NOT EVERY BANACH SPACE CONTAINS AN IMBEDDING OF l_p OR c_0

B. S. Tsirel'son

In the geometric theory of Banach spaces the following problem has been posed (see, for instance, [1]): does every infinite-dimensional Banach space contain a subspace which is linearly homeomorphic to one of the spaces l_p, $1 \le p < +\infty$, or c_0?

In this work we give a negative answer to this question. In fact, we will construct a reflexive Banach space which does not contain any infinite-dimensional subspace which is linearly homeomorphic to a uniformly convex space. A more complete formulation of the result stated in our theorem requires the following definition.

Definition 1. A Banach space X is said to be finitely universal if there exists $C > 1$ such that for each finite-dimensional normed space E there exists a subspace $F \subset X$ of the same dimension and an invertible operator $T: E \to F$ such that $\|T\| \cdot \|T^{-1}\| < C$.

In this definition we can restrict our attention, without changing the result, to spaces E of the form l_∞^N (which denotes an N-dimensional space with norm $\max(|\lambda_1|, \ldots, |\lambda_N|)$). In fact, if E is a finite-dimensional normed space, then for each $\varepsilon > 0$ the space can be ε-isometrically imbedded in l_∞^N (where N is sufficiently large); it suffices to choose a finite collection of linear functionals $g_1, \ldots, g_N \in E^*$ such that $\max_{1 \leqslant j \leqslant N} |g_j(x)| \leqslant \|x\| \leqslant (1+\varepsilon) \max_{1 \leqslant j \leqslant N} |g_j(x)|$ and to set $Ux = (g_1(x), \ldots, g_N(x))$.

It is clear that this property is invariant with respect to taking linear homeomorphisms.

LEMMA 1. A uniformly convex space cannot be finitely universal.

Proof. Suppose that X is finitely universal with a constant C and is also uniformly convex. There exists $\theta \in (0, 1)$ such that if $\|x\| = 1$, $\|y\| = 1$, $\|(x-y)/2\| \geqslant 1/C$, then $\|(x+y)/2\| \le \theta$. Suppose that we are given an invertible operator $T: l_\infty^N \to F \subset X$; we will show that $\|T\| \cdot \|T^{-1}\| > \min(C, \theta^{2-N})$. (This will immediately lead to a contradiction if we choose N sufficiently large and T such that $\|T\| \cdot \|T^{-1}\| \le C$.) For $N = 1$ the inequality is trivial; for other values of N we use proof by induction. Suppose that $\|T\| \cdot \|T^{-1}\| \leqslant \min(C, \theta^{2-N})$. We assume that l_∞^{N-1} is imbedded in l_∞^N in the natural way, and we consider the restriction U of T to l_∞^{N-1}; it suffices to show that $\|U\| \le \theta \|T\|$. This follows easily from the fact that each $z \in l_\infty^{N-1}$, $\|z\| \le 1$, can be written in the form $z = (x+y)/2$, $x, y \in l_\infty^N$, $\|x\| \leqslant 1$, $\|y\| \leqslant 1$, $\|(x-y)/2\| \geqslant 1$.

THEOREM. There exists a reflexive Banach space in which each infinite-dimensional subspace is finitely universal.

The proof of this theorem will be given after several lemmas. In Lemma 2 we construct a weakly compact set K in c_0 which has some special properties. Then we show that its closed convex hull V also has these properties. Finally we consider a Banach space X for which V is the unit ball and we show that this is the required space.

It will be convenient to speak of elements of the space m (and other spaces of sequences) as functions on the set of natural numbers and to use terminology like "pointwise convergence." The operator which takes the points $1, \ldots, n$ to zero (that is, the operator of pointwise multiplication by the characteristic function $\chi_{[n+1, +\infty)}$) will be denoted by P_n.

Leningrad State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 8, No. 2, pp. 57-60, April-June, 1974. Original article submitted July 20, 1973.

Definition 2. A collection $(x_1, \ldots, x_N)$ of elements of the space m does not contain inverses if for arbitrary i, j such that $1 \leq i < j \leq N$ and arbitrary k, l such that $x_i(k) \neq 0$ and $x_j(l) \neq 0$ we have $k < l$.

It is clear that in such a case the x_i are automatically finite except perhaps for one of them, namely, the last nonzero one. There exist i and n such that $x_i = P_n(x_1 + \ldots + x_N)$.

We formulate properties which we will later verify for some sets A lying in m.

(1) A is contained in the unit ball. Each basis vector e_j (equal to 1 in position j and 0 elsewhere) belongs to A.

(2) $\forall x \in A\ \forall y \in m\ (|y| \leqslant |x| \Rightarrow y \in A)$. (The moduli and the inequalities are pointwise.)

Properties (1) and (2) are preserved if we take the closed and the convex hull.

(3) If $(x_1, \ldots, x_N)$ is a collection of elements in A (of arbitrary order N) which does not contain inverses, then $\frac{1}{2} P_N(x_1 + \ldots + x_N) \in A$.

If A has properties (1)-(3), then its closure (in the topology of pointwise convergence) also has these properties.

(4) $\forall x \in A\ \exists n\ 2P_n(x) \in A$.

Repeated application of (4) gives us the following property:

(4^a) $\forall x \in A\ \forall q\ \exists n\ 2^q P_n(x) \in A$; from ($4^a$) and (1) it follows that $A \subset c_0$. Property (4) is not necessarily preserved when we take closures.

LEMMA 2. There exists a weakly compact set $K \subset c_0$ which has properties (1)-(4).

Proof. Let A be the smallest set having properties (1)-(3). In other words, let $A = A_1 \cup A_2 \cup \ldots$, $A_1 = \{\alpha e_j\colon j > 1,\ |\alpha| \leqslant 1\}$ (the e_j are basis vectors), and $A_{n+1} = A_n \cup B_n$ where B_n is the set of all elements of the form $\frac{1}{2} P_N(x_1 + \ldots + x_N)$ where $(x_1, \ldots, x_N)$ is an arbitrary collection of elements of A_n which does not contain inverses.

We let K be the closure of A in m in the topology of pointwise convergence; K has properties (1)-(3) and we will show that it also has property (4). Let $x \in K$; we assume that x is not finite, for otherwise the proof is trivial. We select $x^{(s)}$ in A such that $x^{(s)} \to x$ pointwise. For large s the function $x^{(s)}$ is nonzero at the point $k_0 = \min\{k : x(k) \neq 0\}$ and also at some other point, and therefore $x^{(s)} \notin A_1$. But then $x^{(s)}$ belongs to some B_n, that is, $x^{(s)} = \frac{1}{2} P_N(x_1^{(s)} + \ldots + x_{N_s}^{(s)})$, where $(x_1^{(s)}, \ldots, x_{N_s}^{(s)})$ is a collection of elements of A which does not contain inverses. It is essential that $N_s \leq k_0$ (since $x^{(s)}(k_0) \neq 0$). We can assume therefore that N_s does not depend on s, $N_s = N$, where if necessary we can pass to a subsequence $\{x^{(s_1)}, x^{(s_2)}, \ldots\}$. We can also assume that each of the N sequences $\{x_i^{(s)}\}_{s=1}^{\infty}$ converges pointwise to some x_i where $x_i \in K$. If we take the limit we find that $x = \frac{1}{2} P_N(x_1 + \ldots + x_N)$; here the collection $(x_1, \ldots, x_N)$ does not contain inverses (as a limit of a sequence of collections which do not contain inverses). Then there exist i and n such that $x_i = P_n(x_1 + \ldots + x_N)$; hence, $2P_{\max(n,N)}(x) = P_N P_n(x_1 + \ldots + x_N) = P_N(x_i) \in K$.

Thus K has properties (1)-(4) and hence is contained in c_0. In a ball in c_0 the weak topology coincides with the topology of pointwise convergence, and in this topology K is compact.

LEMMA 3. Suppose that $K \subset c_0$ is a weakly compact set which has properties (1)-(4). Then its closed convex hull V is an absolutely convex weakly compact set in c_0 and has properties (1)-(4).

Proof. It is known that the closed convex hull of a weakly compact set in a Banach space is weakly compact. Absolute convexity of V follows from the fact that $x \in K \Rightarrow -x \in K$. We note that it makes no difference in which topology we close the convex hull M of K: the norm, weak, or pointwise topology. We have to show two things: that M has property (3) and that V has property (4).

Suppose that $(x_1, \ldots, x_N)$ does not contain inverses and each x_i is a convex combination of elements of K: $x_i = \alpha_i^{(1)} x_i^{(1)} + \ldots + \alpha_i^{(n)} x_i^{(n)}$, $x_i^{(s)} \in K$. Since K has property (2), we can assume that $x_i(k) = 0 \Rightarrow x_i^{(s)}(k) = 0$; then for each set of indices $s_1, \ldots, s_N$ in [1, n] the collection $(x_1^{(s_1)}, \ldots, x_N^{(s_N)})$ does not contain inverses and therefore $\frac{1}{2} P_N(x_1^{(s_1)} + \ldots + x_N^{(s_N)}) \in K$. It is easy to see that $x_1 + \ldots + x_N$ can be written as a convex combination of elements of the form $x_1^{(s_1)} + \ldots + x_N^{(s_N)}$; consequently, $\frac{1}{2} P_N(x_1 + \ldots + x_N) \in M$.

We now establish property (4). We set $D_n = \{x \in K\colon 4P_n(x) \in K\}$; $\{D_n\}_{n=1}^{\infty}$ is an ascending sequence of weakly closed subsets of K whose union equals K. Let $x_0 \in V$; it is known (see [2], theorem 5.6.4) that there exists a probability measure μ on K (weakly Borel regular) such that for each linear functional f on c_0

we have $f(x_0) = \int_K f\,d\mu$. Since $\mu D_n \to 1$, there exists n_0 such that $\mu D_{n_0} \geq 3/4$; then $2P_{n_0}(x_0) \in V$, since otherwise there would exist a function f such that $f(x_0) > 1$ and $\forall x\,(2P_{n_0}(x) \in V \Rightarrow |f(x)| \leqslant 1)$, and this would yield the contradiction

$$1 < \int_{D_{n_0}} f\,d\mu + \int_{K \setminus D_{n_0}} f\,d\mu \leqslant \frac{1}{2}\mu D_{n_0} + 2\mu(K \setminus D_{n_0}) \leqslant \frac{1}{2} + \frac{1}{2}.$$

We recall that by a normed block-system with respect to the basis $\{e_j\}_1^\infty$ we mean a sequence of the form $\left\{\sum_{j=n_i}^{n_{i+1}-1} \lambda_j e_j\right\}_{i=1}^\infty$ for which the norm of each term equals one.

LEMMA 4. Suppose that an absolutely convex weakly compact set $V \subset c_0$ has properties (1)-(4). We let X denote the linear hull of V equipped with a norm in which V is the unit ball. Then X is a reflexive Banach space; the sequence of unit vectors $\{e_j\}_1^\infty$ forms an unconditional basis in X, and the conjugate system of functions $\{e_j^*\}_1^\infty$ is an unconditional basis in the dual space X*. If $\{x_i\}_1^\infty$ is a normed block-system with respect to the basis $\{e_j\}_1^\infty$, then $\|P_N(\lambda_1 x_1 + \ldots + \lambda_N x_N)\|_X \leqslant 2 \max_{1 \leqslant i \leqslant N} |\lambda_i|$ for arbitrary N and $\lambda_1, \ldots, \lambda_N$.

Proof. Completeness of X follows from the fact that V is closed and bounded in c_0. In order to show that $\{e_j\}$ is a basis in X we just have to verify that $\forall x \in X\ \|P_n x\|_X \underset{n}{\to} 0$, and this follows from (4^a). By virtue of property (2) this basis is unconditional. If $\{x_i\}$ is a normed block-system, then for each N the collection $(x_1, \ldots, x_N)$ does not contain inverses; property (3) implies that $\frac{1}{2} P_N(x_1 + \ldots + x_N) \in V$; from property (2) we now obtain the required inequalities for $\|P_N \sum \lambda_i x_i\|_X$.

We will show that $\{e_j^*\}$ is an unconditional basis in X*. We only have to show that $\forall f \in X^*\ \|P_n^* f\| \underset{n}{\to} 0$. Suppose the contrary, that is, $\forall n\ \|P_n^* f\| > \varepsilon$; we choose a finite x_1 such that $\|x_1\| = 1$ and $f(x_1) > \varepsilon$; then we choose x_2, which is "further to the right" than x_1, with the same properties. We continue this process and obtain a normed block-system $\{x_i\}$ such that $\forall i\ f(x_i) > \varepsilon$. Clearly P_N does not affect $x_{N+1} + \ldots + x_{2N}$ and therefore $\|x_{N+1} + \ldots + x_{2N}\| \leq 2$ in spite of the fact that $f(x_{N+1} + \ldots + x_{2N}) > N\varepsilon$; this contradicts the fact that f is continuous.

We still have to show that X is reflexive, that is, that V is compact in the weak topology $\sigma(X, X^*)$. We will show that the topologies $\sigma(X, X^*)$ and $\sigma(c_0, c_0^*)$ coincide on V. For each $f \in X^*$ the restriction $f|_V$ is continuous in the topology $\sigma(c_0, c_0^*)$ since $f|_V$ is the uniform limit of a sequence $(f_n|_V)$, $f_n \in c_0^*$; for instance, we can let $f_n = f(e_1)e_1^* + \ldots + f(e_n)e_n^* = f - P_n^* f$. [We could also make use of conditions for reflexivity of a space with a basis (see [3]).]

Proof of the Theorem. We have to show that every infinite-dimensional subspace Y of the space X considered in the preceding lemma is finitely universal. If $\{x_i\}$ is a normed block-system with respect to the basis $\{e_j\}$ in X, then the subspace $X_0 \subset X$ generated by this system is finitely universal since for arbitrary N and $\lambda_1, \ldots, \lambda_N$ we have $\max_{1 \leqslant i \leqslant N} |\lambda_i| \leqslant \|\lambda_1 x_{N+1} + \ldots + \lambda_N x_{2N}\| \leqslant 2 \max_{1 \leqslant i \leqslant N} |\lambda_i|$. Therefore it suffices to choose $\{x_i\}$ such that X_0 is linearly homeomorphic to some subspace $X_1 \subset Y$. This is done in the usual fashion; namely, we successively choose y_i from Y and x_i such that $\|y_i - x_i\| \leq 2^{-i-1}$ and we apply the Krein–Mil'man–Rutman theorem on stability of minimal systems [4] according to which there exists an invertible linear operator S taking X_0 onto the subspace $X_1 \subset Y$ generated by the system $\{y_i\}$ such that $Sx_i = y_i$. The theorem is proved.

Remark 1. The constant in the definition of finite universality is obtained simultaneously for all infinite-dimensional subspaces of the space we constructed and can be chosen arbitrarily close to 2; it can be reduced if in the construction of K we replace $\frac{1}{2} P_N(x_1 + \ldots + x_N)$ by $(1 - \varepsilon) P_N(x_1 + \ldots + x_N)$. For each $\varepsilon > 0$ there exists a reflexive Banach space in which each infinite-dimensional subspace is finitely universal with a constant $\leq 1 + \varepsilon$.

Remark 2. We can show that for each infinite-dimensional subspace $Z \subset X^*$ and each N there exist an N-dimensional subspace $F \subset Z$ and an invertible operator $T: l_1^N \to F$ such that $\|T\| \times \|T^{-1}\| < 3$. From this it follows that X* does not contain an infinite-dimensional subspace which is linearly isomorphic to a uniformly convex space.

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