5 Stability

5a Discrete case

Fourier-Walsh coefficients, introduced in Sect. 3c for an arbitrary dyadic coarse factorization,

$$f = \sum_{M\in\mathcal{C}[i]} \hat f_M \tau_M = \hat f_\emptyset + \sum_{m\in\frac1i\mathbb{Z}} \hat f_{\{m\}}\tau_m + \sum_{m_1,m_2\in\frac1i\mathbb{Z},m_1<m_2} \hat f_{\{m_1,m_2\}}\tau_{m_1}\tau_{m_2} + \dots$$

help us to examine stability of a function f, as explained below. Imagine another array of random signs $(\tau'_m)_{m\in\frac1i\mathbb{Z}}$ (also independent equiprobable ±1) correlated with the array $(\tau_m)_{m\in\frac1i\mathbb{Z}}$,

$$\mathbb{E}\,\tau_m\tau'_m = \rho \quad \text{for each } m\in\frac1i\mathbb{Z}\,;$$

$\rho\in[-1,+1]$ is a parameter. Other correlations vanish. That is, the joint distribution of all τ_m and τ'_m is the product (over $m\in\frac1i\mathbb{Z}$) of (copies of) such a four-atom distribution:

		τ_m: -1	τ_m: $+1$
τ'_m	-1	$\frac{1+\rho}{4}$	$\frac{1-\rho}{4}$
	$+1$	$\frac{1-\rho}{4}$	$\frac{1+\rho}{4}$

Denoting by $\tilde\Omega[i]$ the product of such four-point probability spaces, we have a natural measure preserving map $\alpha:\tilde\Omega[i]\to\Omega[i]$; as before, $\Omega[i]$ is the product of two-point probability spaces. In addition, we have another measure preserving map $\alpha':\tilde\Omega[i]\to\Omega[i]$;

$$\tau_m\circ\alpha = \tau_m\,,\quad \tau_m\circ\alpha' = \tau'_m\,;$$

we use the same 'τ_m' for denoting a coordinate function on $\Omega[i]$ and $\tilde\Omega[i]$.

For products

$$\tau_M = \prod_{m\in M}\tau_m\,,\quad M\in\mathcal{C}[i] = \{M\subset\tfrac1i\mathbb{Z} : |M|<\infty\}$$

we have

$$\mathbb{E}\,\tau_M\tau'_M = \rho^{|M|}\,,\quad \tau_M\circ\alpha = \tau_M\,,\quad \tau_M\circ\alpha' = \tau'_M\,,$$

where $|M|$ is the number of elements of M. Therefore

$$\mathbb{E}\,(f\circ\alpha)(g\circ\alpha') = \sum_M \hat f_M\hat g_M\rho^{|M|} = \langle g,\rho^{\mathbf{N}[i]}f\rangle\,,$$

$$\rho^{\mathbf{N}[i]} : L_2[i]\to L_2[i]\,,\quad \rho^{\mathbf{N}[i]}\tau_M = \rho^{|M|}\tau_M\,,\quad \rho^{\mathbf{N}[i]}f = \sum_M \rho^{|M|}\hat f_M\tau_M\,.$$

The Hermite operator $\rho^{\mathbf{N}[i]}$ is a function of a self-adjoint operator $\mathbf{N}[i]$ such that $\mathbf{N}[i]\tau_M = |M|\tau_M$ for $M\in\mathcal{C}[i]$.

Every bounded function $\varphi : \mathcal{C}[i] \to \mathbb{R}$ acts on $L_2[i]$ by the operator $f \mapsto \sum_{M\in\mathcal{C}[i]} \varphi(M)\hat{f}_M\tau_M$. A commutative operator algebra is isomorphic to the algebra of functions. The operator $\rho^{\mathbf{N}[i]}$ corresponds to the function $M \mapsto \rho^{|M|}$. (In some sense, the unbounded operator $\mathbf{N}$ corresponds to the unbounded function $M \mapsto |M|$.)

A function $\varphi : \mathcal{C}[i] \to \{0,1\}$, the indicator of a subset of $\mathcal{C}[i]$, corresponds to a projection operator. Say, for the (indicator of) the set $\{\emptyset\}$, the operator projects to the one-dimensional space of constants (the expectation). For the set $\{M : M \cap (-\infty,0) = \emptyset\}$, the operator is the conditional expectation, $\mathbb{E}\big(\cdot\,\big|\,\mathcal{F}_{0,\infty}[i]\big)$.

The function $M \mapsto |M|$ is the sum (over $m \in \frac{1}{i}\mathbb{Z}$) of localized functions $M \mapsto |M \cap \{m\}|$. The latter is the indicator of the set $\{M : M \ni m\}$, corresponding to the projection operator $1 - \mathbb{E}\big(\cdot\,\big|\,\mathcal{F}_{\frac{1}{i}\mathbb{Z}\setminus\{m\}}\big)$. Thus,

$$\mathbf{N}f = \sum_m \big(f - \mathbb{E}\big(f\,\big|\,\mathcal{F}_{\frac{1}{i}\mathbb{Z}\setminus\{m\}}\big)\,.$$

The operator $\rho^{\mathbf{N}[i]}$ may be interpreted as the conditional expectation w.r.t. the sub-σ-field $\alpha^{-1}(\mathcal{F})$ generated by $\tau_m \circ \alpha$, $m \in \frac{1}{i}\mathbb{Z}$:

$$\mathbb{E}\big(f\circ\alpha'\,\big|\,\alpha^{-1}(\mathcal{F})\big) = (\rho^{\mathbf{N}[i]}f)\circ\alpha \quad \text{for } f \in L_2[i]\,.$$

We may imagine that our data τ_m are an unreliable copy of true data τ'_m; each sign τ_m is either correct (with probability $(1+\rho)/2$) or inverted (with probability $(1-\rho)/2$). If ρ is close to 1, our knowledge of τ'_M is satisfactory for moderate $|M|$ (when $\rho^{|M|} \approx 1$) but very bad for large $|M|$ (when $\rho^{|M|} \approx 0$). The place of a given function f between the two extremes is indicated by the number $\|f - \rho^{\mathbf{N}}f\|$.

5a1 Example. In the Brownian coarse factorization (recall 3b2),

$$\sup_i \|f[i] - \rho^{\mathbf{N}[i]}f[i]\| \to 0 \quad \text{for } \rho \to 1$$

for all $f \in L_2(\mathcal{A})$. It follows easily from convergence of operators (recall 2c and 3d2):

$$\operatorname{Lim}_{i\to\infty} \rho^{\mathbf{N}[i]} = \rho^{\mathbf{N}[\infty]}\,,$$

$$\rho^{\mathbf{N}[\infty]}f = \sum_{n=0}^{\infty} \rho^n \int\!\!\cdots\!\!\int_{t_1<\dots<t_n} \hat{f}(\{t_1,\dots,t_n\})\,dB(t_1)\dots dB(t_n)\,.$$

Convergence of operators follows from (2a6). The same holds for Example 3b4.

5a2 Example. A very different situation appears in Example 3b5. The second Brownian motion B_2 (or rather, its discrete approximation) is not linear but quadratic in random signs τ_m, $m \in \frac{1}{i}\mathbb{Z}$. It is twice less stable:

$$\mathbf{N}[i]f^{(2)}_{s,t}[i] = 2f^{(2)}_{s,t}[i]\,; \qquad \operatorname{Lim}_{i\to\infty} \rho^{\mathbf{N}[i]} = \rho^{2\mathbf{N}[\infty]}\,,$$

if $\mathbf{N}[\infty]$ is defined in the same way as in 5a1. For B_3 it is $\rho^{3\mathbf{N}[\infty]}$, and so on. Still, $\sup_i \|f[i] - \rho^{\mathbf{N}[i]}f[i]\| \to 0$ for $\rho \to 1$. For B_λ, however, the change is dramatic. Namely,

$$\mathbf{N}[i]f^{(\lambda)}_{s,t}[i] = \operatorname{entier}(\lambda\sqrt{i})f^{(\lambda)}_{s,t}[i]\,; \qquad \operatorname{Lim}_{i\to\infty} \rho^{\mathbf{N}[i]} = 0^{\mathbf{N}[\infty]}$$

for all $\rho \in (-1,+1)$; here $0^{\mathbf{N}[\infty]} = \lim_{\rho\to 0} \rho^{\mathbf{N}[\infty]}$ is the orthogonal projection to the one-dimensional subspace of constants (just the expectation). The same holds for Example 3b6.

5b Continuous case

We start with the *Brownian* continuous factorization $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\big)$. Using Wiener-Itô decomposition of $L_2(\Omega,\mathcal{F},P)$,

$$f = \sum_{n=0}^{\infty} \underbrace{\int\cdots\int_{t_1<\dots<t_n} \hat f(\{t_1,\dots,t_n\})\,dB(t_1)\dots dB(t_n)\,,}_{\text{belongs to } n\text{-th Wiener chaos}} \quad \hat f \in L_2(\mathcal{C}_{\text{finite}})\,,$$

we can define a self-adjoint operator $\mathbf{N}: L_2 \to L_2$ such that for each n, $\mathbf{N}f = nf$ for all f of n-th Wiener chaos. Accordingly, $\rho^{\mathbf{N}} f = \rho^n f$ for these f. Informally, $\mathbf{N}(dB(t_1)\dots dB(t_n)) = ndB(t_1)\dots dB(t_n)$.

Every bounded Borel function φ on $\mathcal{C}_{\text{finite}}$ acts on $L_2(\Omega,\mathcal{F},P)$ by the operator

(5b1) $$f \mapsto \sum_{n=0}^{\infty} \int\cdots\int_{t_1<\dots<t_n} \varphi\big(\{t_1,\dots,t_n\}\big)\hat f\big(\{t_1,\dots,t_n\}\big)\,dB(t_1)\dots dB(t_n)\,.$$

The operator $\rho^{\mathbf{N}}$ corresponds to the function $M \mapsto \rho^{|M|}$. (In some sense, the unbounded operator $\mathbf{N}$ corresponds to the unbounded function $M \mapsto |M|$.) The decomposition $|M| = |M \cap (-\infty,t)| + |M \cap (t,\infty)$ (it holds for μ_f-almost all M) leads to the operator decomposition $\mathbf{N} = \mathbf{N}_{-\infty,t} + \mathbf{N}_{t,\infty}$. Informally, $\mathbf{N}_{-\infty,t}\big(dB(t_1)\dots dB(t_n)\big) = kdB(t_1)\dots dB(t_n)$ and $\mathbf{N}_{t,\infty}\big(dB(t_1)\dots dB(t_n)\big) = (n-k)dB(t_1)\dots dB(t_n)$ whenever $t_1 < \dots < t_k < t < t_{k+1} < \dots < t_n$. Accordingly, $\rho^{\mathbf{N}} = \rho^{\mathbf{N}_{-\infty,t}} \otimes \rho^{\mathbf{N}_{t,\infty}}$.

A function $\varphi : \mathcal{C}_{\text{finite}} \to \{0,1\}$, the indicator of a Borel subset of $\mathcal{C}_{\text{finite}}$, corresponds to a projection operator. Say, for the (indicator of the) set $\{\emptyset\}$ the operator projects to the one-dimensional space of constants (the expectation). For the set $\{M : M \cap (-\infty,0) = \emptyset\}$ the operator is the conditional expectation, $\mathbb{E}\big(\cdot \,\big|\, \mathcal{F}_{0,\infty}\big)$.

The function

$$\varphi_{s,t}(M) = \begin{cases} 1 & \text{if } M \cap (s,t) \ne \emptyset, \\ 0 & \text{if } M \cap (s,t) = \emptyset \end{cases}$$

acts by the operator $\mathbf{1} - \mathbb{E}\big(\cdot \,\big|\, \mathcal{F}_{(-\infty,s)\cup(t,\infty)}\big)$.

For a finite set $L = \{s_1,\dots,s_n\} \subset \mathbb{R}$, $s_1 < \dots < s_n$, the function $\varphi_L(M) = \varphi_{s_1,s_2}(M) + \dots + \varphi_{s_{n-1},s_n}(M)$ counts intervals (s_j,s_{j+1}) that intersect M. Clearly, $\varphi_L(M) \le |M|$, and

$$\varphi_L(M) \uparrow M \quad \text{for } \mu_f\text{-almost all } M$$

if $L_1 \subset L_2 \subset \dots$ are chosen so that their union is dense in $\mathbb{R}$. Accordingly,

$$\mathbf{N}_{L_n} \uparrow \mathbf{N}\,,$$

(5b2) $$\mathbf{N}_{\{s_1,\dots,s_n\}} = \sum_{j=1}^{n-1} \Big(\mathbf{1} - \mathbb{E}\big(\cdot \,\big|\, \mathcal{F}_{(-\infty,s_j)\cup(s_{j+1},\infty)}\big)\Big)\,.$$

The operator $\mathbf{N}$ is expressed in terms of the factorization only, irrespective of Wiener-Itö decomposition. It gives us a bridge to *arbitrary* continuous factorizations.

5b3 Lemma. For every continuous factorization $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\big)$, every finite sets $L_1 \subset L_2 \subset \dots$ whose union is dense in $\mathbb{R}$, and every $\lambda \in [0,\infty)$, the limit

$$U_\lambda = \lim_n \exp(-\lambda \mathbf{N}_{L_n})\,,$$

where $\mathbf{N}_L$ is defined by (5b2), exists in the strong operator topology, and does not depend on the choice of $L_1, L_2, \dots$ Also,

$$U_\lambda U_\mu = U_{\lambda+\mu} \quad \text{for all } \lambda,\mu \in [0,\infty)\,.$$

For a proof, see [10, (2.4)].

In the Brownian factorization we know that $U_\lambda = \exp(-\lambda\mathbf{N})$, $\mathbf{N} = \lim_n \mathbf{N}_{L_n}$. In general, however, the semigroup $(U_\lambda)_{\lambda\ge 0}$ is discontinuous at $\lambda = 0$ (and $\mathbf{N}$ is ill-defined).

In fact, every bounded Borel function φ on $\mathcal{C}$ acts on $L_2(\Omega,\mathcal{F},P)$ by an operator [7, Sect. 2], though in general we have no explicit formula like (5b1). Once again, a commutative operator algebra is isomorphic to the algebra of functions. The operator U_λ corresponds to the function

$$\varphi(M) = e^{-\lambda|M|} \qquad (e^{-\infty} = 0)\,.$$

If μ_f is concentrated on $\{M \in \mathcal{C} : |M| = \infty\}$ then $U_\lambda f = 0$ for all $\lambda > 0$. Of course, $U_0 f = f$ anyway.

5b4 Lemma. For every continuous factorization $\big((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\big)$ there exists a sub-σ-field $\mathcal{F}_{\text{stable}} \subset \mathcal{F}$ such that for all $f \in L_2(\Omega,\mathcal{F},P)$,

$$f \in L_2(\mathcal{F}_{\text{stable}}) \quad\Longleftrightarrow\quad \|f - U_\lambda f\| \xrightarrow[\lambda\to 0]{} 0\,.$$

For a proof, see [10, Lemma 2.5].

In order to understand probabilistic meaning of U_λ, consider first $\rho^{\mathbf{N}_L}$, $L = \{s_1,\dots,s_n\}$, $s_1 < \dots < s_n$. We have

$$\Omega = \Omega_{-\infty,s_1} \times \Omega_{s_1,s_2} \times \dots \times \Omega_{s_{n-1},s_n} \times \Omega_{s_n,\infty}$$

or rather, $(\Omega,\mathcal{F},P) = (\Omega_{-\infty,s_1},\mathcal{F}_{-\infty,s_1},P_{-\infty,s_1}) \times \dots$, but let me use the shorter notation. Each $\omega \in \Omega$ may be thought of as a sequence $(\omega_{-\infty,s_1},\omega_{s_1,s_2},\dots\omega_{s_{n-1},s_n},\omega_{s_n,\infty})$ of local portions of data. Imagine another portion of data $\omega'_{s_1,s_2} \in \Omega_{s_1,s_2}$, either equal to ω_{s_1,s_2} (with probability ρ), or independent of it (with probability $1-\rho$). The joint distribution of ω_{s_1,s_2} and ω'_{s_1,s_2} is a convex combination of two probability measures on $\tilde\Omega_{s_1,s_2} = \Omega_{s_1,s_2} \times \Omega_{s_1,s_2}$. One measure is concentrated on the diagonal, it is the image of P_{s_1,s_2} under the map $\Omega_{s_1,s_2} \ni \omega_{s_1,s_2} \mapsto (\omega_{s_1,s_2},\omega_{s_1,s_2}) \in \tilde\Omega_{s_1,s_2}$; this measure encounters with the coefficient ρ. The other measure is the product measure $P_{s_1,s_2} \otimes P_{s_1,s_2}$; it encounters with the coefficient $1-\rho$.

Similarly we introduce $\tilde\Omega_{s_2,s_3},\dots,\tilde\Omega_{s_{n-1},s_n}$ and construct $\tilde\Omega = \Omega_{-\infty,s_1} \times \tilde\Omega_{s_1,s_2} \times \dots \times \tilde\Omega_{s_{n-1},s_n} \times \Omega_{s_n,\infty}$ (the factors being equipped with corresponding measures). You see, it is the same idea as in Sect. 5a. Again, we have two measure preserving maps $\alpha,\alpha' : \tilde\Omega \to \Omega$. It appears that

$$\mathbb{E}\big(\, f\circ\alpha' \,\big|\, \alpha^{-1}(\mathcal{F})\,\big) = (\rho^{\mathbf{N}_L} f)\circ\alpha \quad \text{for } f \in L_2(\Omega,\mathcal{F},P)\,.$$

This is the probabilistic interpretation of $\rho^{\mathbf{N}_L}$; each portion of data is either correct (with probability ρ), or wrong (with probability $1-\rho$).[24] However, portions are not small yet. The limit $n \to \infty$ makes them infinitesimal, and turns $\rho^{\mathbf{N}_L}$ into U_λ, where ρ and λ are related by $\rho = e^{-\lambda}$.

The interpretation above motivates the term 'stable' for $f \in L_2(\mathcal{F}_{\text{stable}})$.

5b5 Lemma. For every $f \in L_2(\Omega, \mathcal{F}, P)$,

$$\bigl(\forall \lambda > 0 \;\; U_\lambda f = 0\bigr) \quad \Longleftrightarrow \quad \mathbb{E}\bigl(\, f \,\big|\, \mathcal{F}_{\text{stable}} \bigr) = 0 \,.$$

For a proof, see [10, Lemma 2.14].

Such functions may be called sensitive.

The space $L_2(\Omega, \mathcal{F}, P)$ decomposes into the direct sum of two subspaces, stable and sensitive.

A continuous factorization is called *classical* (or *stable*), if $\mathcal{F}_{\text{stable}} = \mathcal{F}$.

A noise is called classical, if its continuous factorization is classical.

Two limiting cases of U_λ are projections. Namely, $U_\infty = \lim_{\lambda\to\infty} U_\lambda$ is the expectation, and $U_{0+} = \lim_{\lambda \to 0+} U_\lambda$ is $\mathbb{E}\bigl(\,\cdot\,\big|\,\mathcal{F}_{\text{stable}}\bigr)$. Restricting the 'perturbation of local data' to a given interval (s,t) we get operators $U_\lambda^{(s,t)}$. These correspond to functions $\mathcal{C} \ni M \mapsto \exp(-\lambda |M \cap (s,t)|)$ and satisfy

$$\begin{gathered} U_\lambda^{(s,t)} U_\mu^{(s,t)} = U_{\lambda+\mu}^{(s,t)}\,; \quad U_\lambda^{(r,s)} U_\lambda^{(s,t)} = U_\lambda^{(r,t)}\,; \\ U_\infty^{(s,t)} = \mathbb{E}\bigl(\,\cdot\,\big|\,\mathcal{F}_{-\infty,s} \otimes \mathcal{F}_{t,\infty}\bigr)\,; \\ U_{0+}^{(s,t)} = \mathbb{E}\bigl(\,\cdot\,\big|\,\mathcal{F}_{-\infty,s} \otimes \mathcal{F}_{s,t}^{\text{stable}} \otimes \mathcal{F}_{t,\infty}\bigr)\,. \end{gathered} \tag{5b6}$$

Note that (5b2) may be written as

$$\mathbf{N}_{\{s_1,\dots,s_n\}} = \bigl(\mathbf{1} - U_\infty^{(s_1,s_2)}\bigr) + \dots + \bigl(\mathbf{1} - U_\infty^{(s_{n-1},s_n)}\bigr)\,. \tag{5b7}$$

5c Back to discrete: two kinds of stability

The operator equality $\operatorname{Lim} \rho^{\mathbf{N}[i]} = \rho^{\mathbf{N}[\infty]}$ holds for some dyadic coarse factorizations (recall 5a1) but fails for some others (recall 5a2). Nothing like that happens for spectral measures; $\mu_f[i] \to \mu_f[\infty]$ always (see 3c5 and 3d). However, the operator $\rho^{\mathbf{N}[i]}$ corresponds to the function $\mathcal{C}[i] \ni M \mapsto \rho^{|M|}$ treated as an element of $L_\infty(\mu_f[i])$, and the operator $\rho^{\mathbf{N}[\infty]}$ corresponds to the function $\mathcal{C}[\infty] \ni M \mapsto \rho^{|M|}$ treated as an element of $L_\infty(\mu_f[\infty])$. How is it possible? Where is the origin of the clash between discrete and continuous?

The origin is, discontinuity of functions $M \mapsto \rho^{|M|}$ and $M \mapsto |M|$ w.r.t. the Hausdorff topology on $\mathcal{C}$.

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[24] This time, $\rho \in [0,1]$ rather than $[-1,1]$. The relation to the approach of Sect. 5a is expressed by the equality

$$\frac{1+\rho}{2}\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + \frac{1-\rho}{2}\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} = \begin{pmatrix} (1+\rho)/4 & (1-\rho)/4 \\ (1-\rho)/4 & (1+\rho)/4 \end{pmatrix} = \rho \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + (1-\rho)\begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}.$$

5c1 Example. Return to the equality $\mathbf{N}[i]f^{(2)}_{s,t}[i] = 2f^{(2)}_{s,t}[i]$ for $f^{(2)}_{s,t}[i] = i^{-1/2}\sum \tau_m\tau_{m+(1/i)}$ (see 5a2 and 3b5). The spectral measure of $f^{(2)}_{s,t}[i]$ is concentrated on two-point sets $M \subset \frac{1}{i}\mathbb{Z}$, namely, on pairs of two adjacent points $\{m, m+(1/i)\}$. However, $f^{(2)}_{s,t}[\infty]$ is just a Brownian increment; its spectral measure is concentrated on single-point sets. Now we see what happens; two close points merge in the limit! Multiplicity of spectral points eludes the continuous model.

The effect becomes dramatic for $f^{(\lambda)}_{s,t}[i]$; everything is stable in the continuous model ($i = \infty$), however, everything is asymptotically sensitive (for $i \to \infty$) in the discrete model. A finite spectral set on the continuum hides infinite multiplicity of each point.

Conformity between discrete and continuous can be restored by modifying the idea of stability introduced in Sect. 5a. Instead of inverting each τ_m (with probability $(1-\rho)/2$) independently of others, we may invert blocks $\tau_{s[i]}, \tau_{s[i]+(1/i)}, \dots, \tau_{t[i]}$ where coarse instants s, t satisfy $t[\infty] - s[\infty] = \varepsilon$. Each block is inverted with probability $(1-\rho)/2$, independently of other blocks. Ultimately we let $\varepsilon \to 0$, but the order of limits is crucial: $\lim_{\varepsilon\to0}\lim_{i\to\infty}(\dots)$. This way, we can define (in discrete time setup) *block stability* and *block sensitivity*, equivalent to stability and sensitivity (resp.) of the refinement. In contrast, the approach of Sect. 5a leads to what may be called *micro-stability* and *micro-sensitivity* (for discrete time only).

The function $\mathcal{C} \ni M \mapsto \rho^{|M|}$ is not continuous, but it is upper semicontinuous. Therefore, every micro-stable function is block stable, and every block sensitive function is micro-sensitive.

5c2 Example. The function $g_{s,t}$ of Example 3b6 is micro-sensitive but block stable. The same holds for all coarse random variables in that dyadic coarse factorization. It holds also for the second construction of Example 3b5 (I mean $f^{(\lambda)}_{s,t}$).

5d Permutation, not replacement

A different idea of stability/sensitivity emerges from a different perturbation of random signs (or other local data). Instead of replacing (or inverting) a small fraction of random signs, we may rearrange them a little.

In the discrete-time setup, we divide $\frac{1}{i}\mathbb{Z}$ into intervals of length ε, and apply a random permutation within each interval. Every such (ε-local) permutation σ acts on $\Omega[i]$ by a measure preserving transformation, thus, on $L_2[i]$ by a unitary operator $U_\sigma[i]$. We have a natural probability measure (just the uniform distribution) on the set of all these permutations; denote it $\mu_\varepsilon[i]$.

For the Brownian dyadic coarse factorization, for every $f \in L_2(\mathcal{A})$,

$$\sup_i \int \| f[i] - U_{\sigma[i]} f[i] \| \, d\mu_\varepsilon[i](\sigma[i]) \xrightarrow[\varepsilon\to0]{} 0\,, \tag{5d1}$$

which implies a weaker property,

$$\sup_i \left\| f[i] - \int U_{\sigma[i]} f[i] \, d\mu_\varepsilon[i](\sigma[i]) \right\| \xrightarrow[\varepsilon\to0]{} 0\,.$$

The latter property (and the more so, the former one) characterizes the Brownian coarse factorization among all dyadic coarse factorizations! (I omit the proof.)

In the continuous-time setup, we consider a noise (not just a continuous factorization). Arbitrary measure preserving maps $\mathbb{R} \to \mathbb{R}$ do not act on $(\Omega, \mathcal{F}, P)$ (unless the noise is classical),[25] but piecewise linear maps σ (having derivative $+1$ on each piece) act on $(\Omega, \mathcal{F}, P)$ by measure preserving transformations, therefore they act on $L_2(\Omega, \mathcal{F}, P)$ by unitary operators U_σ. Once again, we divide $\mathbb{R}$ into intervals of length ε, and require σ to map each interval into itself. Of course, there is no 'uniform distribution' on such σ. Rather, we subdivide each ε-interval into smaller intervals, of equal length δ, and use a random permutation of these δ-subintervals. This way, we have probability measures $\mu_{\varepsilon,\delta}$ on maps σ.

If f is stable (that is, $f \in L_2(\mathcal{F}_{\text{stable}})$, recall 5b), then

$$\int \| f - U_\sigma f \| \, d\mu_{\varepsilon,\delta}(\sigma) \to 0 \qquad \text{(5d2)}$$

for $\varepsilon \to 0$, $\delta \to 0$, $\delta < \varepsilon$. The converse is also true. Moreover, if f is sensitive (that is, $\mathbb{E}\big(f \,\big|\, \mathcal{F}_{\text{stable}} \big) = 0$, recall 5b5), then f and $U_\sigma f$ are asymptotically independent when $\delta \to 0$, for every ε. (I omit the proof.)

Note that the Poisson (continuous) factorization is classical; everything satisfies (5d2) in this case. However, it cannot be obtained as the refinement of a *dyadic* coarse factorization satisfying (5d1).

[25] Another characterization of classical noises!