6 Generalizing Wiener chaos

6a First chaos, decomposable processes, stability

We consider an arbitrary continuous factorization. As was mentioned in Sect. 5b, Borel functions on $\mathcal{C}$ act on $L_2(\Omega, \mathcal{F}, P)$ by linear operators, and indicators of Borel subsets of $\mathcal{C}$ act by orthogonal projections to subspaces.

In particular, for the Brownian factorization, only $\mathcal{C}_{\text{finite}}$ is relevant. The set $\{M \in \mathcal{C}_{\text{finite}} : |M| = n\}$ corresponds to the subspace called n-th Wiener chaos.

In general, we may define n-th chaos as the subspace of $L_2(\Omega, \mathcal{F}, P)$ that corresponds to $\{M \in \mathcal{C} : |M| = n\}$. These subspaces are orthogonal, however, they do not span the whole $L_2(\Omega, \mathcal{F}, P)$, unless the noise is classical.

6a1 Proposition. (a) The subspace of $L_2(\Omega, \mathcal{F}, P)$, spanned by n-th chaos spaces for $n = 0, 1, 2, \dots$, is equal to $L_2(\mathcal{F}_{\text{stable}})$.

(b) The sub-σ-field generated by the first chaos is equal to $\mathcal{F}_{\text{stable}}$.

For a proof, see [10, (2.7) and Th. 2.12].

A random variable $X \in L_2(\Omega, \mathcal{F}, P)$ belongs to the first chaos if and only if

$$X = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{-\infty,t} \right) + \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{t,\infty} \right) \quad \text{for all } t \in \mathbb{R}\,.$$

For such X, letting $X_{s,t} = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{s,t} \right)$ we get a *decomposable process*, that is, a family $(X_{s,t})_{s \le t}$ of random variables such that $X_{s,t}$ is $\mathcal{F}_{s,t}$-measurable and $X_{r,s} + X_{s,t} = X_{r,t}$ whenever $r \le s \le t$. This way we get decomposable processes satisfying $\mathbb{E}\,|X_{s,t}|^2 < \infty$ and $\mathbb{E}\,X_{s,t} = 0$. Waiving these additional conditions we get a larger set of processes, but the sub-σ-field generated by these processes is still $\mathcal{F}_{\text{stable}}$. We may also consider complex-valued *multiplicative decomposable processes;* it means that $X_{s,t} : \Omega \to \mathbb{C}$ is $\mathcal{F}_{s,t}$-measurable and $X_{r,s} X_{s,t} = X_{r,t}$. The generated sub-$\sigma$-field is $\mathcal{F}_{\text{stable}}$, again. The same holds under the restriction $|X_{s,t}| = 1$ a.s. See [11, Th. 1.7].

Dealing with a noise (rather than factorization) we may restrict ourselves to stationary Brownian and Poisson decomposable processes. 'Stationary' means $X_{r,s} \circ \alpha_t = X_{r-t,s-t}$. 'Brownian' means $X_{s,t} \sim \mathrm{N}(0, t-s)$. 'Poisson' means $X_{s,t} \sim \mathrm{Poisson}(\lambda(t-s))$ for some $\lambda \in (0, \infty)$. The generated sub-σ-field is still $\mathcal{F}_{\text{stable}}$. See [7, Lemma 2.9]. (It was written for the Brownian component, but works also for the Poisson component.)

For a finite set $L = \{s_1, \dots, s_n\} \subset \mathbb{R}$, $s_1 < \dots < s_n$, introduce an operator Q_L on the space $L_2^0 = \{X \in L_2(\Omega, \mathcal{F}, P) : \mathbb{E}\,X = 0\}$ by

$$Q_L = \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{-\infty,s_1} \right) + \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{s_1,s_2} \right) + \dots + \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{s_{n-1},s_n} \right) + \mathbb{E}\left(\cdot \,\middle|\, \mathcal{F}_{s_n,\infty} \right)\,.$$

6a2 Proposition. If finite sets $L_1 \subset L_2 \subset \dots$ are such that their union is dense in $\mathbb{R}$, then operators Q_{L_n} converge in the strong operator topology to the orthogonal projection from L_2^0 onto the first chaos.

The proof is left to the reader. Hint: Q_L corresponds to the set of all nonempty $M \in \mathcal{C}$ contained in one of the $n+1$ intervals.

Stochastic analysis gives us another useful tool for calculating the first chaos, pioneered by Jon Warren [12, Th. 8]. Let $(B_{s,t})_{s\le t}$ be a decomposable Brownian motion, that is, a decomposable process such that $B_{s,t} \sim \mathrm{N}(0, t-s)$. One says that B has the *representation property,* if every $X \in L_2(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}\, X = 0$ is equal to a stochastic integral,

$$X = \int_{-\infty}^{+\infty} H(t)\, dB_{0,t}\,,$$

where H is a predictable *process* w.r.t. the filtration $(\mathcal{F}_{-\infty,t})_{t\in\mathbb{R}}$.

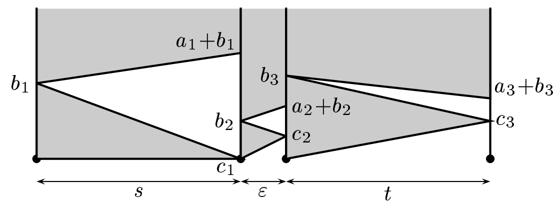
6a3 Proposition. If B has the representation property then the first chaos is equal to the set of all *linear* stochastic integrals

$$\int_{-\infty}^{+\infty} \varphi(t)\, dB_{0,t}\,, \qquad \varphi \in L_2(\mathbb{R})\,.$$

It follows that $\mathcal{F}_{\text{stable}}$ is generated by B.

6a4 Example. For the sticky noise (see Sect. 4), the process $\big(a(s,t)\big)_{s\le t}$ is a decomposable Brownian motion having the representation property. Therefore it generates $\mathcal{F}_{\text{stable}}$. On the other hand we know (recall 4h3) that $a(\cdot,\cdot)$ does not generate the whole σ-field. So, the sticky noise is not classical (Warren [12]).

The approach of 6a2 is also applicable. Let $\varphi : G_3 \to [-1,+1]$ be a Borel function, and $0 < t-\varepsilon < t < 1$. We consider $\varphi(\xi_{0,1}) = \varphi(\xi_{0,t-\varepsilon}\xi_{t-\varepsilon,t}\xi_{t,1})$ (you know, $\xi_{t-\varepsilon,t} = f_{a(t-\varepsilon,t),b(t-\varepsilon,t),c(t-\varepsilon,t)}$), and compare it with $\varphi(\xi_{0,t-\varepsilon}\tilde\xi_{t-\varepsilon,t}\xi_{t,1})$, where $\tilde\xi_{t-\varepsilon,t} = f_{a(t-\varepsilon,t),b(t-\varepsilon,t),0}$.



It appears that

$$\|\varphi(\xi_{0,t-\varepsilon}\xi_{t-\varepsilon,t}\xi_{t,1}) - \varphi(\xi_{0,t-\varepsilon}\tilde\xi_{t-\varepsilon,t}\xi_{t,1})\|_{L_2} = O(\varepsilon^{3/4}) = o(\sqrt{\varepsilon})\,,$$

provided that t is bounded away from 1 (otherwise we get $O(\varepsilon^{3/4}(1-t)^{-1/2})$ with an absolute constant). Taking into account that $\tilde\xi_{t-\varepsilon,t}$ is measurable w.r.t. the σ-field generated by $a(\cdot,\cdot)$ we conclude that the projection of $\varphi(\xi_{0,1})$ onto the first chaos is measurable w.r.t. the σ-field generated by $a(\cdot,\cdot)$. See Sect. 7b for the rest.

6b Higher levels of chaos

We still consider an arbitrary continuous factorization. Any Borel subset of $\mathcal{C}$ determines a subspace of $L_2(\Omega,\mathcal{F},P)$. However, the subset $\mathcal{C}_{\text{finite}} \subset \mathcal{C}$ is special; the corresponding subspace, being equal to $L_2(\mathcal{F}_{\text{stable}})$ by 6a1(a), is of the form $L_2(\mathcal{F}_1)$ for a sub-σ-field $\mathcal{F}_1 \subset \mathcal{F}$. Moreover, sub-$\sigma$-fields $\mathcal{F}^{\text{stable}}_{s,t} = \mathcal{F}_{s,t} \cap \mathcal{F}_{\text{stable}}$ form a continuous factorization (of the quotient probability space $(\Omega,\mathcal{F},P)/\mathcal{F}_{\text{stable}}$). That is a special case of the following result.

6b1 Proposition. Let $\mathcal{C}_1 \subset \mathcal{C}$ be an ideal, that is,

$$M_1 \subset M_2,\ M_2 \in \mathcal{C}_1 \quad \Longrightarrow \quad M_1 \in \mathcal{C}_1\,,$$
$$M_1, M_2 \in \mathcal{C}_1 \quad \Longrightarrow \quad (M_1 \cup M_2) \in \mathcal{C}_1\,.$$

Then:

(a) The corresponding (to $\mathcal{C}_1$) subspace of $L_2(\Omega, \mathcal{F}, P)$ is of the form $L_2(\mathcal{F}_1)$ for some sub-σ-field $\mathcal{F}_1 \subset \mathcal{F}$.

(b) The family $(\mathcal{F}_{s,t} \cap \mathcal{F}_1)_{s \le t}$ is a continuous factorization (of the quotient probability space $(\Omega, \mathcal{F}, P)/\mathcal{F}_1$).

I omit the proof.

If a noise is given (rather than a factorization), and $\mathcal{C}_1$ is shift invariant, that is,

$$M \in \mathcal{C}_1 \quad \Longrightarrow \quad (M + t) \in \mathcal{C}_1 \quad \text{for all } t\,,$$
$$M + t = \{m + t : m \in M\}\,,$$

then the new object pointed out in 6b1(b) is also a noise (a subnoise of the given noise).

The two most important ideals are

$$\mathcal{C}_{\text{finite}} \subset \mathcal{C}_{\text{countable}} \subset \mathcal{C}\,,$$
$$\mathcal{C}_{\text{finite}} = \{M \in \mathcal{C} : |M| < \aleph_0\}\,,$$
$$\mathcal{C}_{\text{countable}} = \{M \in \mathcal{C} : |M| \le \aleph_0\}\,.$$

A countable compact set M contains at least one limit point (in other words, accumulation point). The set

$$M' \subset M\,, \quad M' \in \mathcal{C}\,,$$

of all limit points of M may be finite or countable. In the latter case, it has its own limit points, these form $M'' = (M')' \subset M'$. And so on. We may classify sets of $\mathcal{C}_{\text{countable}}$ according to the number of steps ($M \mapsto M' \mapsto M'' \mapsto \dots$) until a finite set is reached, and the number of points of the last (finite) set. In general, the hierarchy is transfinite, numbered by all countable ordinals (the so-called Cantor-Bendixson hierarchy), but let us not climb too high.

We define the n-th *superchaos* as the subspace of $L_2(\Omega, \mathcal{F}, P)$ corresponding to $\{M \in \mathcal{C} : |M'| = n\}$.

Similarly to (6a1), the subspace spanned by n-th chaos spaces and n-th superchaos spaces for all n is equal to $L_2(\mathcal{F}_2)$ for some sub-σ-field $\mathcal{F}_2 \subset \mathcal{F}$, and $\mathcal{F}_2$ is generated by $\mathcal{F}_{\text{stable}}$ and the first superchaos.

Similarly to (5b2) and (5b7) we may 'count' points of M' by the operator

$$\mathbf{N}'_{\{s_1,\dots,s_n\}} = \sum_{j=1}^{n-1} \Big(\mathbf{1} - \mathbb{E}\,\big(\,\cdot\,\big|\, \mathcal{F}_{-\infty,s_j} \otimes \mathcal{F}^{\text{stable}}_{s_j,s_{j+1}} \otimes \mathcal{F}_{s_{j+1},\infty}\,\big)\Big) =$$
$$= \big(\mathbf{1} - U_{0+}^{(s_1,s_2)}\big) + \dots + \big(\mathbf{1} - U_{0+}^{(s_{n-1},s_n)}\big)\,,$$

or rather its limit $\mathbf{N}' = \lim_n \mathbf{N}'_{L_n}$. Further, similarly to 5b3 we may define

$$V_\lambda = \lim_n \exp(-\lambda \mathbf{N}'_{L_n}) \,.$$

This way, an ordinal hierarchy of operators may be constructed. It corresponds to Cantor-Bendixson hierarchy of countable compact sets.

Similarly to 6a2 we consider

$$\begin{aligned} Q'_{\{s_1,\dots,s_n\}} X = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{-\infty,s_1} \otimes \mathcal{F}^{\text{stable}}_{s_1,\infty} \right) + \mathbb{E}\left(X \,\middle|\, \mathcal{F}^{\text{stable}}_{-\infty,s_1} \otimes \mathcal{F}_{s_1,s_2} \otimes \mathcal{F}^{\text{stable}}_{s_2,\infty} \right) + \dots + \\ + \mathbb{E}\left(X \,\middle|\, \mathcal{F}^{\text{stable}}_{-\infty,s_{n-1}} \otimes \mathcal{F}_{s_{n-1},s_n} \otimes \mathcal{F}^{\text{stable}}_{s_n,\infty} \right) + \mathbb{E}\left(X \,\middle|\, \mathcal{F}^{\text{stable}}_{-\infty,s_n} \otimes \mathcal{F}_{s_n,\infty} \right) \end{aligned}$$

for $X \in L_2(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}} \right) = 0$.

6b2 Proposition. If finite sets $L_1 \subset L_2 \subset \dots$ are such that their union is dense in $\mathbb{R}$, then $Q'_{L_n} X$ converge to the orthogonal projection of X onto the first superchaos.

6b3 Example. For the sticky noise, consider such a random variable X: the number of random chords $[s,t] \times \{x\}$ such that $s > 0$ and $t > 1$. In other words (see 4i),

$$X = |\{ x : \sigma_1(x) \in \Pi \cap (0,\infty) \}| \,.$$

The conditional distribution of X given the Brownian path $B(\cdot) = a(0,\cdot)$ is Poisson(λ) with $\lambda = a(0,1) + b(0,1) = B(1) - \min_{[0,1]} B(\cdot)$, which is easy to guess from the discrete counterpart (see (4c10)). That is a generalization of a result of 4h3. In fact, the conditional distribution of the set $\{ x : \sigma_1(x) \in \Pi \cap (0,\infty) \}$, given the Brownian path, is the Poisson point process of intensity 1 on $[-b(0,1), a(0,1)]$, which is a result of Warren [12]. Taking into account that the σ-field generated by $B(\cdot)$ is $\mathcal{F}_{\text{stable}}$ (recall 6a4), we get $\mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}} \right) = a(0,1) + b(0,1)$. The random variable

$$Y = X - \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}} \right) = X - a(0,1) - b(0,1)$$

is sensitive, that is, $\mathbb{E}\left(Y \,\middle|\, \mathcal{F}_{\text{stable}} \right) = 0$. I claim that Y belongs to the first superchaos.

The proof is based on 6b2. Given $0 < s_1 < \dots < s_n < 1$, we have to check that Y can be decomposed into a sum $Y_0 + \dots + Y_n$ such that each Y_j is measurable w.r.t. $\mathcal{F}^{\text{stable}}_{0,s_j} \otimes \mathcal{F}_{s_j,s_{j+1}} \otimes \mathcal{F}^{\text{stable}}_{s_{j+1},1}$. Here is the needed decomposition:

$$\begin{aligned} X_j &= |\{ x : \sigma_1(x) \in \Pi \cap (s_j, s_{j+1}) \}| \,, \\ Y_j &= X_j - \mathbb{E}\left(X_j \,\middle|\, \mathcal{F}_{\text{stable}} \right) \,. \end{aligned}$$

We apply a small perturbation on $(0,s_j)$ and $(s_{j+1},1)$ but not (s_j,s_{j+1}). The set $\Pi \cap (s_j,s_{j+1})$ remains unperturbed. The function σ_1 is perturbed, but only a little; being a function of $B(\cdot)$, it is stable.

So, Y belongs to the first superchaos, and X belongs to the first superchaos plus $L_2(\mathcal{F}_{\text{stable}})$. It means that μ_X is concentrated on sets M such that $|M'| \le 1$.

The same holds for random variables $X_u = |\{ x : x \le u,\, \sigma_1(x) \in \Pi \cap (0,\infty) \}|$, for any u. They all are measurable w.r.t. the σ-field generated by the first superchaos and $\mathcal{F}_{\text{stable}}$. The random variable $c(0,1)$ is a (nonlinear!) function of these X_u (recall 4i). We see that the first superchaos and $\mathcal{F}_{\text{stable}}$ generate the whole σ-field $\mathcal{F}$. Every spectral set (of every random variable) has only a finite number of limit points.

6b4 Example. Another nonclassical noise, discovered and investigated by Warren [13], see also Watanabe [14], may be called the noise of splitting. It is the scaling limit of the model of 1e1; see also 8c. Spectral measures of most interesting random variables are described explicitly! A spectral set contains a single limit point, and two sequences converging to the point from the left and from the right.

Again, every spectral set (of every random variable) has only a finite number of limit points.

6b5 Question. We have no example of a noise whose spectral sets M are countable and M' is not always finite. Can it happen, at all? Can it happen for the refinement of a dyadic coarse factorization (as defined in 3b1)?

6c Black noise

6c1 Definition. A noise is *black,* if its stable σ-field $\mathcal{F}_{\text{stable}}$ is degenerate. In other words: its first chaos contains only 0.

Why 'black'? Well, the white noise is called 'white' since its spectral density is constant. It excites harmonic oscillators of all frequencies to the same extent. For a black noise, however, the response of any linear sensor is zero!

What could be a physically reasonable nonlinear sensor able to sense a black noise? Maybe, a fluid could do it, which is hinted at by the following words by Shnirelman [6, p. 1263] about a paradoxical motion of an ideal incompressible fluid: '... very strong external forces are present, but they are infinitely fast oscillating in space and therefore are indistinguishable from zero in the sense of distributions. The smooth test functions are not "sensitive" enough to "feel" these forces.'

The very idea of black noises, nonclassical factorizations etc. was suggested to me by Anatoly Vershik in 1994.

6c2 Proposition. A noise is black if and only if the spectral measure μ_X of every random variable $X \in L_2$ is concentrated on (the set of all) perfect sets $M \in \mathcal{C}$ (that is, M having no isolated points), except for the empty set.

For a proof see [7, Corollary 2.11].

Existence of black noises was proven first by Tsirelson and Vershik [11, Sect. 5]. A simpler and more natural example is described in the next section. Another example is found by Watanabe [15].

If all spectral sets are finite or countable (as in 6b3, 6b4), such a noise cannot contain a black subnoise.

6c3 Question. If a noise contains no black subnoise, does it follow that all spectral sets are countable?

Perfect sets may be classified, say, by Hausdorff dimension. For any $\alpha \in (0, 1)$, sets $M \in \mathcal{C}$ of Hausdorff dimension $\le \alpha$ are a shift invariant ideal, which leads to a subnoise (recall 6b1), and to a 'chaos subspace number α'. A continuum of such chaos subspaces (not in a single noise, of course) could occur, corresponding to different 'levels of stability'. Namely, in terms of 5b, we may consider $\rho \approx 1 - n^{-\alpha}$, that is,

$$U_\lambda^{(\alpha)} = \lim_n \exp\bigl(-\lambda n^{-\alpha} \mathbf{N}_{L_n}\bigr)$$

where L_n divides $(0,1)$ into intervals of length $1/n$. For now, however, I know of perfect spectral sets of Hausdorff dimension $1/2$ only.

6c4 Question. Can a noise have perfect spectral sets of Hausdorff dimension other than $1/2$?

6c5 Question. Can a black noise emerge as the refinement of a dyadic coarse factorization (as defined in 3b1)?