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QUANTUM ANALOGUES OF THE BELL INEQUALITIES. THE CASE OF TWO SPATIALLY SEPARATED DOMAINS

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One investigates inequalities for the probabilities and mathematical expectations which follow from the postulates of the local quantum theory. It turns out that the relation between the quantum and the classical correlation matrices is expressed in terms of Grothendieck's known constant. It is also shown that the extremal quantum correlations characterize the Clifford algebra (i.e., canonical anticommutative relations).

The Bell inequalities are inequalities for probabilities that are valid in any local deterministic theory with hidden parameters (briefly and not entirely exactly, these theories will be said to be classical), but need not be true in quantum theory; see, for example, the surveys [1-4], and also [5, pp. 190-193], and [6]. The domain of the probability distributions, admissible in the classical theories, lends itself to a mathematical description [7] and this description is model-independent, i.e., it is not connected with any concrete physical mechanisms. On the other hand, for the probability distributions, admissible in the quantum theory, one considers usually only certain special cases; moreover, in the mentioned paper [7] one has expressed scepticism regarding the possibility of a model-independent approach to quantum probabilities. However, such an approach is possible (it has been communicated by the author in [9]) for a very general situation, allowing many domains in space—time,
(and chronologically ordered domains. The case of two spatially separated)
both in spatially separated domains is considered in more detail in this paper; one proves certain theorems, regarding this case, which have been communicated in [9]. Then we carry out a comparison of the quantum case with the classical one; here, unexpectedly, there arises the Grothendieck constant, known from the geometry of Banach spaces. It turns out that the quantum correlations exceed the classical ones at most 1.782 times. In the last section we present some preliminary results for the case of three spatially separated domains.

1. Some Facts about Clifford Algebras

This auxiliary section prepares the technical means, used in the subsequent sections.

By a Clifford algebra $C(n)$ we mean a C*-algebra, generated by the Hermitian generators $X_1, \ldots, X_n$ and by the relations $X_k^2 = \mathbb{1}$, $X_k X_\ell + X_\ell X_k = 0$ for $k, \ell = 1, \ldots, n$, $k \neq \ell$. Setting $X(x) = \sum x_k X_k$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we achieve an explicit $O(n)$-invariance: $X^2(x) = |x|^2 \cdot \mathbb{1}$ for $x \in \mathbb{R}^n$. It is known that for $\cancel{C(n)}$ n even, the C^*-algebra $C(n)$ is isomorphic to the C^*-algebra of matrices of order $2^{n/2}$, while for n odd, to the direct sum of two matrix algebras, each of order $2^{\frac{n-1}{2}}$ (see, for example, [10], Subsection 17.3). Consequently, for even n, $C(n)$ has, within the accuracy of a unitary equivalence, a unique

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irreducible representation (in the space of dimension $2^{\frac{n}{2}}$), while for odd n , two non-equivalent irreducible representations (in the space of dimension $2^{\frac{n-1}{2}}$).

We consider the tensor product $C(n) \otimes C(n)$ of two Clifford algebras. We are interested in the following Hermitian element of the algebra $C(n) \otimes C(n)$:

$$A = \frac{1}{n}\left(X_1 \otimes X_1 + \ldots + X_n \otimes X_n\right).$$

This can be given another definition, possessing an explicit $O(n)$-invariance, namely, A is the mean of $X(x) \otimes X(x)$ over all unit vectors x. We need the spectrum of the operator $\pi(A)$ in the irreducible representation π of the algebra $C(n) \otimes C(n)$. (For n even there is one such representation, while for n odd there are four.)

LEMMA 1.1. For n even, the spectrum of the operator $\pi(A)$ consists of the numbers $1 - 2\frac{k}{n}$ with multiplicities C_n^k, $k = 0, 1, \ldots, n$. For n odd, in two of the four irreducible representations, the spectrum of the operator $\pi(A)$ consists of the numbers $1 - 4\frac{k}{n}$ with multiplicities C_n^{2k} $(k = 0, 1, \ldots \frac{n-1}{2})$ and, in the other two representations, of the numbers $-1 + 4\frac{k}{n}$ with multiplicities C_n^{2k} $(k = 0, 1, \ldots, \frac{n-1}{2})$.

Proof. First we consider the joint spectrum of a collection of n commuting operators $\pi(X_1 \otimes X_1), \ldots, \pi(X_n \otimes X_n)$. Each point of this joint spectrum has the form $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_k = \pm 1$ since $(X_k \otimes X_k)^2 = \mathbb{1}$. First we assume that n is even and we prove that each such point θ belongs in fact to the spectrum and has multiplicity 1. We make use of the fact that the group $O(n) \times O(n)$ acts by automorphisms on the algebra $C(n) \otimes C(n)$. In the space of the representations we have only a projective action of this group but even this is sufficient for our purpose. We take only the subgroup $1 \times O(n)$ and in it only the commutative subgroup of transformations of the form $1 \times \tau$ where $\tau = diag(\tau_1, \ldots, \tau_n)$ is a diagonal matrix, $\tau_k = \pm 1$. Such a transformation maps $X_k \otimes X_k$ into $\tau_k X_k \otimes X_k$ and, correspondingly, $(\theta_1, \ldots, \theta_n)$ into $(\tau_1\theta_1, \ldots, \tau_n\theta_n)$. From this it is clear that all the points θ have the same multiplicity. This multiplicity is equal to 1 since the number 2^n of points is equal to the dimension of the space of representations. It remains to note that the spectrum of the operator A consists of points of the form, $\frac{1}{n}(\theta_1 + \ldots + \theta_n)$.

The case of an odd n is considered in a similar manner, except that instead of the group $O(n)$ one applies $SO(n)$ in connection with which one imposes on τ the condition $\tau_1 \ldots \tau_n = 1$, and the set of 2^n points θ is decomposed into two sets of 2^{n-1} points each in accordance with the two values of the product $\theta_1 \ldots \theta_n = \pm 1$, now we take into account that the space of the representation has dimension 2^{n-1}. The remaining details are left to the reader.

In the sequel we shall be interested only in one point of the spectrum: the unity. We can see that it belongs in fact to the spectrum and its multiplicity is equal to 1.

LEMMA 1.2. Assume that π is an irreducible representation of the algebra $C(n) \otimes C(n)$ and let ψ be such that $\pi(A)\psi = \psi$, $|\psi| = 1$, A being as above. Then

$$\langle \pi(X(x) \otimes X(y))\psi, \psi \rangle = \langle x, y \rangle,$$

$$\langle \pi(X(x) \otimes \mathbb{1})\psi, \psi \rangle = 0,$$

$$\langle \pi(\mathbb{1} \otimes X(y))\psi, \psi \rangle = 0$$

for all $x, y \in \mathbb{R}^n$.

Proof. We know that the mean of $\langle \pi (X(x) \otimes X(x)) \psi, \psi \rangle$ over all unit vectors x is equal to $\langle \pi(A)\psi, \psi \rangle = 1$; taking into account that $\langle \pi (X(x) \otimes X(x)) \psi, \psi \rangle \leqslant 1$ for all such x we obtain that

$$\langle \pi (X(x) \otimes X(x)) \psi, \psi \rangle = 1$$

for all unit vectors x. In order to obtain from here the relation

$$\langle \pi (X(x) \otimes X(y)) \psi, \psi \rangle = \langle x, y \rangle$$

one has to show only that

$$\langle \pi (X(x) \otimes X(y)) \psi, \psi \rangle = \langle \pi (X(y) \otimes X(x)) \psi, \psi \rangle,$$

in fact, the symmetric bilinear form is uniquely determined by its quadratic form. We note that the operator A is invariant under the automorphism of the algebra $C(n) \otimes C(n)$, which takes $X(x) \otimes \mathbb{1}$ into $\mathbb{1} \otimes X(x)$ and conversely. For n even, to this automorphism there corresponds a unitary operator in the space of the representations; taking into account that the equality $\pi(A)\psi = \psi$ determines the vector ψ within the accuracy of a scalar factor, we obtain the required symmetry of the bilinear form. We leave to the reader the case of odd n; we only note that for $n = 4m+1$ the above mentioned automorphism is suitable, while for $n = 4m+3$ one has to apply the automorphism which takes $X(x) \otimes \mathbb{1}$ into $-\mathbb{1} \otimes X(x)$ and conversely. It remains to show that $\langle \pi (X(x) \otimes \mathbb{1}) \psi, \psi \rangle = 0$. But this can be easily obtained from the $SO(n)$-invariance.

LEMMA 1.3. Assume that the representation π of the algebra $C(n) \otimes C(n)$ is such that there exists a cyclic (in an other terminology: totalizing) vector ψ with the property $\pi(A)\psi = \psi$. Then the representation π is either irreducible, or it is a sum of two non-equivalent irreducible representations (of course, the second case is possible only for odd n).

Proof. The existence of irreducible subrepresentations does not cause any doubt in view of the finite-dimensionality of the algebra. One has to prove only that the representation π cannot contain two equivalent irreducible subrepresentations. We assume that the opposite: π is equivalent to the sum $\pi_1 \oplus \pi_1 \oplus \pi_2$ and π_1 is irreducible; then correspondingly $\psi = \psi_1 \oplus \lambda\psi_1 \oplus \psi_2$, in fact the property $\pi_1(A)\psi_1 = \psi_1$ determines the vector ψ_1 within the accuracy of a scalar factor. But the space of vectors of the form $\xi \oplus \lambda\xi \oplus \zeta$ (with fixed λ and arbitrary ξ, ζ) is invariant relative to $\pi_1 \oplus \pi_1 \oplus \pi_2$, this contradicts the cyclicity of the vector ψ.

2. Quantum Realizability of Correlation Matrices

An $m \times n$ matrix C with real elements is said to be a quantum realiz~~ed~~able correlation matrix if it satisfies the first (or the second) conditions of the subsequent theorem. For the set of all quantum realized correlation $m \times n$ matrices we introduce the notation $\mathrm{Cor}(m,n)$. The fourth (or the fifth) condition of the subsequent theorem yields a simpler geometric description of such matrices.

THEOREM 2.1. For any $m \times n$ matrix $C = \{c_{k\ell}\}$ with real elements, the following five statements are equivalent:

(1) There exists a C^*-algebra $\mathfrak{A}$ with identity, Hermitian elements $A_1,\dots,A_m$, $B_1,\dots,B_n \in \mathfrak{A}$ and a state f on $\mathfrak{A}$ such that for any k, ℓ $A_k B_\ell = B_\ell A_k$; we have:

$$-\mathbb{1} \leqslant A_k \leqslant \mathbb{1}\ ; \quad -\mathbb{1} \leqslant B_\ell \leqslant \mathbb{1}\ ;$$

$$f(A_k B_\ell) = c_{k\ell}\ .$$

(2) There exist Hermitian operators $A_1,\dots,A_m,B_1,\dots,B_n$ and a density matrix W in the Hilbert space H of finite or countable dimension ~~ality~~, such that for any k,ℓ, one has $A_k B_\ell = B_\ell A_k$; the spectrum of each of the operators A_k, B_ℓ lies on $[-1,+1]$; $\mathrm{Tr}(A_k B_\ell W) = c_{k\ell}$.

(3) The same as (2) and, in addition, $A_k^2 = \mathbb{1}$, $B_\ell^2 = \mathbb{1}$, $\mathrm{Tr}(A_k W)=0$, $\mathrm{Tr}(B_\ell W)=0$ for all k, ℓ , and $H=H_1\otimes H_2$, $A_k = A_k^{(1)}\otimes \mathbb{1}^{(2)}$, $B_\ell = \mathbb{1}^{(1)}\otimes B_\ell^{(2)}$, where $A_k^{(1)}$, $B_\ell^{(2)}$ some operators in H_1, H_2 respectively, $\mathbb{1}^{(1)}$, $\mathbb{1}^{(2)}$ are the identity operators; and all the anticommutators $A_{k_1}^{(1)}A_{k_2}^{(1)} + A_{k_2}^{(1)}A_{k_1}^{(1)}$, $B_{\ell_1}^{(2)}B_{\ell_2}^{(2)} + B_{\ell_2}^{(2)}B_{\ell_1}^{(2)}$ are scalar operators (i.e., multiples of the identity operator) and the spaces H, H_1, H_2 are finite-dimensional, moreover

$$2\log_2 \dim H_1 \leqslant \begin{cases} m & \text{for even } m, \\ m+1 & \text{for odd } m, \end{cases}$$
$$2\log_2 \dim H_2 \leqslant \begin{cases} n & \text{for even } n, \\ n+1 & \text{for odd } n. \end{cases}$$

(4) In the Euclidean space of dimension $m+n$, there exist unit vectors $x_1,\dots,x_m$, $y_1,\dots,y_n$ such that $\langle x_k, y_\ell\rangle = c_{k\ell}$ for all k, ℓ .

(5) In the Euclidean space of dimension $\min(m,n)$, there exist vectors $x_1,\dots x_m$, $y_1,\dots y_n$ such that $|x_k| \leqslant 1$, $|y_\ell| \leqslant 1$ and $\langle x_k, y_\ell\rangle = c_{k\ell}$ for all k, ℓ .

Proof. Clearly, (3) $\Rightarrow$ (2) $\Rightarrow$ (1). We show that (1) $\Rightarrow$ (5). The algebra $\mathfrak{A}$ can be considered as a real linear space with inner product $\langle X,Y\rangle = \mathrm{Re}\, f(Y^*X)$ for $X,Y\in\mathfrak{A}$ (the corresponding quadratic form may be degenerate but that does not change the facts); then we have $\langle A_k, A_k\rangle \leqslant 1$, $\langle B_\ell, B_\ell\rangle \leqslant 1$ and $\langle A_k, B_\ell\rangle = c_{k\ell}$ for all k, ℓ . Thus, all the requirements of part (5) are satisfied except that ~~and, in addition,~~ the dimension of the space has to be at most $\min(m,n)$. This can be easily achieved by the orthogonal projection of the vectors x_k onto the subspace generated by the vectors y_ℓ , or in the other way around.

We show that (5) $\Rightarrow$ (4). One has to make the vectors to be unit vectors. For this, it is sufficient that instead of x_k and y_ℓ one should take $x_k + x_k'$ and $y_\ell + y_\ell'$ respectively, where x_k' and y_ℓ' are chosen to be orthogonal to each other and to the vectors x_k, y_ℓ , then $\langle x_k + x_k', y_\ell + y_\ell'\rangle = \langle x_k, y_\ell\rangle = c_{k\ell}$, and for a proper choice of the lengths of the vectors x_k', y_ℓ' we obtain $|x_k + x_k'| = 1$, $|y_\ell + y_\ell'| = 1$. Of course, this construction may require one to go into a space of larger dimension; but afterwards one can restrict oneself to the space generated by the vectors x_k, y_ℓ ; this dimension does not exceed $m+n$.

Thus (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (5) $\Rightarrow$ (4). It remains to prove that (4) $\Rightarrow$ (3). Consider the vectors x_k, y_ℓ from $\mathbb{R}^{m+n}$. We consider the Clifford algebra $C(m+n)$. According to what has been proved in Sec. 1, there exists a state f on the algebra $C(m+n)\otimes C(m+n)$ such that

$$f(X(x)\otimes X(y)) = \langle x, y\rangle ,$$
$$f(X(x)\otimes \mathbb{1}) = 0 ,$$
$$f(\mathbb{1}\otimes X(y)) = 0$$

for all $x, y \in \mathbb{R}^{m+n}$. Setting $A_K = X(x_K)\otimes \mathbb{1}$, $B_\ell = \mathbb{1}\otimes X(y_\ell)$, we obtain $f(A_K B_\ell) = c_{K\ell}$, $f(A_K) = 0$, $f(B_\ell) = 0$. The operators $X(x_K)$ generate in $C(m+n)$ a subalgebra, isomorphic to $C(m')$ with some $m' \leqslant m$; similarly, $X(y_\ell) \in C(n') \subset C(m+n)$ $n' \leqslant n$. We restrict the state f from the algebra $C(m+n)\otimes C(m+n)$ to the algebra $C(m')\otimes C(n')$. We take a faithful representation of the algebra $C(m')$ into the space H_1 whose dimension is equal to $2^{\frac{m'}{2}}$ for even m' and to $2^{\frac{m'+1}{2}}$ for odd m'; we represent similarly $C(n')$ into H_2; the tensor product gives a faithful representation of $C(m')\otimes C(n')$ into $H = H_1 \otimes H_2$. The state f on $C(m')\otimes C(n')$ is realized by a density matrix in H. All the requirements of part (3) hold.

The proved Theorem 2.1 reduces the question of the quantum realizability of a correlation matrix to the simpler question of the existence of finite-dimensional vectors, but does not give an explicit solution for this problem. For the simplest case when $m=2$ and $n=2$ an explicit solution is given by the following theorem.

THEOREM 2.2. A 2×2 matrix c with real elements is a quantum realized correlation matrix if and only if $|c_{K\ell}| \leqslant 1$ for all K, ℓ and if at least one of the following two inequalities holds:

$$0 \leqslant (c_{12}c_{21} - c_{11}c_{22})(c_{11}c_{12} - c_{21}c_{22})(c_{11}c_{21} - c_{12}c_{22}) \leqslant$$
$$\leqslant \frac{1}{4}\Big(\sum_{K,\ell} c_{K\ell}^2\Big)^2 - \frac{1}{2}\sum_{K,\ell} c_{K\ell}^4 - 2\prod_{K,\ell} c_{K\ell} ,$$
$$0 \leqslant 2 \max_{K,\ell} c_{K\ell}^4 - \big(\max_{K,\ell} c_{K\ell}^2\big)\Big(\sum_{K,\ell} c_{K\ell}^2\Big) + 2\prod_{K,\ell} c_{K\ell} .$$

The proof is elementary (in fact, one has four vectors in the plane) but cumbersome; we shall not give it. The same refers to the following theorem, giving the explicit solution of the dual problem.

THEOREM 2.3. Assume that there is a given 2×2 matrix γ with real elements, and let

$$M = \sup \sum_{K,\ell} \gamma_{K\ell}\, c_{K\ell} ,$$

where the supremum is taken over all quantum realized 2×2 correlation matrices c. Then

(1) if $\prod_{K,\ell} \gamma_{K\ell} \geqslant 0$, then $M = \sum_{K,\ell} |\gamma_{K,\ell}|$;

(2) if $\prod_{K,\ell} \gamma_{K\ell} < 0$ and $(\min_{K,\ell} |\gamma_{K\ell}|)(\sum_{K,\ell} |\gamma_{K\ell}|^{-1}) \leqslant 2$, then
$$M = \sum_{K,\ell} |\gamma_{K\ell}| - 2 \min_{K,\ell} |\gamma_{K\ell}| ;$$

(3) if $\prod_{K,\ell} \gamma_{K\ell} < 0$ and $(\min_{K,\ell} |\gamma_{K\ell}|)(\sum_{K,\ell} |\gamma_{K\ell}|^{-1}) \geqslant 2$, then
$$M = \sqrt{\Big(\sum_{K,\ell} \gamma_{K\ell}^2\Big) + \Big(\prod_{K,\ell} |\gamma_{K\ell}|\Big)\Big(\sum_{K,\ell} \gamma_{K\ell}^{-2}\Big)} .$$

3. Representation of Extremal Correlations

As one can easily see, the set $\mathrm{Cor}(m,n)$ of all quantum realized $m\times n$ correlation matrices, introduced in the previous section, is a closed, bounded, centrally symmetric, convex body in the space of $m\times n$ matrices. We consider the set $\mathrm{Ex}\,\mathrm{Cor}(m,n)$ of all extremal (in other terminology: extreme) points of the set $\mathrm{Cor}(m,n)$; by a known general theorem, from the geometry of convex sets (see, at least [10]), $\mathrm{Cor}(m,n)$ is the closed convex hull of the set $\mathrm{Ex}\,\mathrm{Cor}(m,n)$.

Let C be an $m\times n$ matrix. By a C-system of vectors we mean any collection of vectors $x_1,\dots,x_m,y_1,\dots,y_n$ lying in some Euclidean space and satisfying the conditions $|x_k|\leqslant 1$, $|y_\ell|\leqslant 1$, $\langle x_k,y_\ell\rangle=c_{k\ell}$ for all k , ℓ .

Clearly, a C-system of vectors exists if and only if $C\in\mathrm{Cor}(m,n)$. By the rank of a C-system of vectors we mean the dimension of the linear hull of the vectors $x_1,\dots,x_m$, $y_1,\dots,y_n$.

LEMMA 3.1. Let $C\in\mathrm{Ex}\,\mathrm{Cor}(m,n)$. Then all C-systems of vectors are isometric to each other and all have the following properties:

(1) $|x_k|=1$, $|y_\ell|=1$ for all k , ℓ .

(2) The linear hull of the vectors $x_1,\dots,x_m$ coincides with the linear hull of the vectors $y_1,\dots,y_n$.

(3) For any quadratic form Q , defined on the linear hull of the vectors $x_1,\dots,$ $x_m, y_1,\dots,y_n$, from the inequalities

$$Q(x_1)=0,\dots,\ Q(x_m)=0,\ \ Q(y_1)=0,\dots,\ Q(y_n)=0$$

there follows that $Q=0$ identically.

(4) The rank r of any C-system of vectors satisfies the inequalities

$$r\leqslant m,\qquad r\leqslant n\quad \text{and } r\leqslant -\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+2(m+n)}\ .$$

Proof. (a). For any C-system of vectors the property (1) holds. Indeed, assume, for example, $|x_1|<1$, then for sufficiently small vectors Δx we obtain $|x_1+\Delta x|\leqslant 1$ and $|x_1-\Delta x|\leqslant 1$. Defining the matrix ΔC by the equalities

$$(\Delta C)_{k\ell}=\begin{cases}\langle\Delta x, y_\ell\rangle & \text{for } k=1,\\ 0 & \text{otherwise},\end{cases}$$

we obtain $C-\Delta C\in\mathrm{Cor}(m,n)$ and $C+\Delta C\in\mathrm{Cor}(m,n)$, which for $\Delta C\neq 0$ contradicts the assumption that C is an extremal point.

(b) For any C-system of vectors, every vector y_ℓ lies in the subspace spanned by the vectors $x_1,\dots,x_m$. Indeed, otherwise, replacing y_ℓ by its projection onto the mentioned subspace, we would obtain a C-system of vectors, contradicting what has been proved at part (a). In the same way, every vector x_k lies in the subspace spanned by the vectors y_n .

(c) All C-systems of vectors are isometric to each other. Indeed, let $x_1',\dots,x_m',$ $y_1',\dots,y_n'$ be a C-system and assume that $x_1'',\dots,x_m'', y_1'',\dots,y_n''$ is also a C-system. We consider their direct (orthogonal) sum: $x_k=\frac{1}{\sqrt2}(x_k'\oplus x_k'')$, $y_\ell=\frac{1}{\sqrt2}(y_\ell'\oplus y_\ell'')$. We have $|x_k|^2=\frac12(|x_k'|^2+|x_k''|^2)$, $|y_\ell|^2=\frac12(|y_\ell'|^2+|y_\ell''|^2)$, $\langle x_k,y_\ell\rangle=\frac12(\langle x_k',y_\ell'\rangle+\langle x_k'',y_\ell''\rangle)$, from where is clear that $x_1,\dots,x_m,y_1,\dots,y_n$ is again a C-system. According to what has been proved at part (b), each vector y_ℓ is a linear combination of the vectors x_k : $y_\ell = \sum_k a_{k\ell} x_k$. We have $\frac{1}{\sqrt{2}}(y'_\ell \oplus y''_\ell) = \frac{1}{\sqrt{2}} \sum_k a_{k\ell} x'_k \oplus \frac{1}{\sqrt{2}} \sum_k a_{k\ell} x''_k$, whence $y'_\ell = \sum_k a_{k\ell} x'_k$ and $y''_\ell = \sum_k a_{k\ell} x''_k$. Then

$$\langle y'_\ell, y'_{\ell_1} \rangle = \langle \sum_k a_{k\ell} x'_k, y'_{\ell_1} \rangle = \sum_k a_{k\ell} c_{k\ell_1} ;$$

similarly, $\langle y''_\ell, y''_{\ell_1} \rangle = \sum_k a_{k\ell} c_{k\ell_1}$; thus, $\langle y'_\ell, y'_{\ell_1} \rangle = \langle y''_\ell, y''_{\ell_1} \rangle$ for all ℓ, ℓ_1. In the same way one proves that $\langle x'_k, x'_{k_1} \rangle = \langle x''_k, x''_{k_1} \rangle$ for all k, k_1. Finally, $\langle x'_k, y'_\ell \rangle = c_{k\ell} = \langle x''_k, y''_\ell \rangle$.

(d) We show part (3). We assume the opposite: there exists a form Q, not identically zero and vanishing at x_k, y_ℓ. On the subspace generated by these vectors, we consider a new metric $|\cdot|_1$, defined by the equality $|z|_1^2 = |z|^2 + \varepsilon Q(z)$; ε is chosen to be small so that the right-hand side be positive definite. We have $|x_k|_1 = 1$, $|y_\ell|_1 = 1$. We define a matrix ΔC in terms of the symmetric bilinear form, corresponding to the quadratic form Q : $(\Delta C)_{k\ell} = Q(x_k, y_\ell)$, then $\langle x_k, y_\ell \rangle_1 = (C + \varepsilon \Delta C)_{k\ell}$. We can see that $C + \varepsilon \Delta C \in \mathrm{Cor}(m, n)$ for all small ε, both positive and negative. For $\Delta C \neq 0$, this contradicts the assumption that C is an extremal point.

(e) Part (4) follows from part (3), already proved. Indeed, the space of quadratic forms on an r-dimensional space has dimension $\frac{1}{2} r(r+1)$; part (3) shows that any of these forms is uniquely defined by a collection of $m+n$ of its values; thus, $\frac{1}{2} r(r+1) \leqslant m+n$, whence $r \leqslant -\frac{1}{2} + \sqrt{\frac{1}{4} + 2(m+n)}$. The inequalities $r \leqslant m$, $r \leqslant n$ are obvious from the previously proved part (2). The lemma is proved.

Assume that C is an $m \times n$ matrix with real elements. By an operator representation of the C-correlations we mean any collection $(H, W, A_1, \ldots, A_m, B_1, \ldots, B_n)$, consisting of a Hilbert space H of finite or countable dimension, a density matrix W in H, and the Hermitian operators $A_1, \ldots, A_m, B_1, \ldots, B_n$ in H, which satisfy the conditions:

the spectrum of each of the operators A_k, B_ℓ lies on $[-1, +1]$, $A_k B_\ell = B_\ell A_k$, $\mathrm{Tr}(A_k B_\ell W) = c_{k\ell}$ for all k, ℓ.

Clearly, an operator representation of the C-correlations exists if and only if $C \in \mathrm{Cor}(m, n)$.

We define in the obvious manner the unitary equivalence of two such representations (for this we require that not only A_k, B_ℓ should "coincide," but also W).

If the density matrix W is one-dimensional, then such a representation is said to be pure.

If in H there exists a projection P, commuting with all A_k, B_ℓ and such that $PWP = W$, then one can define in an obvious manner a subrepresentation of the given representation. If, however, there is no such P (other than the identity, of course), then the representation is said to be nondegenerate. Clearly, every operator representation of the C-correlations contains a unique nondegenerate subrepresentation.

An operator representation of the C-correlations is said to be factorial if the von Neumann algebra, generated by the operators $A_1, \ldots, A_m, B_1, \ldots, B_n$, is factorial.

Having two operator representations of the C -correlations

$$(H^{(i)}, W^{(i)}, A_1^{(i)}, \ldots, A_m^{(i)}, B_1^{(i)}, \ldots, B_n^{(i)}), \quad i = 1, 2,$$

and given positive coefficients α_1, α_2 such that $\alpha_1 + \alpha_2 = 1$. we can construct the direct sum of the given representations:

$$H = H^{(1)} \oplus H^{(2)}, \quad W = \alpha_1 W^{(1)} \oplus \alpha_2 W^{(2)}$$
$$A_K = A_K^{(1)} \oplus A_K^{(2)}, \quad B_\ell = B_\ell^{(1)} \oplus B_\ell^{(2)}.$$

In addition, instead of $\alpha_1 W^{(1)} \oplus \alpha_2 W^{(2)}$ one can take any density matrix W in $H = H_1 \oplus H_2$ such that $P^{(1)} W P^{(1)} = \alpha_1 W^{(1)} \oplus 0$, $P^{(2)} W P^{(2)} = 0 \oplus \alpha_2 W^{(2)}$ (here $P^{(1)}, P^{(2)}$ are the projections onto $H_1 \oplus 0$, $0 \oplus H_2$, respectively); any representation obtained in this manner will be called a pseudodirect sum of the two given representations.

The operator representation of the C-correlations is said to be Clifford if all the anticommutators $A_{K_1} A_{K_2} + A_{K_2} A_{K_1}, B_{\ell_1} B_{\ell_2} + B_{\ell_2} B_{\ell_1}$ are scalar (i.e., multiples of the identity operator).

As shown by Theorem 2.1, Clifford representations exist for all $C \in Cor(m, n)$.

<u>THEOREM 3.1.</u> Let $C \in ExCor(m, n)$. Then any nondegenerate operator representation of the C-correlations is Clifford.

<u>THEOREM 3.2.</u> Let $C \in ExCor(m, n)$ and assume that the rank of any C-system of vectors is an even number. Then:

(1) All the pure, nondegenerate operator representations of C-correlations are unitarily equivalent.

(2) Any non-pure, nondegenerate operator representation of C-correlations decomposes into the direct sum of a finite or countable family of pure nondegenerate representations.

<u>THEOREM 3.3.</u> Let $C \in ExCor(m, n)$ and assume that the rank of any C-system of vectors is an odd number. Then:

(1) There exists exactly two unitarily nonequivalent, pure, factorial, nondegenerate operator representations of C -correlations.

(2) Any non-pure, factorial, nondegenerate operator representation of C -correlations decomposes into a direct sum of a finite or a countable number of unitarily equivalent to each other, pure, factorial, nondegenerate representations.

(3) Any nonfactorial, nondegenerate operator representation of C -correlations decomposes into a pseudodirect sum of two factorial, nondegenerate representations.

<u>Proof of Theorem 3.1.</u> The algebra of operator in H can be considered as a real linear space with inner product $\langle X, Y \rangle = \mathrm{Re}\,\mathrm{Tr}(Y^* X W)$. Moreover, one has to take into account that the equality $\langle X, X \rangle = 0$ means only $XW = 0$ and not necessarily $X = 0$. We have

$$\langle A_K, A_K \rangle \leq 1, \ \langle B_\ell, B_\ell \rangle \leq 1, \ \langle A_K, B_\ell \rangle = c_{K\ell};$$

thus, we have obtained a C-system of vectors. By virtue of Lemma 3.1, all such systems are isometric. Let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be a C-system of vectors, lying in $\mathbb{R}^r$, where r is the rank of any C-system. From the mentioned lemma we know that every vector x_K

is a linear combination of vectors y_ℓ: $x_K = \sum_\ell a_{K\ell} y_\ell$. In view of the isometry we obtain from here $A_K W = \sum_\ell a_{K\ell} B_\ell W$. Taking into account the nondegeneracy of the representation, we can see that the closed linear hull of the vectors of the form $B_{\ell_1}\dots B_{\ell_\nu} W\psi$ (ν runs through $0, 1, \dots$; $\ell_1, \dots, \ell_\nu$ run through $1, \dots, n$; ψ runs through H) coincides with the space H. Indeed, it is invariant not only relative to all B_ℓ but also relative to all A_K. In fact, $A_K B_{\ell_1}\dots B_{\ell_\nu} W\psi = B_{\ell_1}\dots B_{\ell_\nu} A_K W\psi = \sum_\ell a_{K\ell} B_{\ell_1}\dots B_{\ell_\nu} B_\ell W\psi$. We can see now that for the operators X, commuting with $B_1, \dots, B_n$, the equality $\langle X, X\rangle = 0$ (or $XW = 0$) implies $X = 0$; in fact $XB_{\ell_1}\dots B_{\ell_\nu} W\psi = B_{\ell_1}\dots B_{\ell_\nu} X W\psi = 0$. The same holds also for the operators which commute with $A_1, \dots, A_m$.

From the above mentioned lemma we also know that $|x_K| = 1$; in view of the isometry we obtain from here $\mathrm{Re}\,\mathrm{Tr}(A_K^* A_K W) = 1$, i.e., $\mathrm{Tr}((\mathbb{1} - A_K^2)W) = 0$; taking into account that $\mathbb{1} - A_K^2 \geqslant 0$, we conclude that $(\mathbb{1} - A_K^2)W = 0$. But $\mathbb{1} - A_K^2$ commutes with $B_1, \dots, B_n$; thus, $A_K^2 = \mathbb{1}$ for all K. Similarly, $B_\ell^2 = \mathbb{1}$ for all ℓ. Then we have

$$(A_K + \sum_\ell a_{K\ell} B_\ell)(A_K - \sum_\ell a_{K\ell} B_\ell) W = 0,$$

i.e., $(\mathbb{1} - (\sum_\ell a_{K\ell} B_\ell)^2) W = 0$, whence $(\sum_\ell a_{K\ell} B_\ell)^2 = \mathbb{1}$ for all K.

Finally, we make use of part (3) of the above mentioned lemma. It shows that any quadratic form on $\mathbb{R}^r$ is uniquely determined by its values at the points $x_1, \dots, x_m, y_1, \dots, y_n$. This holds for quadratic forms with real values; but then this is automatically true also for forms with values in any linear space, in particular, for forms with operator values. We define a quadratic form Q on the space $\mathbb{R}^r$, having operator values, by the following equality

$$Q(\sum \beta_\ell y_\ell) = (\sum \beta_\ell B_\ell)^2 \text{ for any } \beta_1, \dots, \beta_n.$$

This is well defined since for those β_ℓ for which $\sum \beta_\ell y_\ell = 0$, we have $\sum \beta_\ell B_\ell W = 0$ and, therefore, $\sum \beta_\ell B_\ell = 0$. The form Q has the properties

$$Q(y_\ell) = B_\ell^2 = \mathbb{1},$$

$$Q(x_K) = Q(\sum_\ell a_{K\ell} y_\ell) = (\sum_\ell a_{K\ell} B_\ell)^2 = \mathbb{1};$$

thus, at the points $x_1, \dots x_m, y_1, \dots, y_n$ it coincides with the form $Q_1(z) = |z|^2 \cdot \mathbb{1}$. In this case, these forms coincide identically: $(\sum \beta_\ell B_\ell)^2 = |\sum \beta_\ell y_\ell|^2 \cdot \mathbb{1}$ for any $\beta_1, \dots, \beta_n$. From here, $B_{\ell_1} B_{\ell_2} + B_{\ell_2} B_{\ell_1} = 2\langle y_{\ell_1}, y_{\ell_2}\rangle \cdot \mathbb{1}$. Similarly, we obtain $A_{K_1} A_{K_2} + A_{K_2} A_{K_1} = 2\langle x_{K_1}, x_{K_2}\rangle \cdot \mathbb{1}$.

Proof of Theorem 3.2. From the given proof of Theorem 3.1 we can see how one constructs a Clifford representation: one selects a C-system of vectors $x_1, \dots x_m, y_1, \dots y_n$ in the space $\mathbb{R}^r$, where r is the rank of such a system, one constructs the tensor product $C(r) \otimes C(r)$ of two Clifford algebras, one considers some representation π of the algebra $C(r) \otimes C(r)$, and one introduces the operators $A_K = \pi(X(x_K) \otimes \mathbb{1})$, $B_\ell = \pi(\mathbb{1} \otimes X(y_\ell))$. The density matrix W defines on the algebra $C(r) \otimes C(r)$ a state f such that $f(X(x_K) \otimes X(y_\ell)) = c_{K\ell} = \langle x_K, y_\ell\rangle$ and, consequently, $f(X(x) \otimes X(y)) = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^r$. From this it is clear that $f(A)=1$, where the element A of the algebra $C(r)\otimes C(r)$ is defined in Sec. 1.

We prove part (1). The one-dimensional density matrix W corresponds to some unit vector $\psi\in H$. The nondegeneracy of the given representation means that ψ is a cycle vector for the representation π of the algebra $C(r)\otimes C(r)$. Applying Lemma 1.3, we conclude that the representation π is irreducible. Thus, it is uniquely defined within the accuracy of a unitary equivalence. It remains to take into account that the equality $\langle A\psi,\psi\rangle=1$ determines uniquely the vector ψ within the accuracy of a scalar factor.

We prove part (2). The representation π of the factor $C(r)\otimes C(r)$ has automatically (within the accuracy of a unitary equivalence) the form $\pi(u)=\pi_1(u)\otimes \mathbb{1}$, where π_1 is an irreducible representation in the space H_1, $\mathbb{1}$ is the identity operator in H_2 , and $H=H_1\otimes H_2$. The density matrix W has the property $\mathrm{Tr}((\pi_1(A)\otimes \mathbb{1})W)=1$. We know (Lemma 1.1) that the eigenvalue 1 of the operator $\pi_1(A)$ has multiplicity 1. From this there follows that $W=W_1\otimes W_2$, where W_1 is a one-dimensional density matrix in H_1 corresponding to the one-dimensional eigenspace for $\pi_1(A)$, and W_2 is some density matrix in H_2 . We have $W_2=\sum_n a_n W_{2,n}$ where $a_n>0$, $\sum_n a_n=1$, $W_{2,n}$ are one-dimensional density matrices, corresponding to orthogonal one-dimensional subspaces $H_{2,n}$. We obtain the desired decomposition: $H=\bigoplus_n (H_1\otimes H_{2,n})$.

The proof of Theorem 3.3 differs little from the proof of Theorem 3.2; we leave it to the reader.

4. Comparison with the Classical Case

The classical case can be characterized by the fact that all the operators $A_1,\dots,A_m$, $B_1,\dots,B_n$ commute. Instead of the commuting operators and density matrices one can consider random variables. Thus, we are interested in matrices C of dimension $m\times n$ represented in the following form:

$$c_{k\ell}=\mathbb{E}A_k B_\ell ;$$

where $\mathbb{E}$ is the mathematical expectation, $A_1,\dots,A_m, B_1,\dots,B_n$ are random variables with $|A_k|\leqslant 1$ and $|B_\ell|\leqslant 1$ with probability 1 for all k, ℓ. It is easy to see that the set of such matrices is a convex polyhedron; its vertices are matrices of rank 1: $c_{k\ell}=a_k b_\ell$, where all a_k, b_ℓ are equal to ± 1.

Proceeding in the vein of the previous section, we introduce for this polyhedron the notation $\mathrm{Cor}_1(m,n)$ in connection with the fact that its extreme points correspond to the systems of vectors of rank 1. Of course, $\mathrm{Cor}_1(m,n)\subset \mathrm{Cor}(m,n)$. The noncoincidence of these sets is a fundamental fact; exactly this is responsible for the existence of the scientific orientation connected with Bell's inequalities.

In Bell's pioneering investigations one has indicated a point of $\mathrm{Cor}(2,2)$ and a hyperplane separating this point from $\mathrm{Cor}_1(2,2)$. In subsequent investigations one has gradually outlined the set $\mathrm{Cor}_1(2,2)$; its complete explicit description has appeared in [7] and [8]. Apparently, the investigation of the set $\mathrm{Cor}(2,2)$, as well as that of $\mathrm{Cor}(m,n)$, has begun in the author's paper [9].

It is natural to ask how much larger is $Cor(m,n)$ than $Cor_1(m,n)$. Some attempt to approach this question can be noticed in [7], where to each of the Bell inequalities one associates a numerical characteristic f, showing how strongly it is violated in the quantum theory; however, the latter is presented only by a spin correlation experiment according to Bohm's scheme. We propose to pose this problem in the following manner: which is the smallest number $K(m,n)$ having the property that

$$Cor(m,n) \subset K(m,n) \cdot Cor_1(m,n) ,$$

i.e., such that $(1/K(m,n))C$ belongs to $Cor_1(m,n)$ for any matrix C from $Cor(m,n)$. It is easy to see that $K(m,n)$ is an increasing function of m, n; there arises the question whether the limit $K = \lim_{m,n\to\infty} K(m,n)$ is finite or not.

As long as the set $Cor(m,n)$ is determined in terms of C^*-algebras or in terms of operators in a Hilbert space, the question of the constant K from a mathematical point of view is rather special. But Theorem 2.1 shows that the set $Cor(m,n)$ can be determined in a significantly simpler manner in terms of vectors in a Euclidean space. Then it becomes clear that the constant K is nothing else but Grothendieck's well known constant K_G, investigated by mathematicians from 1956 up to now!

Regarding the problem of the Grothendieck constant, we refer the reader to [11]. It is proved there that

$$K_G \leqslant \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1.782 .$$

The constant $K_G(k)$, considered there, differs from the above introduced $K(m,n)$. For given m, n, we select r in such a manner that for any $C \in Ex\,Cor(m,n)$ the rank of the C-system of vectors does not exceed r; then $K(m,n) \leqslant K_G(r)$. As shown in Sec. 3, r can be chosen so that $r \leqslant m$, $r \leqslant n$,

$$r \leqslant -\frac{1}{2} + \sqrt{\frac{1}{4} + 2(m+n)} .$$

For small r, in [11] one has obtained: $K_G(2) = \sqrt{2}$; $K_G(3) < 1.517$; $K_G(4) \leqslant \pi/2$. We note that

$$\sqrt{2} \leqslant K(2,2) \leqslant K(2,n) \leqslant K_G(2) = \sqrt{2} ,$$

from where $K(2,n) = \sqrt{2}$ for all $n \geqslant 2$. In this sense one can assert that the quantum realized correlation matrix $\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, well known among specialists in the Bell inequalities, is optimal (by its "nonclassical" character) not only among the 2×2 matrices but also among the $2\times n$ matrices. A restriction of the class of quantum systems influences the realizability of correl. matrices. We define the set $Cor_\nu(m,n)$ as the closed convex hull of the set of all $m\times n$ matrices C of the form $C_{k\ell} = \langle x_k, y_\ell \rangle$, where x_k, y_ℓ are unit vectors of the ν-dimensional Euclidean space. In other words, $Cor_\nu(m,n)$ consists of matrices C of the form $C_{k\ell} = \mathbb{E}\langle A_k, B_\ell \rangle$, where A_k, B_ℓ are random vectors with values in the ν-dimensional Euclidean space and $|A_k| \leqslant 1$, $|B_\ell| \leqslant 1$ with probability 1. We have $Cor_\nu(m,n) \subset K_G(\nu) \cdot Cor_1(m,n)$. If to the condition (1) of Theorem 2.1 one adjoins the requirement that the linear span of the observables $A_1, \dots, A_m$ should have dimension at most ν, then the corresponding correlations matrices are in $Cor_\nu(m,n)$; this is clear from the proof of the mentioned theorem. In correlation experiments, related with the Bell inequalities, for the observables one applied frequently the projections of the spin onto certain directions and one makes use of particles spin 1/2. The linear span of these observables is not greater than three-dimensional (in fact all of them are represented by Hermitian 2×2 matrices with zero trace). The correlation matrices obtained in this way belong to $Cor_3(m,n)$ consequently, they exceed the classical ones at most $K_G(3) < 1{,}517$ times. If, however, as it is frequently done, we restrict ourselves to the projections of the spin onto directions lying in one selected plane, or one considers experiments with polarization photons and the observables are taken so that each of them is equal to -1 in some state with plane polarization and to $+1$ in another such state, then in a similar manner we obtain two-dimensional span of the observables, the set $Cor_2(m,n)$ and the constant $K_G(2)=\sqrt{2}$.

In conclusion, we rewrite in our notations the explicit description of the sets $Cor_1(2,2)$ and $ABC_1(2,2)$ obtained in [7, 8]. For this, we use the notation $ABC_1(m,n)$ for the set of those collections (a,b,c) consisting of an m-vector a, an n-vector b, and an $m\times n$-matrix c which can be represented in the form $a_k=\mathbb{E}A_k$, $b_\ell=\mathbb{E}B_\ell$, $c_{k\ell}=\mathbb{E}A_kB_\ell$; as before, the random variables A_k, B_ℓ are subjected to the conditions $|A_k|\leqslant 1$, $|B_\ell|\leqslant 1$.

The set $ABC_1(2,2)$ is defined by the system eight double inequalities

$$-1+|a_k+b_\ell|\leqslant c_{k\ell}\leqslant 1-|a_k-b_\ell|,$$

$$|c_{11}+c_{12}+c_{21}+c_{22}-2c_{k\ell}|\leqslant 2\ ;$$

where $k=1,2$; $\ell=1,2$.

The set $Cor_1(2,2)$ can be defined by the single inequality

$$|c_{11}+c_{12}+c_{21}+c_{22}|+|c_{11}+c_{12}-c_{21}-c_{22}|+|c_{11}-c_{12}+c_{21}-c_{22}|+|c_{11}-c_{12}-c_{21}+c_{22}|\leqslant 4.$$

One can get rid of the absolute value symbols and obtain 16 linear inequalities; from them eight inequalities are trivial and the other eight are known as the Bell/CH, CHSH, etc. inequalities (CHSH stands for Clauser, Horne, Shimony, Holt.)

5. The Case of Three Domains

To three spatially separated domains there correspond three collections of observables $A_1,\dots,A_n$; $B_1,\dots,B_p$; $C_1,\dots,C_q$, and all the commutators $[A_k,B_\ell]$, $[B_\ell,C_m]$, $[C_m,A_k]$ vanish. In every state f we define ~~dual~~ pair correlations $f(A_kB_\ell)$, $f(B_\ell C_m)$, $f(C_mA_k)$, generating three matrices, and triple correlations $f(A_kB_\ell C_m)$ generating a trivalent tensor. Here we shall consider only dual correlations. Clearly, each of the three correlation matrices must be quantum realized in the sense of the definition given before for the case of two domains; this, of course, is a necessary but not sufficient condition for the joint quantum realization of a triple ~~domains~~ of correlation matrices.

It turns out that for three domains the situation is entirely different from the case of two domains. Vectors in the Euclidean space are not suitable anymore and the quadratic operator inequalities do not work; in a series of cases the quantum boundaries coincide unexpectedly with the classical ones. The fact is that the presence of a quantum correlation between two objects restricts their possible correlations with any other objects; and if two objects are connected by a "full" correlation, then each of them emerges, with respect to any third object, as the classical one; in fact its state is "somehow known," i.e., in some sense it is subjected to measurement; and the measurement acts in a destructive manner on the quantum correlations. What has been said is an attempt to comment on the precise statements, formulated below. Their proof will be published later.

Thus, in some C^*-algebra there are given Hermitian elements A_1, A_2, B_1, B_2, C_1, C_2, and $A_k^2 \le \mathbb{1}$, $B_\ell^2 \le \mathbb{1}$, $C_m^2 \le \mathbb{1}$, $[A_k, B_\ell] = 0$, $[B_\ell, C_m] = 0$, $[C_m, A_k] = 0$ for $k, \ell, m = 1, 2$. We introduce the notation

$$K_\varphi(A,B) = A_1 B_1 \cos\varphi + A_1 B_2 \sin\varphi - A_2 B_1 \sin\varphi + A_2 B_2 \cos\varphi;$$

φ is an arbitrary parameter, running through $[0, 2\pi]$. In a similar manner one defines $K_\varphi(B,C)$ and $K_\varphi(C,A)$. Taking into account the results of Sec. 2, it is easy to see that the quantum bounds for $K_\varphi(A,B)$ are equal to ± 2, i.e., $-2\cdot\mathbb{1} \le K_\varphi(A,B) \le 2\cdot\mathbb{1}$ and the constants -2 and $+2$ are sharp. The classical bounds for $K_\varphi(A,B)$ are $\pm 2\max(|\cos\varphi|, |\sin\varphi|)$.

Proposition 5.1. For any φ, the quantum bounds for the observable

$$K_\varphi(A,B) + K_\varphi(B,C) + K_\varphi(C,A)$$

coincides with its classical bounds.

Remark. The mentioned classical bounds are easily computed; the upper and lower bounds are equal, respectively, to the largest and the smallest of the three numbers

$$-2\cos\varphi + 4\sin\varphi; \quad 6\cos\varphi; \quad -2\cos\varphi - 4\sin\varphi.$$

Proposition 5.2. Under an appropriate choice of the C^*-algebra $\mathfrak{A}$ and observables $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathfrak{A}$, we have: for any $\varphi \in [0, 2\pi]$ there exists a linear functional f on $\mathfrak{A}$ satisfying the conditions

$$f(\mathbb{1}) = 1,$$

$$f\left((a_1 A_1 + a_2 A_2 + b_1 B_1 + b_2 B_2 + c_1 C_1 + c_2 C_2 + d\cdot\mathbb{1})^2\right) \ge 0$$

for any a_k, b_ℓ, c_m, d and such that

$$f\left(K_\varphi(A,B) + K_\varphi(B,C) + K_\varphi(C,A)\right) = 6\max\left(\cos\left(\varphi + \frac{2\pi}{3}\right),\ \cos\varphi,\ \cos\left(\varphi - \frac{2\pi}{3}\right)\right).$$

Remark. We see that for some φ, the indicated value violates the boundaries mentioned in Proposition 5.1 (and in the remark following it); this is connected with the fact that the functional f is not a state. Thus, in the case of three domains, as opposed to the case of two domains, one cannot obtain sharp inequalities by using the positivity of the squares of only those observables which are linear combinations of the observables A_k, B_ℓ, C_m. We also note that the inner product of the unit vectors in the Euclidean space allow us to reach the boundary mentioned in Proposition 5.2; thus, they may violate the quantum boundaries, which again differs from the case of two domains.

Proposition 5.3. For any state f we have the inequality

$$\left(\frac{1}{2} f(B_\ell C_m)\right)^2 + \left(\frac{1}{2} f(K_{\pi/4}(A,B))\right)^2 \le 1$$

for any ℓ, m.

Remark. If $f(K_{\pi/4}(A,B))$ attains its maximal value, equal to 2, then necessarily $f(B_\ell C_m) = 0$.

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