [Shi], [Ev], and reappearing as a by-product of the main results of the section, is shown on Fig. 1(a). For a given angle $\alpha \in (0,\pi)$, an instant $t \in (0,\infty)$ will be called an α-minimum, if

$$|B_2(s) - B_2(t)| \sin\tfrac{\alpha}{2} \le \bigl(B_1(s) - B_1(t)\bigr)\cos\tfrac{\alpha}{2}$$

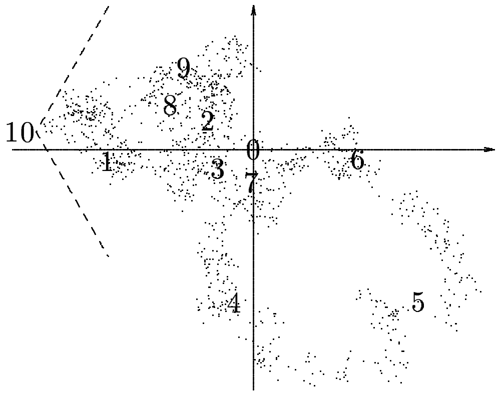

Fig. 1(a). An α-minimum of a two-dimensional Brownian motion, $\alpha = \pi/3$. The shown α-minimum is the first one after $t = 1$.

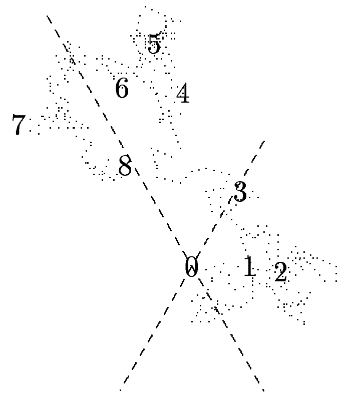

Fig. 1(b). Soon after an α-minimum, $\alpha = \pi/3$. The shown α-minimum is the last one before $t = 1$.

for all $s \in [0, t]$. Here (B_1, B_2) is a two-dimensional Brownian motion. Each $\omega \in \Omega$ determines the set (maybe empty) of α-minima. The question is: are there α-minima? The answer is positive for $\alpha < \pi/2$ and negative for $\alpha \geq \pi/2$ ("with probability 1" is implied, as usual); see Lemma 4.6 and the paragraph after it.

There is one more by-product (see Lemma 4.12 and the paragraph after it), admitting a nice geometric reformulation presented below. I am grateful to Marc Yor for the reformulation [Y1], and to Krzysztof Burdzy for pointing out that the geometric statement follows easily from a result of Evans [Ev, Th. 1(ii)]. Let $\alpha < \pi/2$. For each α-minimum t, each of the two inequalities

$$\pm\big(B_2(s) - B_2(t)\big) \sin\tfrac{\alpha}{2} \leq \big(B_1(s) - B_1(t)\big) \cos\tfrac{\alpha}{2}$$

is violated for some $s \in (t, t+\varepsilon)$, no matter how small $\varepsilon > 0$ is. After an α-minimum, the trajectory cannot be sustained in a positive time within one of the two half-planes shown on Fig. 1(b). The fact is evident (and still holds for $\alpha = 0$) for a predictable α-minimum (chosen in real time, without anticipating the future), but the statement is much stronger: it holds for all α-minima. Note that the statement ceases to hold at $\alpha = 0$. A 0-minimum is a t such that $B_1(t) = \min\{B_1(s) : s \in [0, t]\}$. Take the last 0-minimum before a given instant (say, 1), then $0 < B_1(s) - B_1(t)$ for all $s \in (t, t+\varepsilon)$

if ε is small enough.

The two statements, presented by the above two paragraphs, will be proved anew. Afterwards the proofs will be generalized, giving new results (Lemmas 4.11, 4.12). Both statements can be reformulated in terms of two correlated Brownian motions $B_1(t)\cos(\alpha/2) \pm B_2(t)\sin(\alpha/2)$. Their instant correlation is constant: $\rho = \cos\alpha$. The same situation was faced in section 2: the self-joining $J(\rho)$ over a Brownian motion B produces two ρ-correlated copies $B\circ\pi_1$ and $B\circ\pi_2$ of B. The process

$$R(t) = B(t) - \inf_{s\in[0,t]} B(s)$$

is a reflecting Brownian motion (as mentioned in sect. 1), and an α-minimum is an instant t such that $R_1(t) = R_2(t) = 0$. Here $R_k = R\circ\pi_k$ are ρ-correlated reflected Brownian motions: $\langle R_1, R_2\rangle(t) = \rho t$. An α-minimum is nothing but a common zero of R_1 and R_2.

There is a simpler example of correlated reflecting Brownian motions: $|B_1| = |B|\circ\pi_1$ and $|B_2| = |B|\circ\pi_2$. However, their instant correlation is not a constant, it is sometimes $(+\rho)$, sometimes $(-\rho)$,

$$\big\langle |B_1|, |B_2| \big\rangle(t) = \rho \int_0^t \operatorname{sgn}\big(B_1(s)\big) \operatorname{sgn}\big(B_2(s)\big)\, ds\,.$$

There are no common zeros for $|B_1|$ and $|B_2|$ (irrespective of $\alpha \in (0,\pi)$), since a two-dimensional Brownian motion never returns to the origin. Our digression is now finished, and we return to a more formal style.

A reflecting Brownian motion $R(t)$ is considered on a filtered probability space $(\Omega, \mathcal{F}, P)$. That is, R is a process of class Σ_+ (see 3.1) whose martingale part is a Brownian motion,

$$R = M_R + V_R\,, \quad M_R \in \mathcal{M}_{\mathrm{loc}}\,, \quad \langle M_R, M_R\rangle(t) = t\,, \quad V_R(t) = \tfrac{1}{2} L_t(R)\,.$$

The reader may restrict himself to the case $\mathcal{F} = \mathcal{F}_R$ (that is, $\mathcal{F}$ is generated by R), but it is not easier than the general case.

Let $J = ((\tilde\Omega, \tilde{\mathcal{F}}, \tilde P), \pi_1, \pi_2)$ be a self-joining over $(\Omega, \mathcal{F}, P)$. We are interested in common zeros for the processes $R_1 = R\circ\pi_1$ and $R_2 = R\circ\pi_2$.

For any C^2-smooth function $f : [0,\infty)\times[0,\infty) \to \mathbb{R}$, the martingale version of Itô's formula gives

(4.1)

$$f\big(R_1(t), R_2(t)\big) \overset{\mathrm{m}}{=} \frac{1}{2}\int_0^t \Delta f\big(R_1(s), R_2(s)\big) ds + \int_0^t f_{12}\big(R_1(s), R_2(s)\big)\rho(s) ds +$$
$$+ \frac{1}{2}\bigg(\int_0^t f_1\big(0, R_2(s)\big)\, dL_s(R_1) + \int_0^t f_2\big(R_1(s), 0\big)\, dL_s(R_2) \bigg)\,,$$

where $f_k(x_1,x_2) = (\partial/\partial x_k) f(x_1,x_2)$, $f_{kl}(x_1,x_2) = (\partial^2/\partial x_k \partial x_l) f(x_1,x_2)$, $\Delta f = f_{11} + f_{22}$, and $\rho(s)$ is the instant correlation between $R_1(s)$ and $R_2(s)$,

$$\langle R_1, R_2\rangle(t) = \int_0^t \rho(s)\, ds\,.$$

Note that $|\rho(s)| \le \rho_{\max}(J)$.

If the smoothness of f is violated at some points, the formula (4.1) can be applied up to a stopping time provided that the process is stopped before something ill-defined really enters the formula.

4.2 LEMMA. *For any $\rho \in (0,1)$ there is a continuous function $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:*

(a) $f(x,y) = f(y,x)$ *for all* $x,y \ge 0$.

(b) f *has continuous partial derivatives of first and second order on* $(0,\infty)\times(0,\infty)$; *the derivatives have continuous extensions to* $[0,\infty)\times[0,\infty)\setminus\{(0,0)\}$.

(c) *For all* $(x,y) \in (0,\infty)\times(0,\infty)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2\rho\frac{\partial^2}{\partial x \partial y}\right) f(x,y) = 0\,.$$

(d) *For all* $x > 0$

$$\frac{\partial}{\partial y}\bigg|_{y=0+} f(x,y) = 0\,.$$

(e) $f(0,0) = 0$, *and* $f(x,y) > 0$ *for* $(x,y) \ne (0,0)$.

(f) *For any* $x,y,r \ge 0$

$$f(rx,ry) = r^{(2-2a)/(2-a)} f(x,y)\,,$$

where $a \in (0,1)$ *is defined by*

$$\cos\frac{\pi a}{2} = \rho\,. \tag{4.3}$$

Proof. Take

$$f(x,y) = \mathrm{Re}\left(Z^{(2-2a)/(2-a)}(x,y)\right) \tag{4.4}$$

where

$$Z(x,y) = (x+y)\sin\frac{\pi a}{4} + i(x-y)\cos\frac{\pi a}{4}\,, \tag{4.5}$$

and, of course, $Z^p(x,y)$ means $(Z(x,y))^p$. Note that

$$|\arg Z(x,y)| \le \frac{\pi}{2} - \frac{\pi a}{4} = \frac{\pi}{2}\cdot\frac{2-a}{2}\,,$$
$$\left|\arg Z^{(2-2a)/(2-a)}(x,y)\right| \le \frac{\pi}{2}(1-a) < \frac{\pi}{2}\,.$$

It follows that $f(x,y) > 0$ for $(x,y) \neq (0,0)$.

Being the real part of a holomorphic function of $z = Z(x,y)$, the function f is harmonic in the coordinates $u = \operatorname{Re} Z(x,y) = (x+y)\sin(\pi a/4)$ and $v = \operatorname{Im} Z(x,y) = (x-y)\cos(\pi a/4)$,

$$\begin{aligned} 0 &= \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) f\left(\frac{u}{2S} + \frac{v}{2C}, \frac{u}{2S} - \frac{v}{2C}\right) \\ &= \left(\frac{1}{4S^2} + \frac{1}{4C^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Big|_{\substack{x=\frac{u}{2S}+\frac{v}{2C} \\ y=\frac{u}{2S}-\frac{v}{2C}}} f(x,y) \\ &\qquad + 2\left(\frac{1}{4S^2} - \frac{1}{4C^2}\right)\frac{\partial^2}{\partial x \partial y}\Big|_{\substack{x=\dots \\ y=\dots}} f(x,y)\,, \end{aligned}$$

where $C = \cos(\pi a/4)$, $S = \sin(\pi a/4)$. It means that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2\underbrace{(C^2 - S^2)}_{\rho}\frac{\partial^2}{\partial x \partial y}\right) f(x,y) = 0\,.$$

It remains to check (d),

$$\begin{aligned} \frac{\partial}{\partial y}\Big|_{y=0+} Z^{(2-2a)/(2-a)}(x,y) &= \frac{2-2a}{2-a} Z^{-a/(2-a)}(x,0) \cdot \frac{\partial}{\partial y}\Big|_{y=0+} Z(x,y) \\ &= \frac{2-2a}{2-a}\left(x \exp\left(i\frac{\pi}{2}\frac{2-a}{2}\right)\right)^{-a/(2-a)} \cdot \exp\left(-i\frac{\pi}{2}\frac{2-a}{2}\right) \\ &= \frac{2-2a}{2-a} x^{-a/(2-a)} i\,; \end{aligned}$$

the real part vanishes. □

4.6 Lemma. *Let the two copies $R_k = R \circ \pi_k$ of the reflecting Brownian motion R be ρ-correlated for some (constant, nonrandom) $\rho \in (0,1)$, that is, $\langle R_1, R_2 \rangle(t) = \rho t$. Then (with probability 1) there is a (random) $t > 0$ such that $R_1(t) = R_2(t) = 0$.*

Proof. Consider the process $X(t) = f(R_1(t+1), R_2(t+1))$ where f is given by Lemma 4.2, and the stopping time $S = \min\{t : X(t) = 0\}$ (it is meant that $S = +\infty$ if no such t exists). The stopped process $X(t \wedge S)$ is a local martingale due to (4.1). In general, a trajectory of a continuous local martingale has only two possible types of behavior for $t \to \infty$; either it tends to a limit, or it is unbounded *both from above and from below* (see [E, 4.45]). Our process $X(t \wedge S)$ is bounded from below (by 0), therefore it is bounded from above, which is impossible unless $S < \infty$. □

The main part of the by-product shown on Fig. 1(a) is now proved: α-minima exist for any $\alpha < \pi/2$. In order to prove their absence for $\alpha \ge \pi/2$ it suffices to consider the case $\alpha = \pi/2$. Here, R_1 and R_2 are independent. Take $f(x,y) = \log(x^2+y^2)$, $X(t) = f(R_1(t_0+t), R_2(t_0+t))$, and $S = \min\{t : R_1(t_0+t) = R_2(t_0+t) = 0\}$, then X is a local martingale till S. Its trajectory cannot tend to $-\infty$, which means that S cannot be finite.

What happens near the critical point $\pi/2$? Let $0 < \pi/2 - \alpha \ll 1$, then the correlation coefficient $\rho = \cos\alpha$ is positive and small; (4.3) gives $(\pi/2)a = \alpha$. The value $X(0) = f(R_1(1), R_2(1))$ is typically close to 1, since the exponent $(2-2a)/(2-a)$ of (4.4) is a small positive number. Assume that $X(0) = 1$, then X can reach 0 earlier or later than 2, chances being 50/50. However, $f(x,y) = 2$ means that $x+y$ is a huge number, about $2^{(2-a)/(2-2a)}$. Thus, the time t when X reaches 2 is typically huge: $1/\log t \asymp 1-a \asymp \pi/2-\alpha$, and the same applies for the time of reaching 0. (Unlike percolation, there are no critical exponents at the critical point $\alpha = \pi/2$.) Fig. 1(a) uses an angle $\alpha = \pi/3$ not close to $\pi/2$, since otherwise the simulation would delay the paper for an exponentially long time!

4.7 Lemma. *Let R_1, R_2, ρ be as in Lemma 4.6, and f be the function defined by* (4.3–4.5). *Then the process $X(t) = f(R_1(t), R_2(t))$ is a submartingale.*

Proof. For any $\varepsilon > 0$ define

$$f_\varepsilon(x,y) = f(x+\varepsilon, y+\varepsilon)\,, \qquad X_\varepsilon(t) = f_\varepsilon\big(R_1(t), R_2(t)\big)\,.$$

For $\varepsilon \to 0$ we have $f_\varepsilon \to f$ uniformly, therefore $X_\varepsilon \to X$ uniformly. It suffices to prove that each X_ε is a submartingale.

The function f_ε being C^2-smooth, (4.1) and 4.2(c) give $X_\varepsilon \overset{\mathrm{m}}{=} V_{f_\varepsilon}$, where

(4.8)
$$V_{f_\varepsilon}(t) = \frac{1}{2}\int_0^t (f_\varepsilon)_1(0, R_2(s))\, dL_s(R_1) + \frac{1}{2}\int_0^t (f_\varepsilon)_2\big(R_1(s), 0\big)\, dL_s(R_2)\,.$$

In order to prove that V_{f_ε} increases, it suffices to check that $(f_\varepsilon)_1(0,y) \ge 0$ and $(f_\varepsilon)_2(x,0) \ge 0$ for all $x, y \ge 0$. We have

$$\begin{aligned}
&(f_\varepsilon)_2(x,0) = \frac{\partial}{\partial y}\bigg|_{y=0} \operatorname{Re} Z^{(2-2a)/(2-a)}(x+\varepsilon, y+\varepsilon) \\
&\quad = \frac{2-2a}{2-a}\operatorname{Re}\big(Z^{-a/(2-a)}(x+\varepsilon,\varepsilon)\cdot\big(-i\exp i\tfrac{\pi a}{4}\big)\big)\,; \\
&Z(x+\varepsilon,\varepsilon) = (x+2\varepsilon)\sin\tfrac{\pi a}{4} + ix\cos\tfrac{\pi a}{4} = x\cdot i\exp\big(-i\tfrac{\pi a}{4}\big) + 2\varepsilon\sin\tfrac{\pi a}{4}\,; \\
&Z^{-a/(2-a)}(x+\varepsilon,\varepsilon)\cdot\exp\big(i\tfrac{\pi a}{4}\big) = \big(Z(x+\varepsilon,\varepsilon)\cdot\exp\big(-i\tfrac{\pi a}{4}\cdot\tfrac{2-a}{a}\big)\big)^{-a/(2-a)}
\end{aligned}$$

$$= \Big(\big(x \cdot i \exp\big(-i\tfrac{\pi a}{4}\big) + 2\varepsilon \sin \tfrac{\pi a}{4} \big) \cdot \big(-i \exp\big(i\tfrac{\pi a}{4}\big)\big)^{-a/(2-a)}$$
$$= \Big(x - 2i\varepsilon \sin \tfrac{\pi a}{4} \exp\big(i\tfrac{\pi a}{4}\big) \Big)^{-a/(2-a)} ;$$

so,

$$\begin{aligned} f_2(x+\varepsilon,\varepsilon) &= (f_\varepsilon)_2(x,0) = \\ &= \frac{2-2a}{2-a} \operatorname{Im} \Big(\big(x - 2i\varepsilon \sin \tfrac{\pi a}{4} \exp\big(i\tfrac{\pi a}{4}\big)\big)^{-a/(2-a)} \Big) , \end{aligned} \tag{4.9}$$

which is positive, since

$$\arg \big(x - 2i\varepsilon \sin \tfrac{\pi a}{4} \exp(i\tfrac{\pi a}{4})\big) \in \big[-\tfrac{\pi}{2} + \tfrac{\pi a}{4}, 0 \big] .$$

So, the process V_{f_ε} increases. It remains to check that the local martingale $X_\varepsilon - V_{f_\varepsilon}$ is a martingale. The gradient of f_ε is bounded, therefore $\langle X_\varepsilon - V_{f_\varepsilon}, X_\varepsilon - V_{f_\varepsilon}\rangle(t) \le \text{const} \cdot t$ with a constant depending on ε and $\rho_{\max}$ but not ω nor t. □

It may seem strange that the limiting procedure $\varepsilon \to 0$ succeeds while the estimation for the quadratic variation depends on ε. However, the convergence $X_\varepsilon \to X$ is without doubt, irrespective of the estimation. The process X belongs to the class Σ_+ (see 3.1), $X = M + V$, $V = \lim_{\varepsilon\to 0} V_{f_\varepsilon}$, $M = \lim_{\varepsilon \to 0}(X_\varepsilon - V_{f_\varepsilon})$ in the ucp-topology. It is natural to define V_f by $V_f = V$. The process M is a martingale (not only a local martingale), which follows from the fact that

$$\mathbb{E} \sup_{s\in[0,t]} |X(s)| < \infty \tag{4.10}$$

by the Doob-Meyer decomposition theorem (see [Me2, the theorem in Sect. 11]).

The above lemma states, in fact, that the function $-f$ is excessive for the two-dimensional Markov process (R_1, R_2). However, we cannot restrict ourselves to Markov processes; we have to consider (R_1, R_2) whose instant correlation coefficient $\rho(t,\omega)$ is a random process. Fortunately, the same function fits.

4.11 LEMMA. *Let R_1, R_2 be two copies of the reflecting Brownian motion such that their instant correlation coefficient $\rho(t) = \rho(t,\omega)$ satisfies the inequality*

$$\rho(t) \le \rho_{\max} \quad \textit{for all } t, \textit{ with probability 1}$$

for a given number $\rho_{\max} \in (0,1)$. Let f be the function defined by (4.4–4.5) with a such that $\cos(\pi a/2) = \rho_{\max}$. Then the process $X(t) = f(R_1(t), R_2(t))$ is a submartingale.

Proof. The same as the proof of Lemma 4.7, but one more term appears:

$$\int_0^t \bigl(\rho(s)-\rho_{\max}\bigr) f_{12}\bigl(R_1(s),R_2(s)\bigr)\,ds\,.$$

In order to prove that the above process increases, it is enough to check that $f_{12}(x,y)\le 0$ for all $x,y\in(0,\infty)$. We have

$$\begin{aligned} f_{12}(x,y) &= \frac{\partial^2}{\partial x\partial y}\operatorname{Re} Z^{(2-2a)/(2-a)}(x,y) \\ &= \operatorname{Re}\Biggl(\frac{d^2}{dz^2}\bigg|_{z=Z(x,y)} z^{(2-2a)/(2-a)} \cdot \underbrace{\frac{d}{dx}Z(x,y)\cdot\frac{d}{dy}Z(x,y)}_{=1}\Biggr) \\ &= \frac{2-2a}{2-a}\cdot\Bigl(-\frac{a}{2-a}\Bigr)\cdot\operatorname{Re} Z^{-2/(2-a)}(x,y)\,, \end{aligned}$$

which is negative, since

$$\Bigl| -\frac{2}{2-a}\arg Z(x,y)\Bigr| \le \frac{2}{2-a}\cdot\Bigl(\frac{\pi}{2}-\frac{\pi a}{4}\Bigr) = \frac{\pi}{2}\,. \qquad \square$$

This time the process X does not belong to Σ_+. It has a non-singular drift $\int(\rho-\rho_{\max})f_{12}\,dt$, and also a singular drift concentrated on the common zeros of R_1,R_2. Subtracting both drifts, we get a martingale (see (4.10)).

The singular drift of X is the limit, for $\varepsilon\to 0$, of V_{f_ε} defined by (4.8–4.9). It follows from (4.9) that

$$f_2(x+\varepsilon,\varepsilon)\ge \mathrm{const}_1\cdot\varepsilon^{-a/(2-a)} \quad \text{for } x\le \mathrm{const}_2\cdot\varepsilon$$

with two positive constants depending on a but not ε. We have

$$\begin{aligned} &\tfrac12\mathbb{E}\int_0^t \mathrm{const}_1\cdot\varepsilon^{-a/(2-a)}\mathbf{1}_{(R_1(s)\le\mathrm{const}_2\cdot\varepsilon)}\,dL_s(R_2)\le \mathbb{E}\,X_\varepsilon(t)\,; \\ &\mathbb{E}\int_0^t \mathbf{1}_{(R_1(s)=0)}\,dL_s(R_2) = \lim_{\varepsilon\to0}\mathbb{E}\int_0^t \mathbf{1}_{(R_1(s)\le\mathrm{const}_2\cdot\varepsilon)}\,dL_s(R_2) \\ &= \lim_{\varepsilon\to0} O(\varepsilon^{a/(2-a)}) = 0\,; \end{aligned}$$

thus, the following result is proved.

4.12 Lemma. *Let $R_1,R_2,\rho_{\max}$ be as in Lemma 4.11 Then*

$$\int_0^t \mathbf{1}_{(R_1(s)=0)}\,dL_s(R_2) = 0\,.$$

The second by-product, shown on Fig. 1(b), can be deduced from the lemma by using the so-called first order calculus [RY, VI.4]. The calculus will be used intensively in section 6, while the by-products will not be used.

So, common zeros are rare for two copies of the reflecting Brownian motion whose instant correlation is bounded away from $+1$. It follows (see the end of section 3) that common zeros are rare for two copies of Walsh's Brownian motion, if $\rho_{\max}(J) < 1$.

4.13 Theorem. *A filtration generated by Walsh's Brownian motion is not cozy.*

Proof. Assume the contrary: there is a sequence (J_n) of self-joinings over the filtration, satisfying conditions (a), (b) of Definition 2.4. By Lemma 3.2, condition (b) ensures that

$$\mathbb{E} \int_0^t 1_{(R_1^{(n)} \neq 0)} \, dL(R_2^{(n)}) \to 0 \quad \text{for } n \to \infty \, .$$

By Lemma 4.12, condition (a) ensures that

$$\int_0^t 1_{(R_1^{(n)} = 0)} \, dL(R_2^{(n)}) = 0 \, .$$

Therefore,

$$\mathbb{E} \int_0^t dL(R_2^{(n)}) \to 0 \quad \text{for } n \to \infty \, ,$$

which is evidently impossible, because the integral is positive and does not depend on n, since each $R_2^{(n)}$ is a reflecting Brownian motion. □

4.14 Theorem. *There is no morphism from a filtration generated by a finite or countable collection of independent Brownian motions to a filtration generated by Walsh's Brownian motion.*

Proof. Follows from Theorems 2.7 and 4.13. □

The above theorem appeared for the first time in my preprint "Walsh process filtration is not Brownian", August 1996 (mentioned in [Y2, Epilogue to Chapter 17]). Only a single one-dimensional Brownian motion was considered, but the generalization for a finite or countable collection is straightforward. The proof of Lemma 4.12 was very cumbersome. A much simpler proof, found by Emery in November 1996 and presented in [EY], exerted some influence on my second proof given here. Being much simpler than my first proof, it is nonetheless more complicated than the proof of [EY], but sheds some additional light on the singularity.

The following two sections contain some generalizations. Most of them were found independently by Barlow, Emery, Knight, Song, and Yor

[BEKSoY], their proofs being better than mine due to effective use of so-called splitting multiplicity. This is an integer-valued invariant of a filtration, introduced in [BPY, Def. 4.2], where the problem, to find the splitting multiplicity of the Brownian filtration [BPY, Problem 1], was left open. See also [Y2, p. 108, "M. Barlow's conjecture"]. The problem is solved in [BEKSoY]: the multiplicity is equal to 2 (as expected), irrespective of the dimension of the Brownian motion. In this sense, a boundary is two-sided in the space of Brownian sample paths, or rather in the space $\Omega \times [0, \infty)$ equipped with the optional σ-field.

The results of section 5 will not be used further, while the main result of section 6 (Theorem 6.1) is the clue to the application to harmonic measures we pursue.

5 A Generalization: Change of Measure, or Drift

We have now two examples of non-Brownian (that is, not isomorphic to the filtration $\mathcal{F}_B$ generated by the one-dimensional Brownian motion B) filtrations of instant dimension 1 (identically). The first example [DFSmT] is of the form $(\Omega, \mathcal{F}_B, Q)$ where $(\Omega, \mathcal{F}_B, P)$ is the Brownian filtered probability space and Q is an equivalent (that is, mutually absolutely continuous) measure, $Q \sim P$. The second example is generated by Walsh's Brownian motion.

The following question was asked me by A. Skorokhod in Summer 1996. The first example was already known, the second example was only conjectured. We both expected Walsh's Brownian motion to be a counterexample, and it really is, as follows from Theorem 5.2 below.

5.1 Question [Sk2]. Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space of instant dimension 1. Is there a probability measure Q equivalent to P such that $(\Omega, \mathcal{F}, Q)$ is Brownian?

5.2 Theorem. *Let $(\Omega, \mathcal{F}_Z, P)$ be the filtered probability space generated by Walsh's Brownian motion Z, and Q be a probability measure equivalent to P. Then the filtered probability space $(\Omega, \mathcal{F}_Z, Q)$ is non-cozy.*

This version of Theorem 4.13 is robust under measure changes. It will follow from robust versions of Lemmas 3.2 and 4.12. Let us start with the latter.

5.3 Lemma. *Let R be a reflecting Brownian motion on $(\Omega, \mathcal{F}, P)$, and $Q \sim P$. Let $J = ((\tilde\Omega, \tilde{\mathcal{F}}, \tilde Q), \pi_1, \pi_2)$ be a self-joining over $(\Omega, \mathcal{F}, Q)$ such* *that* $\rho_{\max}(J) < 1$. *Then*

$$\int_0^t 1_{(R_1(s)=0)}\, dL_s(R_2) = 0$$

where $R_k = R \circ \pi_k$.

Proof. We assume $\mathcal{F} = \mathcal{F}_R$ without loss of generality. The measures P and Q (or rather the corresponding filtered probability spaces) form a Girsanov pair as defined in [RY, VIII.1.8]: the density (that is, the Radon-Nikodym derivative) of Q w.r.t. P is of the form

$$\exp\left(\int_0^\infty \Phi(t)\, dB(t) - \frac{1}{2}\int_0^\infty \Phi^2(t)\, dt\right)$$

where Φ is a process on $(\Omega, \mathcal{F}, Q)$ and $B(t) = R(t) - \frac{1}{2}L_t(R)$. The martingale part B of R is a P-Brownian motion, that is, a Brownian motion on $(\Omega, \mathcal{F}, P)$. The process Φ emerges from the local martingale $\int \Phi\, dB$ (denoted by L in [RY]) via the martingale representation theorem for B [RY, V.3.4] and the equality $\mathcal{F}_R = \mathcal{F}_B$ [RY, VI.2.2]. The process

$$B(t) - \int_0^t \Phi(s)\, ds$$

is a Q-martingale by Girsanov's theorem [RY, VIII.1.7], thus the two processes $B \circ \pi_k - (\int \Phi\, ds) \circ \pi_k$ $(k = 1, 2)$ are $\tilde{Q}$-martingales. We want to get a measure $\tilde{P}$ equivalent to $\tilde{Q}$ such that both π_1 and π_2 send $\tilde{P}$ into P (in addition to sending $\tilde{Q}$ into Q) by constructing $\tilde{\Phi}_1, \tilde{\Phi}_2$ such that

$$\exp\left(\int \tilde{\Phi}_1\, d(B \circ \pi_1) + \int \tilde{\Phi}_2\, d(B \circ \pi_2) - \tfrac{1}{2}\int (\tilde{\Phi}_1^2 + \tilde{\Phi}_2^2 + 2\rho\tilde{\Phi}_1\tilde{\Phi}_2)\, ds\right)$$

can serve as the density of $\tilde{Q}$ w.r.t. $\tilde{P}$. Here the process ρ is the instant correlation between $B \circ \pi_1$ and $B \circ \pi_2$. Processes $B \circ \pi_k$ must be $\tilde{P}$-martingales. Girsanov's correction to $B \circ \pi_k$ must be equal to $\int (\Phi \circ \pi_k)\, ds$. That is, we need

$$\int (\Phi \circ \pi_k)\, ds = \left\langle B \circ \pi_k, \int \tilde{\Phi}_1\, d(B \circ \pi_1) + \int \tilde{\Phi}_2\, d(B \circ \pi_2)\right\rangle ;$$

$$\begin{cases} \Phi \circ \pi_1 = \tilde{\Phi}_1 + \rho\tilde{\Phi}_2\,, \\ \Phi \circ \pi_2 = \rho\tilde{\Phi}_1 + \tilde{\Phi}_2\,. \end{cases}$$

The solution exists, since $\rho \neq 1$,

$$\begin{cases} \tilde{\Phi}_1 = \dfrac{1}{1-\rho^2}(\Phi \circ \pi_1 - \rho\Phi \circ \pi_2)\,, \\ \tilde{\Phi}_2 = \dfrac{1}{1-\rho^2}(\Phi \circ \pi_2 - \rho\Phi \circ \pi_1)\,. \end{cases}$$

It is not clear whether or not the corresponding local martingale is a martingale (see [RY, VIII.1.14–16] about such difficulties). For each n introduce a stopping time $S_n = \min\bigl\{t : \bigl|\int\tilde\Phi_1\,d(B\circ\pi_1)+\int\tilde\Phi_2\,d(B\circ\pi_2)\bigr|\ge n\bigr\}$ and a measure $\tilde P_n$ on $\tilde{\mathcal F}_{S_n}$ such that the density of $\tilde Q$ w.r.t. $\tilde P_n$ on $\tilde{\mathcal F}_{S_n}$ is given by the exponential formula written before. By stopping all the processes at S_n we get two processes $R_k = R\circ\pi_k$ on $(\tilde\Omega,\tilde{\mathcal F}_{S_n},\tilde P_n)$ that are stopped reflecting Brownian motions whose instant correlation does not exceed $\rho_{\max}(J)$. (Definition 2.3 can be made explicitly invariant under measure changes by replacing "martingales" with "semimartingales"; quadratic covariations are invariant.) Lemma 4.12 gives

$$\int_0^{t\wedge S_n} 1_{(R_1(s)=0)}\,dL_s(R_2) = 0$$

$\tilde P_n$-almost sure, therefore, $\tilde Q$-almost sure. However, $S_n\to\infty$ for $n\to\infty$. □

5.4 Note. It was shown that any measure change on Ω can be lifted to $\tilde\Omega$ provided that $\rho_{\max}(J)<1$ (though, a localization by stopping times is needed). The fact is true for any filtration, and the above proof can be generalized from the instant dimension 1 to any instant dimension, finite or infinite.

The next lemma is a robust version of Lemma 3.2. Convergence in probability is stated rather than L_1-convergence, since the local time, being P-integrable, need not be Q-integrable.

5.5 Lemma. *Let Z be Walsh's Brownian motion on $(\Omega,\mathcal F,P)$, and $Q\sim P$. Let (J_n) be a sequence of self-joinings $J_n = ((\tilde\Omega_n,\tilde{\mathcal F}_n,\tilde Q_n),\pi_1^{(n)},\pi_2^{(n)})$ over $(\Omega,\mathcal F,Q)$ satisfying condition* (b) *of Definition 2.4:* $\operatorname{Cov}(f\circ\pi_1^{(n)},f\circ\pi_2^{(n)})\to\operatorname{Var}(f)$ *when $n\to\infty$ for any $f\in L_2(\mathcal F(\infty))$. Then for any $t\in[0,\infty)$*

$$\int_0^t 1_{(R_1^{(n)}\ne 0)}\,dL(R_2^{(n)})\to 0 \quad \textit{in probability, for } n\to\infty\,,$$

where $R_k^{(n)}(s) = |Z(s)|\circ\pi_k^{(n)}$.

The proof will be given after two more lemmas. Assuming that $\rho_{\max}(J_n)<1$ for each n, we can lift the equivalence $Q\sim P$ to $\tilde\Omega_n$ similarly to the proof of Lemma 5.3, but it does not help, since the lifted equivalence is not uniform in n. Another idea will be used: the singular drift $1_{(R_1\ne0)}\,dL(R_2)$ of the distance process D_J (see Lemma 3.3) cannot be killed by the regular Girsanov's drift. The two drifts will be separated by a random set on the time axis. The next lemma is a robust version of Lemma 3.3.

5.6 LEMMA. *Let Z be Walsh's Brownian motion on $(\Omega,\mathcal{F},P)$, and $Q\sim P$. Then there is a process Φ on $(\Omega,\mathcal{F},Q)$ such that*

(a) *$\int_0^t \Phi^2(s)\,ds<\infty$ almost sure for any t,*

(b) *for any self-joining J over $(\Omega,\mathcal{F},Q)$, the following process is a sum of a martingale and an increasing process:*

$$X(t)+\int_0^t |\Phi_1(s)|\,ds+\int_0^t |\Phi_2(s)|\,ds\,;$$

here X is as in Lemma 3.3, and $\Phi_k=\Phi\circ\pi_k$.

Proof. Similarly to the proof of Lemma 5.3, we write the density of Q w.r.t. P in the form

$$\exp\left(\int_0^\infty \Phi(t)\,dB(t)-\frac{1}{2}\int_0^\infty \Phi^2(t)\,dt\right)$$

with some Φ satisfying (a), and $B(t)=R(t)-\frac12 L_t(R)$, $R(t)=|Z(t)|$. The martingale representation theorem for Z is used (Theorem 4.1 of [BPY]).

Similarly to the proof of Lemma 3.3, we introduce Y_1,Y_2,Y_3 and $H_1,\dots,H_6$. Still, $A=D_J-\sum_{k=1,\dots,6}\int H_k\,dY_k$ is an increasing process. The process $2X_k-X_1-X_2-X_3+\frac16 L(R)$ is a Brownian motion on $(\Omega,\mathcal{F},P)$. Its Girsanov's correction is $\int \Phi\rho\,dt$ where ρ is its instant correlation with B. Thus, its differential drift on $(\Omega,\mathcal{F},Q)$ is between $\pm|\Phi|$, and its quadratic variation is that of Brownian motion. Accordingly, the differential drift of $Y_k+\frac16 L(R_1)-\frac16 L(R_2)$ is between $\pm(|\Phi|\circ\pi_1+|\Phi|\circ\pi_2)$, and its quadratic variation on any $[s,t]$ does not exceed $4(t-s)$. Both assertions hold for $X-A$, since $X-A=\sum_{k=1}^3\int(H_k-H_{3+k})\,d\big(Y_k+\frac16 L(R_1)-\frac16 L(R_2)\big)$ and $\sum_{k=1}^3|H_k-H_{3+k}|\le 1$. Therefore, $X-A+\int_0^t(|\Phi|\circ\pi_1+|\Phi|\circ\pi_2)\,ds$ is a martingale plus an increasing process. □

5.7 LEMMA. *Let Z, $(\Omega,\mathcal{F},P)$, Q and Φ be as in Lemma 5.6, J_n and $R_k^{(n)}$ as in Lemma 5.5, and H be a locally finite variation process on $(\Omega,\mathcal{F},Q)$ such that $0\le H(t)\le 1$ always. Then for any $M\in[0,\infty)$*

$$\begin{aligned}
&\limsup_{n\to\infty}\mathbb{E}\left(M\wedge\frac16\int_0^t H_2^{(n)}1_{(R_1^{(n)}\neq 0)}\,dL(R_2^{(n)})\right)\\
&\quad\le 2\left(\mathbb{E}\int_0^t H^2(s)\,ds\right)^{1/2}\\
&\qquad+\limsup_{n\to\infty}\mathbb{E}\left(M\wedge\int_0^t H_2^{(n)}(s)\big(|\Phi_1^{(n)}(s)|+|\Phi_2^{(n)}(s)|\big)\,ds\right);
\end{aligned}$$

here $H_2^{(n)}=H\circ\pi_2^{(n)}$ and $\Phi_k^{(n)}=\Phi\circ\pi_k^{(n)}$.

Proof. Consider the processes

$$D_n(t) = \operatorname{dist}\big(Z(t) \circ \pi_1^{(n)},\, Z(t) \circ \pi_2^{(n)}\big)\,,$$

where dist is the "geodesic" metric introduced before Lemma 3.3. There, it was noted that $D_n(t) \to 0$ in L_2, but now $Z(t)$ need not belong to L_2, and we use a weaker statement: for any $t \in [0,\infty)$

$$D_n(t) \to 0 \quad \text{in probability, for } n \to \infty\,.$$

The above pointwise convergence statement follows from the fact that $X \circ \pi_1^{(n)} - X \circ \pi_2^{(n)} \to 0$ in probability, for any random variable X on $(\Omega, \mathcal{F}, Q)$. Really, for $X \in L_2$ it follows from the L_2-convergence, the L_2 being dense in probability. A stronger statement holds: for any $t \in [0,\infty)$

$$\max_{s\in[0,t]} D_n(s) \to 0 \quad \text{in probability, for } n \to \infty\,.$$

The above locally uniform convergence statement follows from the pointwise convergence due to equicontinuity:

$$\begin{aligned}
&\mathbb{P}_{\tilde{Q}_n}\Big(\max_{|r-s|\le\delta} |D_n(r) - D_n(s)| > \varepsilon \Big) \\
&\quad \le \mathbb{P}_{\tilde{Q}_n}\bigg(\sum_{k=1,2} \max_{|r-s|\le\delta} \operatorname{dist}\big(Z(r)\circ\pi_k^{(n)}, Z(s)\circ\pi_k^{(n)}\big) > \varepsilon \bigg) \\
&\quad \le 2\mathbb{P}_Q\Big(\max_{|r-s|\le\delta} \operatorname{dist}\big(Z(r), Z(s)\big) > \tfrac{\varepsilon}{2} \Big)\,;
\end{aligned}$$

the latter does not depend on n and tends to 0 for $\delta \to 0$ (meaning that $r, s \in [0,t]$).

For each n the process D_n on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{Q}_n)$ is a semimartingale, since the following process is a local submartingale by Lemma 5.6:

$$\begin{aligned}
D_n(t) - \frac{1}{6}\int_0^t 1_{(R_1^{(n)}\ne 0)}\, dL(R_2^{(n)}) - \frac{1}{6}\int_0^t 1_{(R_2^{(n)}\ne 0)}\, dL(R_1^{(n)}) \\
+ \int_0^t |\Phi_1^{(n)}(s)|\, ds + \int_0^t |\Phi_2^{(n)}(s)|\, ds\,.
\end{aligned}$$

In fact, $\langle D_n\rangle(t) - \langle D_n\rangle(s) \le 4(t-s)$ for $0 < s < t$ (see the proof of Lemma 5.6). Stochastic integrals $\int H_2^{(n)}\, dD_n$ are well-defined, and for each $t \in [0,\infty)$

$$\int_0^t H_2^{(n)}\, dD_n \to 0 \quad \text{in probability, for } n \to \infty\,,$$

which follows from the locally uniform convergence $D_n \to 0$ and the finite variation property of H via integration by parts: for any $C \in (0,\infty)$,

$t \in [0,\infty)$, and $\varepsilon > 0$

$$\limsup_{n\to\infty} \mathbb{P}_{\tilde{Q}_n}\Big(\Big|\int_0^t H_2^{(n)}\,dD_n\Big|>\varepsilon\Big) \le \limsup_{n\to\infty} \mathbb{P}_{\tilde{Q}_n}\Big(\Big(\overset{t}{\underset{0}{\mathbb{V}}} H_2^{(n)}\Big)\Big(\max_{[0,t]} D_n\Big) > \varepsilon\Big)$$
$$\le \mathbb{P}_Q\Big(\overset{t}{\underset{0}{\mathbb{V}}} H > C\Big) + \underbrace{\limsup_{n\to\infty} \mathbb{P}_{\tilde{Q}_n}\Big(\max_{[0,t]} D_n > \varepsilon/C\Big)}_{=0}\,;$$

the first expression does not depend on C, while the third expression tends to 0 for $C \to \infty$. Here $\overset{t}{\underset{0}{\mathbb{V}}} f(t)$ is the total variation of f on $[0,t]$, including $|f(0)|$ and $|f(t)|$. We have

$$D_n(t) - \tfrac{1}{6}\int_0^t 1_{(R_1^{(n)}\ne 0)}\,dL(R_2^{(n)}) - \tfrac{1}{6}\int_0^t 1_{(R_2^{(n)}\ne 0)}\,dL(R_1^{(n)})$$
$$+ \int_0^t |\Phi_1^{(n)}(s)|\,ds + \int_0^t |\Phi_2^{(n)}(s)|\,ds = M_n(t) + A_n(t)$$

for some martingale M_n such that $\langle M_n\rangle(t) - \langle M_n\rangle(s) \le 4(t-s)$ for $0 < s < t$ and some increasing process A_n. Therefore

$$\tfrac{1}{6}\int_0^t H_2^{(n)} 1_{(R_1^{(n)}\ne 0)}\,dL(R_2^{(n)})$$
$$\le \int_0^t H_2^{(n)}\,dD_n - \int_0^t H_2^{(n)}\,dM_n + \int_0^t H_2^{(n)}\big(|\Phi_1^{(n)}| + |\Phi_2^{(n)}|\big)\,ds\,;$$
$$\mathbb{E}\Big(\int_0^t H_2^{(n)}\,dM_n\Big)^2 = \mathbb{E}\int_0^t \big(H_2^{(n)}\big)^2\,d\langle M_n\rangle$$
$$\le 4\mathbb{E}\int_0^t \big(H_2^{(n)}\big)^2\,ds = 4\mathbb{E}\int_0^t \big(H(s)\big)^2\,ds\,;$$
$$\mathbb{E}\Big(M \wedge \Big|\int_0^t H_2^{(n)}\,dM_n\Big|\Big) \le \mathbb{E}\Big|\int_0^t H_2^{(n)}\,dM_n\Big| \le 2\Big(\mathbb{E}\int_0^t H^2\,ds\Big)^{1/2}\,;$$
$$\mathbb{E}\Big(M \wedge \Big|\underbrace{\int_0^t H_2^{(n)}\,dD_n}_{\to 0 \text{ in prob.}}\Big|\Big) \to 0 \quad \text{for } n \to \infty\,;$$

the statement of the lemma follows immediately. □

Proof of Lemma 5.5. Let H be a locally finite variation process on $(\Omega, \mathcal{F}, Q)$ such that $0 \le H(t) \le 1$ always and $H(t) = 1$ whenever $Z(t) = 0$. Then $dL(|Z|) = H\,dL(|Z|)$, $dL(R_2^{(n)}) = H_2^{(n)}\,dL(R_2^{(n)})$, hence

$$\limsup_{n\to\infty} \mathbb{E}\Big(1 \wedge \int_0^t 1_{(R_1^{(n)}\ne 0)}\,dL(R_2^{(n)})\Big)$$

$$\leq 12\Big(\mathbb{E}\int_0^t H^2(s)ds\Big)^{1/2} + \limsup_{n\to\infty}\mathbb{E}\Big(1\wedge 6\int_0^t H_2^{(n)}\big(|\Phi_1^{(n)}|+|\Phi_2^{(n)}|\big)ds\Big)$$

by Lemma 5.7 (for $M=1/6$). The left-hand side does not depend on H. It suffices to make the right-hand side arbitrarily small by choosing H appropriately.

We cannot take $H(t)=1_{(Z(t)=0)}$, since the process is of locally infinite variation. However, a lot of well-known approximations can be used. In particular (see [RY], Fig. 4 of VI.1), define for any $\delta>0$

$H_\delta(t)=1$ if $|Z(s)|<\delta$ for all s such that $\max\{r\in[0,t]:Z(r)=0\}\leq s\leq t$,
$H_\delta(t)=0$ otherwise,

then H_δ is a locally finite variation process, $H_\delta(t)=1$ whenever $Z(t)=0$, and $H_\delta(t)=0$ whenever $|Z(t)|\geq\delta$. The monotone convergence of H_δ to $1_{(Z=0)}$ for $\delta\to 0$ implies

$$\int_0^t |H_\delta|^2\,ds\to 0\quad\text{for }\delta\to 0\text{ almost sure},$$
$$\mathbb{E}\int_0^t |H_\delta|^2\,ds\to 0\quad\text{for }\delta\to 0\,.$$

It remains to show that

$$\limsup_{n\to\infty}\mathbb{E}\Big(1\wedge 6\int_0^t (H_\delta)_2^{(n)}\big(|\Phi_1^{(n)}|+|\Phi_2^{(n)}|\big)\,ds\Big)\to 0\quad\text{for }\delta\to 0\,.$$

For any $C\in(0,\infty)$, $k=1,2$ and $\varepsilon>0$

$$\begin{aligned}
&\lim_{\delta\to 0}\sup_n \mathbb{P}_{\tilde Q_n}\Big(\int_0^t (H_\delta)_2^{(n)}|\Phi_k^{(n)}|\,ds>\varepsilon\Big)\\
&\quad\leq \lim_{\delta\to 0}\sup_n\bigg(\mathbb{P}_{\tilde Q_n}\bigg(\Big(\int_0^t |\Phi_k^{(n)}|^2\,ds\Big)^{1/2}>C\bigg)\\
&\qquad+\mathbb{P}_{\tilde Q_n}\bigg(\Big(\int_0^t \big|(H_\delta)_2^{(n)}\big|^2\,ds\Big)^{1/2}>\frac{\varepsilon}{C}\bigg)\bigg)\\
&\quad\leq \mathbb{P}_Q\bigg(\Big(\int_0^t |\Phi|^2\,ds\Big)^{1/2}>C\bigg)+\underbrace{\lim_{\delta\to 0}\mathbb{P}_Q\bigg(\Big(\int_0^t |H_\delta|^2\,ds\Big)^{1/2}>\frac{\varepsilon}{C}\bigg)}_{=0};
\end{aligned}$$

the first expression does not depend on C, while the third expression tends to 0 for $C\to\infty$. So, $\int_0^t (H_\delta)_2^{(n)}\big(|\Phi_1^{(n)}|+|\Phi_2^{(n)}|\big)\,ds\to 0$ in probability, uniformly in n, for $\delta\to 0$. $\square$

Proof of Theorem 5.2. Assume the contrary: there is a sequence (J_n) of self-joinings over $(\Omega, \mathcal{F}, Q)$, satisfying conditions (a), (b) of Definition 2.4. By Lemma 5.5, condition (b) ensures that

$$\int_0^t 1_{(R_1^{(n)} \neq 0)} \, dL(R_2^{(n)}) \to 0 \quad \text{in probability, for } n \to \infty \, .$$

By Lemma 5.3, condition (a) ensures that

$$\int_0^t 1_{(R_1^{(n)} = 0)} \, dL(R_2^{(n)}) = 0 \, .$$

Therefore

$$\int_0^t dL(R_2^{(n)}) \to 0 \quad \text{in probability, for } n \to \infty \, ,$$

which is evidently impossible: the integral is always positive, and its distribution does not depend on n, since the distribution of $R_2^{(n)}$ w.r.t. $\tilde{Q}_n$ is the same as the distribution of $R = |Z|$ w.r.t. Q. □

So, the triple point remains an essential singularity in the presence of a non-singular drift.

6 Another Generalization: Asymmetric Triple Point

The process Z on three rays, considered in sections 3–4, is a symmetric case of Walsh's Brownian motion: its distribution is invariant under permutations of the three rays. A nonsingular drift, introduced in section 5, is a peripheral break of symmetry. The central part of the symmetry is the singular drift $L(X_1) + e^{2\pi i/3} L(X_2) + e^{4\pi i/3} L(X_3) = 0$, which means $L(X_1) = L(X_2) = L(X_3)$ (remember the text after Definition 3.1). The change of measure, considered in section 5, left the local times unchanged. Now we turn to the case of three non-overlapping processes X_1, X_2, X_3 of the class Σ_+ (defined by 3.1) that need not satisfy the condition $L(X_1) = L(X_2) = L(X_3)$. That is, the process $Z(t) = X_1(t) + e^{2\pi i/3} X_2(t) + e^{4\pi i/3} X_3(t)$ is driftless when $Z(t) \neq 0$, but may have a singular drift at the origin. The next theorem will be used in the second part of the paper when proving that a boundary is two-sided. A notation: for two positive measures μ_1, μ_2 on $[0, \infty)$ the measure $\mu_1 \wedge \mu_2$ is the greatest μ satisfying both $\mu \le \mu_1$ and $\mu \le \mu_2$. The same for $\mu_1 \wedge \mu_2 \wedge \mu_3$.

6.1 Theorem. *Let X_1, X_2, X_3 be non-overlapping processes of the class Σ_+, defined on $(\Omega, \mathcal{F}, P)$. If $\mathcal{F}$ is cozy then*

$$dL(X_1) \wedge dL(X_2) \wedge dL(X_3) = 0$$

almost sure.

Informally: each time when the process $Z(t)$ goes through the origin, only two rays are active (or even only one). It should not be understood literally, in a topological fashion: the third ray may be visited infinitely often in any neighborhood of the instant, but its instantaneous activity is negligible in the sense that $dL_t(X_k)/dL_t(|Z|) = 0$. Such k exists for $dL(|Z|)$-almost any t.

Postpone the proof of Theorem 6.1 and start with a rather elementary case: $c_1 L(X_1) = c_2 L(X_2) = c_3 L(X_3)$ for some positive numbers c_1, c_2, c_3 (not depending on t, ω). We may take $c_1 X_1$, $c_2 X_2$, $c_3 X_3$ as new X_1, X_2, X_3, which brings us back to $L(X_1) = L(X_2) = L(X_3)$, but the equality $\langle X_1, X_1 \rangle(t) + \langle X_2, X_2 \rangle(t) + \langle X_3, X_3 \rangle(t) = t$ is lost (thus, we get a *time-changed* Walsh's Brownian motion). Fortunately, the equality takes almost no part in section 3, which leads to the following generalization of Lemma 3.2.

6.2 LEMMA. *Let Z be a complex-valued continuous local martingale such that $Z^3(t) \in [0, \infty)$ always, and $Z(0) = 0$. Let (J_n) be a sequence of self-joinings over $\mathcal{F}_Z$ satisfying condition* (b) *of Definition 2.4. Then for any $t \in [0, \infty)$*

$$\int_0^t \mathbf{1}_{(R_1^{(n)} \ne 0)} \, dL(R_2^{(n)}) \to 0 \quad \text{in probability, for } n \to \infty \, .$$

Proof. It suffices to prove the lemma for stopped processes $Z(t \wedge T_m)$ for some stopping times $T_m \to \infty$ (since the convergence in m is uniform in n). Thus, we may assume that the processes Z, $\langle Z, Z \rangle$ and $L(|Z|)$ are bounded in t and ω. Then the arguments of section 3 need only trivial adaptation. □

The trick of substituting $c_1 X_1$, $c_2 X_2$, $c_3 X_3$ for X_1, X_2, X_3 is generalized from constants c_1, c_2, c_3 to processes C_1, C_2, C_3 by the next lemma. Its hypothesis is fulfilled, in particular, by the Brownian filtration, which, however, will be proved only after Lemmas 6.4–6.8.

6.3 LEMMA. *Let $(\Omega, \mathcal{F}, P)$ be such that any complex-valued continuous local martingale Z on $(\Omega, \mathcal{F}, P)$, satisfying $Z^3(t) \in [0, \infty)$ and $Z(0) = 0$, is necessarily zero. Then any non-overlapping processes X_1, X_2, X_3 of the class Σ_+, defined on $(\Omega, \mathcal{F}, P)$, satisfy $dL(X_1) \wedge dL(X_2) \wedge dL(X_3) = 0$ almost surely.*

Proof. Consider the densities (that is, Radon-Nikodym derivatives)

$$H_k = \frac{dL(X_k)}{dL_{\text{all}}} \, , \quad L_{\text{all}} = L(X_1) + L(X_2) + L(X_3) \, .$$

For almost each ω the density is defined for dL_{all}-almost all t. It is possible to define H_k for all t, ω such that H_k is a predictable process, see [JShir, Chap. 1, Prop. 3.13], and

$$L(X_k) = \int H_k \, dL_{\mathrm{all}}, \quad 0 \le H_k \le 1, \quad H_1 + H_2 + H_3 = 1.$$

The infimum of measures corresponds to the infimum of densities:

$$dL(X_1) \wedge dL(X_2) \wedge dL(X_3) = (H_1 \wedge H_2 \wedge H_3) \, dL_{\mathrm{all}}.$$

If it vanishes with probability 1, there is nothing to prove. Otherwise there is $\varepsilon > 0$ such that

$$\int 1_{(H_1 \wedge H_2 \wedge H_3 \ge \varepsilon)} \, dL_{\mathrm{all}} > 0$$

with a positive probability. Define predictable processes C_1^0, C_2^0, C_3^0 by

$$C_k^0 = \frac{1}{H_k} \cdot 1_{(H_1 \wedge H_2 \wedge H_3 \ge \varepsilon)},$$

then $0 \le C_k^0 \le 1/\varepsilon$ and

$$\int C_1^0 \, dL(X_1) = \int C_2^0 \, dL(X_2) = \int C_3^0 \, dL(X_3),$$

the integral being > 0 with a positive probability.

The measure dL_{all} is concentrated on the closed set $\{t : X_1(t) = X_2(t) = X_3(t) = 0\}$ of common zeros. Indeed, if an open time interval (r, s) is free of common zeros (for a given ω), then one of X_k, say X_1, is positive on (r, s). Now, clearly, $dL(X_1)$ vanishes on (r, s) for one reason, while $dL(X_2)$ and $dL(X_3)$ vanish on (r, s) for another reason.

Given t (and ω), consider the latest common zero $g(t)$ before t,

$$g(t) = \max \bigl\{ s \in [0, t] : X_1(s) = X_2(s) = X_3(s) = 0 \bigr\}.$$

Define processes C_1, C_2, C_3 by

$$C_k(t) = C_k^0(g(t));$$

they are predictable (see [RY, VI.4.1]), and still, $0 \le C_k \le 1/\varepsilon$ and

$$\int C_1 \, dL(X_1) = \int C_2 \, dL(X_2) = \int C_3 \, dL(X_3),$$

since $C_k = C_k^0$ on the support of dL_{all}.

We use the first order calculus, presented in [RY, VI.4], especially Proposition 4.5 there: processes $C_k X_k$ belong to Σ_+, and

$$L(C_k X_k) = \int C_k \, dL(X_k).$$

Therefore the process

$$Z = C_1 X_1 + e^{2\pi i/3} C_2 X_2 + e^{4\pi i/3} C_3 X_3$$

is a local martingale satisfying $Z^3(t) \in [0,\infty)$ and $Z(0) = 0$, but not identically zero, which contradicts the hypothesis. □

In order to prove Theorem 6.1 it could be enough to generalize Lemma 4.12 from two correlated copies of a reflecting Brownian motion to two correlated copies of a process of the class Σ_+, that is, a time-changed reflecting Brownian motion. However, it does not work, for two reasons. First, section 4, unlike section 3, depends heavily on the constant diffusion speed, $\langle R, R\rangle(t) = t$, since the specific function $f(x,y)$ of Lemma 4.2 satisfies the specific partial differential equation 4.2(c) with $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ appearing with equal coefficients. The second reason is that the set of zeros of a process of Σ_+ may be of a positive Lebesgue measure, in which case $\int 1_{(X_1=0)}\, dL(X_2)$ does not vanish even for *independent* copies X_1, X_2 of the process.

In fact, we do not really need the integral $\int 1_{(X_1=0)}\, dL(X_2)$ to vanish (nor even to be small). Rather, we need the integral $\int 1_{(X_1\ne 0)}\, dL(X_2)$ to be non-small. The first-order calculus allows (see the next proof) transforming it into the need for non-zero X_1 at the starting point of an excursion of X_2. Of course, X_1 may vanish there, but then X_2 does not vanish at the starting point of an excursion of X_1 (which is equally good for us), or else X_1, X_2 start their excursions simultaneously (which should not happen; we will return to that point).

Formally, a starting point of an excursion of a process X (for a given ω) is an instant t such that $X(t) = 0$ but there is $\varepsilon > 0$ satisfying $X(s) > 0$ for all $s \in (t, t+\varepsilon)$. Clearly, such points are a finite or countable set.

6.4 LEMMA. *Let X be a process of the class Σ_+, satisfying $\sup_\omega \langle X\rangle(t) < \infty$ for all t, defined on $(\Omega, \mathcal{F}, P)$, and (J_n) be a sequence of self-joinings $J_n = ((\tilde\Omega_n, \tilde{\mathcal{F}}_n, \tilde P_n), \pi_1^{(n)}, \pi_2^{(n)})$ over $(\Omega, \mathcal{F}, P)$ satisfying condition* (b) *of Definition 2.4. Consider $X_1^{(n)} = X \circ \pi_1^{(n)}$, $X_2^{(n)} = X \circ \pi_2^{(n)}$, and assume that for any n the two processes $X_1^{(n)}, X_2^{(n)}$ have no common starting points of excursions. Then for any $t \in [0,\infty)$*

$$\tfrac12 \liminf_n \mathbb{E}\biggl(\int_0^t 1_{(X_1^{(n)}>0)}\, dL(X_2^{(n)}) + \int_0^t 1_{(X_2^{(n)}>0)}\, dL(X_1^{(n)}) \biggr) \ge \mathbb{E} X(t)\,.$$

Proof. Fix some n for a while. Define predictable processes C_1, C_2 by

$$C_1(t) = 1_{(X_2>0)}(g_1(t))\,, \qquad C_2(t) = 1_{(X_1>0)}(g_2(t))\,,$$

where $X_k = X_k^{(n)}$, and $g_k(t) = \max\{s \in [0,t] : X_k(s) = 0\}$. The first-order calculus states that $C_k X_k$ belongs to Σ_+ and $L(C_k X_k) = \int C_k \, dL(X_k)$. However, $\int C_1 \, dL(X_1) = \int 1_{(X_2>0)} \, dL(X_1)$, since $g_1(t) = t$ on the support of $dL(X_1)$. So, $L(C_1 X_1) = \int 1_{(X_2>0)} \, dL(X_1)$ and $L(C_2 X_2) = \int 1_{(X_1>0)} \, dL(X_2)$. We claim that

$$C_1 X_1 \vee C_2 X_2 \ge X_1 \wedge X_2 \,,$$

that is, $\max(C_1(t)X_1(t), C_2(t)X_2(t)) \ge \min(X_1(t), X_2(t))$ for all t. Indeed, assume $X_1(t) \wedge X_2(t) > 0$ (otherwise the inequality is trivial), then $g_1(t) \ne g_2(t)$. Assume that $g_1(t) < g_2(t)$ (the other case being similar), then $C_2(t) = 1$, therefore

$$C_1(t)X_1(t) \vee C_2(t)X_2(t) \ge C_2(t)X_2(t) = X_2(t) \ge X_1(t) \wedge X_2(t) \,.$$

The process $C_k X_k - \frac{1}{2} L(C_k X_k)$ is a martingale; we have

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left(\int_0^t 1_{(X_1>0)} \, dL(X_2) + \int_0^t 1_{(X_2>0)} \, dL(X_1) \right) \\
&\quad = \frac{1}{2}\mathbb{E}\big(L(C_2(t)X_2(t)) + L(C_1(t)X_1(t)) \big) = \mathbb{E}\big(C_2(t)X_2(t) + C_1(t)X_1(t) \big) \\
&\quad \ge \mathbb{E}\big(X_1(t) \wedge X_2(t) \big) \,.
\end{aligned}$$

Unfix n and note that $\mathbb{E}|X_1^{(n)}(t) - X_2^{(n)}(t)| \to 0$ for $n \to \infty$:

$$\begin{aligned}
&\frac{1}{2} \liminf_n \mathbb{E}\left(\int_0^t 1_{(X_1^{(n)}>0)} \, dL(X_2^{(n)}) + \int_0^t 1_{(X_2^{(n)}>0)} \, dL(X_1^{(n)}) \right) \\
&\qquad \ge \liminf_n \mathbb{E}\big(X_1^{(n)}(t) \wedge X_2^{(n)}(t) \big) = \liminf_n \mathbb{E} X_1^{(n)}(t) = \mathbb{E} X(t) \,. \qquad \square
\end{aligned}$$

A single difficulty remains: why do excursion starting points never coincide? Hypercontractivity helps us to overcome the difficulty.

6.5 Lemma. *Let a self-joining $J = ((\tilde\Omega, \tilde{\mathcal{F}}, \tilde P), \pi_1, \pi_2)$ over $(\Omega, \mathcal{F}, P)$ satisfy $\rho_{\max}(J) < 1$. Let p, q be such that $1 < p < q$ and $p - 1 = (q-1)\rho_{\max}^2(J)$. Let $X \in L_p(\Omega, \mathcal{F}(\infty), P)$ and $Y \in L_{q'}(\Omega, \mathcal{F}(\infty), P)$, where q' is defined by $\frac{1}{q} + \frac{1}{q'} = 1$. Let $X_1 = X \circ \pi_1$ and $Y_2 = Y \circ \pi_2$. Then $X_1 Y_2 \in L_1(\tilde\Omega, \tilde{\mathcal{F}}(\infty), \tilde P)$, and*

$$\|X_1 Y_2\|_1 \le \|X\|_p \|Y\|_{q'} \,.$$

Proof. Ito's formula is applied to $d(M_1^{1/p}(t) N_2^{1/q'}(t))$, where $M_1 = M \circ \pi_1$, $N_2 = N \circ \pi_2$, $M(t) = \mathbb{E}(|X|^p | \mathcal{F}_t)$, $N(t) = \mathbb{E}(|Y|^{q'} | \mathcal{F}_t)$, and the well-known calculation (see [N, the proof of Theorem 1.4.1], see also [RY, V.3.19]) does the job, since $\rho_{\max}$ majorizes the instant correlation of M_1 and N_2 with no mediation of Brownian motions and representation theorems. $\square$

6.6 LEMMA. *Let a self-joining* $J = ((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \pi_1, \pi_2)$ *over* $(\Omega, \mathcal{F}, P)$ *satisfy* $\rho_{\max}(J) < 1$. *Then*

$$\mathbb{P}(A_1 \cap A_2) \le (\mathbb{P}(A))^{2/(1+\rho_{\max}(J))}$$

for any $A \in \mathcal{F}(\infty)$; *here* $A_1 = \pi_1^{-1}A$, $A_2 = \pi_2^{-1}A$.

Proof. Apply Lemma 6.5 for $X = Y = 1_A$, $p = 1 + \rho_{\max}(J)$, $q = 1 + 1/\rho_{\max}(J)$, then

$$\|X\|_p \|Y\|_{q'} = (\mathbb{P}(A))^{\frac{1}{p}+\frac{1}{q'}},$$

$$\frac{1}{p} + \frac{1}{q'} = \frac{1}{1+\rho_{\max}(J)} + 1 - \frac{\rho_{\max}(J)}{\rho_{\max}(J)+1} = \frac{2}{1+\rho_{\max}(J)}. \qquad \square$$

6.7 LEMMA. *Let a self-joining* $J = ((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \pi_1, \pi_2)$ *over* $(\Omega, \mathcal{F}, P)$ *satisfy* $\rho_{\max}(J) < 1$, *and* $X : \Omega \to \mathbb{R}$ *be a random variable such that* $\mathbb{P}(X = x) = 0$ *for each* $x \ne 0$. *Then*

$$\mathbb{P}(X_1 = X_2 \ne 0) = 0,$$

where $X_1 = X \circ \pi_1$, $X_2 = X \circ \pi_2$.

Proof. Divide $\mathbb{R} \setminus \{0\}$ into subsets A_k satisfying $\mathbb{P}(X \in A_k) \le \varepsilon$, then

$$\mathbb{P}(X_1 = X_2 \ne 0) \le \sum_k \mathbb{P}(X_1 \in A_k,\, X_2 \in A_k)$$

$$\le \sum_k \big(\mathbb{P}(X \in A_k)\big)^{2/(1+\rho_{\max}(j))} \le \varepsilon^{\frac{2}{1+\rho_{\max}(J)}-1}\mathbb{P}(X \ne 0) \to 0 \text{ for } \varepsilon \to 0. \square$$

6.8 LEMMA. *Let* X *be a process of the class* Σ_+, *and* $g(t) = \max\{s \in [0,t] : X(s) = 0\}$. *Then*

$$\mathbb{P}(g(t) = s) = 0$$

whenever $0 \le s < t$.

Proof. Define a predictable process

$$C(t) = 1_{(g(t)=s)},$$

s being fixed; the first-order calculus states that $C(t)X(t)$ belongs to Σ_+ and $L(CX) = \int C\, dL(X)$. However, $g(t) = t$ and $C(t) = 1_{(t=s)}$ on the support of $dL(X)$, therefore $\int C\, dL(X) = \int 1_{(t=s)}\, dL_t(X) = 0$. So, $L(CX) = 0$, which means that CX is a local martingale. Also, $C(0)X(0) = 0$ and $C(t)X(t) \ge 0$. It follows that $C(t)X(t) = 0$. It remains to note that $g(t) = s < t$ implies $C(t)X(t) = X(t) \ne 0$. $\square$

Proof of Theorem 6.1. Assume the contrary. By Lemma 6.3 there is a complex-valued continuous local martingale on $(\Omega, \mathcal{F}, P)$, satisfying $Z^3(t) \in [0, \infty)$, $Z(0) = 0$, and not identically zero. On the other hand, there is a sequence (J_n) of self-joinings over $\mathcal{F}$ satisfying conditions (a), (b) of Definition 2.4. Lemma 6.2 gives

$$\int_0^t 1_{(R_1^{(n)}\neq 0)}\, dL(R_2^{(n)}) \to 0 \quad \text{in probability, for } n \to \infty\,.$$

The process $R(t) = |Z(t)|$ belongs to Σ_+. Consider $g(t) = \max\{s \in [0,t] : R(t) = 0\}$. Lemma 6.8 shows that $\mathbb{P}\,(g(t) = s) = 0$ for all $s \in [0,t)$. Lemma 6.7 ensures that

$$\mathbb{P}\left(g(t)\circ\pi_1^{(n)} = g(t)\circ\pi_2^{(n)} \neq t\right) = 0\,.$$

It means that two processes $R_1^{(n)} = R\circ\pi_1^{(n)}$, $R_2^{(n)} = R\circ\pi_2^{(n)}$ have no common starting points of excursions.

Let T be a stopping time such that $\sup_\omega \langle R,R\rangle(T) < \infty$. Lemma 6.4, applied to the stopped process $R(t\wedge T)$, gives

$$\frac{1}{2}\liminf_n \mathbb{E}\left(\int_0^{t\wedge T} 1_{(R_1^{(n)}\neq 0)},dL(R_2^{(n)}) + \int_0^{t\wedge T} 1_{(R_2^{(n)}\neq 0)} dL(R_1^{(n)})\right) \geq \mathbb{E}R(t\wedge T)\,.$$

On the other hand,

$$\int_0^{t\wedge T} 1_{(R_1^{(n)}\neq 0)}\, dL(R_2^{(n)}) \to 0 \quad \text{for } n\to\infty$$

in probability, therefore in L_1, since the sequence is uniformly integrable: $\int_0^{t\wedge T}\dots \leq L_{t\wedge T}(R_2^{(n)})$, and $L_{t\wedge T}(R_2^{(n)})$ is distributed like $L_{t\wedge T}(R)$, the latter being integrable. It follows that $\mathbb{E}R(t\wedge T) = 0$, so $R(t\wedge T) = 0$. Taking a sequence $T_m \to \infty$ we conclude that $|Z(t)| = R(t) = 0$ identically. $\square$

7 Application to Harmonic Measures

Consider the random process $X(t) = f(B(t))$, where $f : \mathbb{R}^d \to \mathbb{R}$ is a smooth function and B an d-dimensional Brownian motion (starting at the origin). Formally, $1 \leq d < \infty$, though dimensions $d = 1, 2$ are of little interest to us. Ito's formula ensures that X is a continuous semimartingale, and

$$X(t) \stackrel{\mathrm{m}}{=} A_X(t)\,, \quad \text{where} \quad A_X(t) = \frac{1}{2}\int_0^t (\Delta f)(B(s))\, ds$$

(see [Me2, item 14]). Clearly, X is a local martingale if and only if $\Delta f = 0$, that is, f is harmonic (see [Ba, II.1.5–7]). Also, X is a local submartingale if and only if $\Delta f \geq 0$, that is, f is subharmonic (see [Ba, II.6.8]). We observe a correspondence between (one half of) the Laplacian Δ, acting on smooth functions, and the operation $X \mapsto A_X$ of taking the drift (that is, the finite variation part) of a semimartingale.[5]

If a sequence (f_k) of smooth functions converges to some continuous function f uniformly on bounded domains, then the corresponding processes $X_k(t) = f_k(B(t))$ converge to $X(t) = f(B(t))$ in the ucp-topology. Therefore, if f_k are subharmonic, then X is a local submartingale, and

$$A_X(t) = \lim_{k\to\infty} \frac{1}{2}\int_0^t (\Delta f_k)(B(s))\,ds\,.$$

The functions Δf_k need not converge in the space of functions, but converge in the space of Schwartz distributions (generalized functions): for any test function φ, smooth and compactly supported,

$$\int \varphi\Delta f_k\,dx = \int f_k\Delta\varphi\,dx \to \int f\Delta\varphi\,dx \stackrel{\text{def}}{=} \int \varphi\Delta f\,dx \quad \text{for } k\to\infty\,.$$

However, $\frac{1}{2}\Delta f_k\,dx = \mu_k$ is a (positive) measure for each k, thus the convergence of Schwartz distributions is just the weak convergence of measures: $\mu_k \to \mu$, where $\mu = \frac{1}{2}\Delta f$ in the sense of Schwartz distributions. In general, μ_k and μ are σ-finite, they may blow up near infinity, but for any bounded domain U, within U the measures are finite and the weak convergence holds.

So, some measures μ correspond to some drifts A (μ being known as the Revuz measure of A); informally,

$$\frac{dA(t)}{dt} = \frac{d\mu}{dx}(B(t)) \quad \text{for } t\in(0,T)\,, \quad \text{i.e.,} \quad A(t) = \frac{1}{2}\int_0^t \left(\frac{d\mu}{dx}(B(s))\right)ds\,,$$

which cannot be understood literally: neither $d\mu/dx$ nor $dA(t)/dt$ can be evaluated at a point. For $d=1$ the situation is simple: any μ is the Revuz measure for some A (see [RY, X.2.10]), and a measure concentrated at a single point x corresponds to the local time of B at x. For $d>1$ a single point is never visited by a Brownian trajectory, it is a polar set. A set $V\subset\mathbb{R}^d$ is called polar, if

$$B(t)\notin V \quad \text{for all} \quad t>0$$

almost surely ([Ba, II.5.12], [RY, V.2.6]). Any measure μ is a sum of a measure μ_{polar} that lives on a polar set and another measure $\mu_{\text{non-polar}}$ that does not charge any polar set.

[5]The semimartingale X is of the special form $f(B(\cdot))$, and its drift A_X, being of the special form $\int g(B(\cdot))\,ds$, is a so-called continuous additive functional of Brownian motion (see [RY, X]). The martingale part of X is a so-called additive local martingale, see [RY, X.2.25(2)].

Ito's formula for Brownian motion can be generalized for non-smooth functions f on $\mathbb{R}^d$ [Ve], [Br], [Me1]. It holds for any f such that the Schwartz distribution $\frac{1}{2}\Delta f$ is a signed measure, finite on bounded domains. Even if f is not continuous, still $f(B(t))$ is a continuous semimartingale, provided that f is properly "upgraded" from a function defined almost everywhere to a function defined up to a polar set. If $\frac{1}{2}\Delta f = \mu \ne 0$ then X is not a martingale, but μ_{polar} and $\mu_{\text{non-polar}}$ act by different mechanisms. If $\mu = \mu_{\text{polar}} \ne 0$ then X is a local martingale, but not a martingale. If $\mu = \mu_{\text{non-polar}} \ne 0$ then X is a sum of a martingale and a locally finite variation process A_μ. The latter depends on μ rather than f, since if $\Delta f_1 = \Delta f_2$ then $f_1 - f_2$ is harmonic, and $f_1(B) \overset{\text{m}}{=} f_2(B)$. So, $A_{\mu_{\text{polar}}} = 0$; $A_\mu = A_{\mu_{\text{non-polar}}}$; and

$$f(B(t)) \overset{\text{m}}{=} A_\mu(t)\,, \qquad \mu = \tfrac{1}{2}\Delta f\,. \tag{7.1}$$

We need only a special case: f is locally bounded, and $\mu = \frac{1}{2}\Delta f$ is a finite positive compactly supported measure. The local boundedness of f implies $\mu = \mu_{\text{non-polar}}$ (see [Br, Prop. 3.2(1,2)]). For such a measure μ, if $\mu \ne 0$ then $A_\mu \ne 0$. Moreover, for any finite number n of such measures $\mu_1, \dots, \mu_n$

$$\mu_1 \wedge \dots \wedge \mu_n \ne 0 \quad \text{implies} \quad dA_{\mu_1} \wedge \dots \wedge dA_{\mu_n} \ne 0\,, \tag{7.2}$$

since $\mu_1 \wedge \dots \wedge \mu_n$ does not charge any polar set.

Consider a domain (a connected open set) $U \subset \mathbb{R}^d$, bounded and containing the origin. The harmonic measure μ_U of U is defined [Ba, III.2.5] as the distribution of the exit point $B(T_U)$ of the Brownian motion from U; here $T_U = \min\{t \in [0,\infty) : B(t) \notin U\}$. The Newtonian potential f_U of μ_U is

$$f_U(x) = \int K(x-y)\,\mu_U(dy) = \mathbb{E}K\bigl(x - B(T_U)\bigr)$$

where the kernel [Ba, Chap. II, formulas (3.1) and (3.24)][6]

$$K(x) = \begin{cases} \dfrac{c_d}{|x|^{d-2}} \quad \text{for } d > 2\,, \qquad c_d = \dfrac{\Gamma(d/2-1)}{2\pi^{d/2}}\,, \\ \dfrac{1}{\pi}\log\dfrac{1}{|x|} \quad \text{for } d = 2\,, \\ -\,|x| \quad \text{for } d = 1 \end{cases}$$

is chosen such that

$$-\tfrac{1}{2}\Delta K = \delta_0\,, \quad \text{therefore} \quad -\tfrac{1}{2}\Delta f_U = \mu_U$$

[6]There, however, $(2\pi)^{d/2}$ is written, mistakenly, instead of $2\pi^{d/2}$.

(here δ_0 is the unit atom at 0), see [Ba, II.3.3]. The function f_U is harmonic on $\mathbb{R}^d \setminus \partial U$. If $x \in \mathbb{R}^d \setminus (U \cup \partial U)$ then $f_U(x) = K(x)$. Indeed, the function $K(x - \cdot)$ is harmonic on U and continuous on $U \cup \partial U$, therefore $K(x - B(t \wedge T_U))$ is a martingale.

The difference between K and f_U is called the Green function g_U of U,

$$g_U(0,x) = K(x) - \mathbb{E}\,K\big(x - B(T_U)\big)\,,$$

see [Ba, II, formula (3.16)]. The function $g_U(0,\cdot)$ is nonnegative (in fact, strictly positive) and continuous on $U \setminus \{0\}$, has a pole at 0, vanishes on $\mathbb{R}^d \setminus (U \cup \partial U)$, and

$$\tfrac12 \Delta g_U(0,\cdot) = \tfrac12 \Delta K - \tfrac12 \Delta f_U = -\delta_0 + \mu_U\,.$$

The function $g_U(0,\cdot)$ is continuous at all regular points of ∂U, see [Ba, II.1.9 and 12]. See also [Ba, II.5.17] for an example of an irregular point (the Lebesgue thorn). Fortunately, irregular points are a polar set [Ba, II.5.5], which means that $g_U(0,B(t))$ is a continuous process; no "upgrade" is needed for $g_U(0,\cdot)$. Of course, there is nothing special in the point 0. For any bounded domain $U \subset \mathbb{R}^d$ and any $x \in U$

$$g_U(x,B(t)) \overset{\mathrm{m}}{=} A_{\mu_{U,x}}(t)\,; \tag{7.3}$$

here $\mu_{U,x}$ is the exit point distribution for a Brownian motion starting from x. Clearly, the process $g_U(x,B(t))$ belongs to the class Σ_+ defined in 3.1.

7.4 Theorem. *Let U_1, U_2, U_3 be pairwise non-intersecting bounded connected open sets in $\mathbb{R}^d$ ($1 \le d < \infty$), $x_k \in U_k$, and $\mu_k = \mu_{U_k,x_k}$ be the corresponding harmonic measures ($k = 1,2,3$). Then*

$$\mu_1 \wedge \mu_2 \wedge \mu_3 = 0\,.$$

Proof. Introduce a d-dimensional Brownian motion B and three random processes

$$X_k(t) = g_k(x_k, B(t))\,, \qquad k = 1,2,3\,,$$

where g_k is the Green function for U_k. These X_1, X_2, X_3 are non-overlapping processes of the class Σ_+, and (7.3) gives $X_k \overset{\mathrm{m}}{=} A_{\mu_k}$, which can be written in terms of local times: $L(X_k) = 2A_{\mu_k}$. Theorem 6.1 (combined with Lemma 2.5) states that $dL(X_1) \wedge dL(X_2) \wedge dL(X_3) = 0$, that is, $dA_{\mu_1} \wedge dA_{\mu_2} \wedge dA_{\mu_3} = 0$ almost sure. It remains to use (7.2). □

Appendix. A Reformulation in Terms of a Measure in a Linear Space

The Wiener measure $\mathcal{W} = \mathcal{W}[0,1]$ on the space $C_0[0,1]$ of all continuous functions on $[0,1]$, vanishing at 0, is the probability distribution for a one-dimensional Brownian motion $B(t)$ considered for $t \in [0,1]$. In some sense, $\mathcal{W}[0,1] = \mathcal{W}\big[0,\frac{1}{2}\big] \otimes \mathcal{W}\big[0,\frac{1}{2}\big]$. Namely:

$$f \longleftrightarrow (g,h)\,, \qquad f \in C_0[0,1]\,, \quad g,h \in C_0[0,\tfrac{1}{2}]\,,$$
$$g(t) = f(t) \quad \text{and} \quad h(t) = f\big(\tfrac{1}{2}+t\big) - f\big(\tfrac{1}{2}\big) \quad \text{for } t \in \big[0,\tfrac{1}{2}\big]\,;$$
$$f(t) = \begin{cases} g(t) & \text{for } t \in \big[0,\frac{1}{2}\big], \\ g\big(\frac{1}{2}\big) + h\big(t-\frac{1}{2}\big) & \text{for } t \in \big[\frac{1}{2},1\big]; \end{cases}$$
$$f = \alpha(g,h)\,, \qquad g = \beta(f)\,, \quad h = \gamma(f)\,;$$
$$\big(C_0[0,1], \mathcal{W}[0,1]\big) \underset{\xleftarrow[\alpha]{}}{\xrightarrow{(\beta,\gamma)}} \big(C\big[0,\tfrac{1}{2}\big], \mathcal{W}\big[0,\tfrac{1}{2}\big]\big) \times \big(C\big[0,\tfrac{1}{2}\big], \mathcal{W}\big[0,\tfrac{1}{2}\big]\big)\,,$$

the latter being a measure preserving one-to-one correspondence. For a fixed g, the map $h \mapsto \alpha(g,h)$ sends $\mathcal{W}\big[0,\frac{1}{2}\big]$ into a measure $\mathcal{W}_g[0,1]$ on $C_0[0,1]$ concentrated on functions f such that $f|_{[0,1/2]} = g$. Clearly, $\mathcal{W}_g[0,1]$ is the conditional distribution of B under the condition $B|_{[0,1/2]} = g$. In principle, the conditional distribution is defined for *almost* all g, but the above canonical construction defines it naturally for all g. Similarly, $\mathcal{W}_g[0,1]$ is defined for $g \in C_0[0,t_0]$, $t_0 \in [0,1]$.

The canonical choice of conditional measures makes all martingales continuous. That is, for any $X \in L_1(C_0[0,1], \mathcal{W}[0,1])$ we consider

$$M(t,f) = \int X \, d\mathcal{W}_{f_0^t}\,,$$

where $f_0^t = f\big|_{[0,t]}$ is the restriction of f to $[0,t]$, and it appears that $M(\cdot,f)$ is a continuous function on $[0,1]$ for $\mathcal{W}$-almost all f. It is the most straightforward definition of a Brownian martingale for the non-probabilist. You see, $M(t) = \mathbb{E}(X|B_0^t)$. Similarly, a complex-valued random variable X produces a complex-valued Brownian martingale M.

It follows from Theorem 6.1 that a complex-valued Brownian martingale M such that $M^3(t) \in [0,\infty)$ for all t and $M(0) = 0$, must vanish identically. In other words: if $X \in L_1(C_0[0,1], \mathcal{W}[0,1])$ satisfies the conditions

$$\Big(\int X \, d\mathcal{W}_g\Big)^3 \in [0,\infty) \quad \text{for almost all } g \in C_0[0,t_0] \text{ and all } t_0 \in [0,1]\,,$$
$$\int X \, d\mathcal{W} = 0\,,$$

then $X = 0$. The statement is strong enough, it implies the negative answer to Problem 2 of [BPY], even if we restrict ourselves to $X \in L_\infty$ rather than $X \in L_1$.

Consider two linear spaces $\mathcal{X}, \mathcal{Y}$ in duality: $\mathcal{X}$ is the space of all real-valued bounded Borel functions on $C_0[0,1]$, while $\mathcal{Y}$ is the space of all Borel measures on $C_0[0,1]$ (finite positive measures and their differences). Each pair (t,f) of $t \in [0,1]$ and $f \in C_0[0,1]$ determines an element $\mathcal{W}_{f_0^t}$ of $\mathcal{Y}$. Choose t and f at random, independently, according to the Lebesgue measure and Wiener measure respectively, then $\mathcal{W}_{f_0^t}$ is a random element of $\mathcal{Y}$, and $\langle X, \mathcal{W}_{f_0^t} \rangle$ is measurable in (t,f) for any $X \in \mathcal{X}$. Denote by μ the probability distribution of $\mathcal{W}_{f_0^t}$. That is, μ is the image of $\operatorname{mes} \times \mathcal{W}$ ("mes" being the Lebesgue measure on $[0,1]$) under the map $(t,f) \mapsto \mathcal{W}_{f_0^t}$.

We can reformulate the statement in terms of $\mathcal{X}, \mathcal{Y}, \mu$ only, as follows:

If $X_1, X_2 \in \mathcal{X}$ satisfy the conditions

$$\big(\langle X_1, Y\rangle + i\langle X_2, Y\rangle\big)^3 \in [0,\infty) \quad \text{for } \mu\text{-almost all } Y \in \mathcal{Y}\,,$$
$$\int \langle X_k, Y\rangle \, d\mu(Y) = 0 \quad \text{for } k = 1,2\,,$$

then $X_1 = 0$ and $X_2 = 0$.

It seems to be a simple geometric property of a measure in a linear space. Strangely enough, it appears to be a problem already for a single, canonical μ.

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