2 Abstract nonsense of scaling limit

2a More on our limiting procedures

The joint compactification K of $\Omega_1 \uplus \Omega_2 \uplus \dots$, used in 1c, is not quite satisfactory. Return to Example 1d3:

$$f_u(\omega) = \frac{1}{\sqrt{n}} \sum_{k<un} \tau_k(\omega) \quad \text{for } u \in [0,1] \cap \mathbb{Q} \tag{2a1}$$

($\mathbb{Q}$ being the set of rational numbers). The limiting model is the Brownian motion, restricted to $[0,1] \cap \mathbb{Q}$. What about an irrational point, $v \in [0,1] \setminus \mathbb{Q}$? The random variable f_v may be defined on Ω as the limit (say, in L_2) of f_u for $u \to v$, $u \in [0,1] \cap \mathbb{Q}$. On the other hand, f_v is naturally defined on $\Omega_1 \uplus \Omega_2 \uplus \dots$ (by the same formula (2a1)). However, f_v is not a continuous function on the compact space K.[4] Thus, the weak convergence $P_n \to P$ is relevant to f_u but not f_v. Something is wrong!

The wrong thing is the uniform topology used in (1c4)–(1c7). A right topology should take measures P_n into account. We have two ways, 'moderate' and 'radical'.

Here is the 'moderate' way. We choose some appropriate subsets $B_n \subset (\Omega_1 \uplus \Omega_2 \uplus \dots)$, $B_1 \subset B_2 \subset \dots$, such that

$$\inf_i P_i(B_n \cap \Omega_i) \uparrow 1 \quad \text{for } n \to \infty$$

and replace in (1c5)–(1c7) the assumption "$f_n \in C$, $f_n \to f$ uniformly $\Longrightarrow f \in C$" with

$$f_n \in C,\ f_n \to f \text{ uniformly on each } B_n \quad \Longrightarrow \quad f \in C\,. \tag{2a2}$$

2a3 Example. Continuing (2a1) we define B_n by

$$B_n \cap \Omega_i = \left\{ \omega \in \Omega_i : \sup_{0 \le k < l \le i} \frac{\left| \frac{1}{\sqrt{i}} \sum_{j=k}^{l} \tau_j(\omega) \right|}{\left(\frac{l-k}{i} \right)^{1/3}} \le n \right\},$$

then[5]

$$|f_u(\omega) - f_v(\omega)| \le n \cdot |u-v|^{1/3} \quad \text{for } \omega \in B_n \cap \Omega_i$$

if i is large enough (namely, $2/i < |u-v|$). The set C (satisfying (2a2)) generated by f_u for all rational u, contains also f_u for all irrational u.

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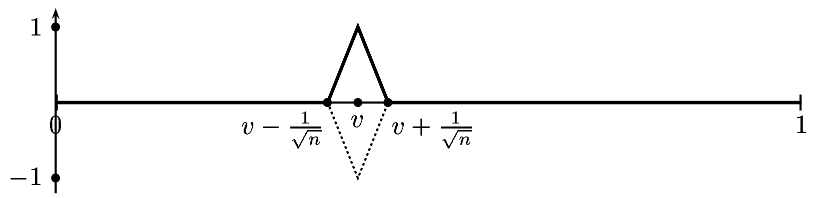
[4]There exist $\omega_n \in \Omega_n$ such that $\lim_n f_u(\omega_n)$ exists for all $u \in [0,1] \cap \mathbb{Q}$, but $\lim_n f_v(\omega_n)$ does not exist.



[5]Of course, $|u-v|^\alpha$ for any $\alpha \in (0, 1/2)$ may be used, not only $|u-v|^{1/3}$.

Similarly to 1c, we may translate (2a2) into the topological language. For each n, the restriction of C to B_n corresponds to a joint compactification (K_n, α_n) of $B_n \cap \Omega_i$. Clearly, $K_{n_1} \subset K_{n_2}$ for $n_1 < n_2$, and $\alpha_{n_1} = \alpha_{n_2}|_{K_{n_1}}$. Thus, we get a *joint σ-compactification*

$$\alpha : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to K_\infty = K_1 \cup K_2 \cup \dots$$

We do not need a topology on the union K_∞ of metrizable compact spaces $K_1 \subset K_2 \subset \dots$[6] We just define $C(K_\infty)$ as the set of all functions $g : K_\infty \to \mathbb{R}$ such that $g|_{K_n}$ is continuous (on K_n) for each n. We have

$$C = \alpha^{-1}\bigl(C(K_\infty)\bigr) ,$$

that is, observables $f \in C$ are functions of the form

$$f = g \circ \alpha , \quad \text{that is, } f(\omega) = g(\alpha(\omega)), \quad g \in C(K_\infty) .$$

If measures $\alpha(P_n)$ weakly converge (w.r.t. bounded functions of $C(K_\infty)$, recall (1c8), (1c9)), we get the limiting model (Ω, P) by taking $\Omega = K_\infty$ and $P = \lim_{n\to\infty} \alpha(P_n)$.

2a4 Example. Continuing 2a3 we see that the limiting measure P exists, and the joint distribution of all f_u (extended to K_∞ by continuity) w.r.t. P is the Wiener measure. The 'uniform' metric on K_∞,

$$\operatorname{dist}(x, y) = \sup_{0 \le u \le 1} |f_u(x) - f_u(y)| ,$$

is continuous on each K_n. Therefore, every function continuous in the 'uniform' metric belongs to $C(K_\infty)$. Our joint σ-compactification is another form of the usual weak convergence of random walks to the Brownian motion.

That was the 'moderate way'. It requires special subsets $B_n \subset (\Omega_1 \uplus \Omega_2 \uplus \dots)$, in contrast to the 'radical way'; basically, the latter allows the sequence of sets B_n to depend on a sequence of functions f_n, see (2a2). In other words, instead of uniform (or 'locally uniform') convergence, we introduce a weaker topology by the metric[7]

(2a5) $$\operatorname{dist}(f, g) = \sup_n \int \frac{|f(\omega) - g(\omega)|}{1 + |f(\omega) - g(\omega)|} \, dP_n(\omega) .$$

If $f_n \in C(K)$ and $\operatorname{dist}(f_n, f) \to 0$ then f_n converge in probability w.r.t. P; thus, f is naturally defined P-almost everywhere.[8]

[6] But if you want, K_∞ may be equipped with the inductive limit topology; that is, $U \subset K_\infty$ is open if and only if for every n, $U \cap K_n$ is open (in K_n). However, the topology usually is not metrizable.

[7] Alternatively, we may restrict ourselves to bounded functions $\Omega_1 \uplus \Omega_2 \uplus \dots \to [-1, +1]$ (applying a transformation like arctan) and use, say,

$$\operatorname{dist}(f, g) = \sup_n \int |f(\omega) - g(\omega)| \, dP_n(\omega) .$$

[8] In fact, every (equivalence class of) P-measurable function can be obtained in that way provided that, for each n, supports of P_n and P do not intersect. It means that every random variable on the limiting probability space is the scaling limit of some function on $\Omega_1 \uplus \Omega_2 \uplus \dots$ (see also 2c8).

Let C be the closure of $C(K)$ in the metric (2a5), then

$$\int \varphi(f_1,\dots,f_d)\,dP_n \xrightarrow[n\to\infty]{} \int \varphi(f_1,\dots,f_d)\,dP$$

for every d, every bounded continuous function $\varphi:\mathbb{R}^d\to\mathbb{R}$, and every $f_1,\dots,f_d\in C$. The joint distribution of $f_1,\dots,f_d$ w.r.t. P_n converges (weakly) to that w.r.t. P. So, the weak convergence $P_n\to P$ is relevant for the whole C (not only $C(K)$). That is the idea of the 'radical way', presented systematically in next subsections (2b, 2c).

Returning again to Example 1d3 we see that f_v (for $v\in[0,1]$) is the limit of f_u (for $u\in[0,1]\cap\mathbb{Q}$) in the metric (2a5); thus, $f_v\in C$ for all $v\in[0,1]$.

However, much more can be said. Not only

$$\operatorname{Lim}\Bigl(\frac{1}{\sqrt{n}}\sum_{an<k<bn}\tau_k(\omega)\Bigr)=\int_a^b dB(t)\,,$$

where 'Lim' means the scaling limit (as explained above), but also

$$\operatorname{Lim}\Bigl(n^{-d/2}\sum_{an<k_1<\dots<k_d<bn}\tau_{k_1}(\omega)\dots\tau_{k_d}(\omega)\Bigr)= \\ =\int\!\!\cdots\!\!\int_{a<t_1<\dots<t_d<b} dB(t_1)\dots dB(t_d)=\frac{1}{n!}H_d\bigl(B(b)-B(a),b-a\bigr)$$

where H_d is the Hermite polynomial (see for instance [5, IV.3.8]). Taking finite linear combinations and their closure in the metric (2a5) we get

(2a6) $$\operatorname{Lim}\Bigl(\sum_{d=0}^{\infty}n^{-d/2}\sum_{0<k_1<\dots<k_d<n}\psi_d\bigl(\tfrac{k_1}{n},\dots,\tfrac{k_d}{n}\bigr)\tau_{k_1}(\omega)\dots\tau_{k_d}(\omega)\Bigr)= \\ =\sum_{d=0}^{\infty}\int\!\!\cdots\!\!\int_{0<t_1<\dots<t_d<1}\psi_d(t_1,\dots,t_d)dB(t_1)\dots dB(t_d)$$

provided that functions ψ_d are Riemann integrable, and vanish for d large enough. The right-hand side is well-defined for all $\psi_d\in L_2$ such that $\sum_d\|\psi_d\|_2^2<\infty$; the scaling limit may be kept by replacing $\psi_d\bigl(\frac{k_1}{n},\dots,\frac{k_d}{n}\bigr)$ with the mean value of ψ_d on the $1/n$-cube centered at $\bigl(\frac{k_1}{n},\dots,\frac{k_d}{n}\bigr)$. Now, $(0,1)$ may be replaced with the whole $\mathbb{R}$; ψ_d is defined on $\Delta_d=\{(x_1,\dots,x_d)\in\mathbb{R}^d:x_1<\dots<x_d\}$. The right-hand side of (2a6) gives us an isometric linear correspondence between $L_2(\Delta_0\uplus\Delta_1\uplus\Delta_2\uplus\dots)$ and $L_2(\Omega,\mathcal{F},P)$, where $(\Omega,\mathcal{F},P)$ is the probability space describing the Brownian motion (on the whole $\mathbb{R}$).

2b Coarse probability space: definition and simple example

2b1 Definition. A *coarse probability space* $\bigl((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^{\infty},\mathcal{A}\bigr)$ consists of a sequence of probability spaces $(\Omega[i],\mathcal{F}[i],P[i])$ and a set $\mathcal{A}$ of subsets of the disjoint union $\Omega[\mathrm{all}]=\Omega(1)\uplus\Omega(2)\uplus\dots$, satisfying the following conditions.

(a) $\forall A \in \mathcal{A}\ \forall i\ (A \cap \Omega[i]) \in \mathcal{F}[i]$.

(b) $\forall A, B \in \mathcal{A}\ \big(A \cap B \in \mathcal{A},\ A \cup B \in \mathcal{A},\ \Omega[\text{all}] \setminus A \in \mathcal{A}\big)$.

(c) $\mathcal{A}$ contains every $A \subset \Omega[\text{all}]$ such that $\forall i\ (A \cap \Omega[i]) \in \mathcal{F}[i]$ and $P[i]\big(A \cap \Omega[i]\big) \to 0$ for $i \to \infty$.

(d) $\big(\cup_{k=1}^{\infty} A_k\big) \in \mathcal{A}$ for every pairwise disjoint $A_1, A_2, \dots \in \mathcal{A}$ such that $\sum_k \sup_i P[i]\big(A_k \cap \Omega[i]\big) < \infty$.

(e) $\lim_i P[i]\big(A \cap \Omega[i]\big)$ exists for every $A \in \mathcal{A}$.

(f) There exists a finite or countable subset $\mathcal{A}_1 \subset \mathcal{A}$ that *generates* $\mathcal{A}$ in the sense that the least subset of $\mathcal{A}$ satisfying (b)–(d) and containing $\mathcal{A}_1$ is the whole $\mathcal{A}$.

A set $\mathcal{A}$ satisfying (a)–(f) will be called a *coarse σ-field*[9] (on the *coarse sample space* $(\Omega[i], \mathcal{F}[i], P[i])_{i=1}^{\infty}$). Each set A belonging to the coarse σ-field $\mathcal{A}$ will be called *coarsely measurable* (w.r.t. $\mathcal{A}$), or a *coarse event*.

2b2 Note. Condition 2b1(c) is equivalent to

(c1) $\forall i\ \mathcal{F}[i] \subset \mathcal{A}$. That is, if a set $A \subset \Omega[\text{all}]$ is contained in some $\Omega[i]$, and is $\mathcal{F}[i]$-measurable, then $A \in \mathcal{A}$.

Also, Condition 2b1(d) is equivalent to each of the following conditions (d1)–(d4). There, we assume that $A \subset \Omega[\text{all}]$, $\forall i\ \big(A \cap \Omega[i]\big) \in \mathcal{F}[i]$, and $\forall k\ A_k \in \mathcal{A}$.

(d1) If $A_k \uparrow A$ (that is, $A_1 \subset A_2 \subset \dots$ and $A = \cup_k A_k$) and $\sup_i P[i]\big((A \setminus A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$.

(d2) If $\sup_i P[i]\big((A \triangle A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$. (Here $A \triangle A_k = (A \setminus A_k) \cup (A_k \setminus A)$.)

(d3) If $A_k \uparrow A$ and $\limsup_i P[i]\big((A \setminus A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$.

(d4) If $\limsup_i P[i]\big((A \triangle A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$.

So, we have 10 equivalent combinations: (c)&(d), (c1)&(d), (c)&(d1), (c1)&(d1), (c)&(d2), ..., (c1)&(d4). (I omit the proof.)

However, "$\sup_i$" in (d) cannot be replaced with "$\limsup_i$".

2b3 Lemma. Let $\mathcal{A}_1$ be a finite or countable set satisfying 2b1(a,e) and

(b1) $\forall A, B \in \mathcal{A}\ \big(A \cap B \in \mathcal{A}\big)$.

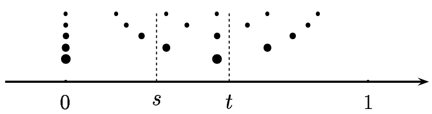
Then the least set $\mathcal{A}$ containing $\mathcal{A}_1$ and satisfying 2b1(b,c,d) is a coarse σ-field.

The proof is left to the reader.

In such a case we'll say that the coarse σ-field $\mathcal{A}$ is *generated* by the set $\mathcal{A}_1$.

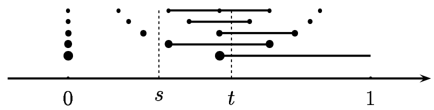
2b4 Example. Let $\Omega[i] = \{0, \frac{1}{i}, \dots, \frac{i-1}{i}\}$, and $P[i]$ be the uniform distribution on $\Omega[i]$. Every interval $(s,t) \subset (0,1)$ gives us a set $A_{s,t} \subset \Omega[\text{all}]$,

$$A_{s,t} \cap \Omega[i] = (s,t) \cap \Omega[i] \,.$$



[9]It is not a σ-field, unless $\mathcal{A}$ contains all sets satisfying 2b1(a).

We take a dense countable set of pairs (s,t) (say, rational s,t) and consider the set $\mathcal{A}_1$ of the corresponding $A_{s,t}$. The set $\mathcal{A}_1$ satisfies the conditions of Lemma 2b3, therefore it generates a coarse σ-field $\mathcal{A}$. In fact, $\mathcal{A}$ consists of all $A = A[1] \uplus A[2] \uplus \dots$ such that sets $A[i] + (0, 1/i) \subset (0,1)$ converge in probability to some $A[\infty] \subset (0,1)$; that is, $\operatorname{mes}\big(A[\infty] \triangle (A[i] + (0,1/i))\big) \to 0$ for $i \to \infty$.



If $A = A_{s,t}$ then, of course, $A[\infty] = (s,t)$.

2b5 Example. Continuing Example 1c1, we take $\Omega[i] = \{-1,+1\}^i$ with the uniform distribution $P[i]$. Given n and $a = (a_1, \dots, a_n) \in \{-1,+1\}^n$, we consider $A_a \subset \Omega[\text{all}]$,

$$A_a \cap \Omega[i] = \{(\tau_1, \dots, \tau_i) : \tau_1 = a_1, \dots, \tau_n = a_n\} \quad \text{for } i \ge n\,.$$

Such sets A_a (for all a and n) are a countable collection $\mathcal{A}_1$ satisfying the conditions of Lemma 2b3, therefore it generates a coarse σ-field $\mathcal{A}$. In fact, $\mathcal{A}$ consists of all $A = A[1] \uplus A[2] \uplus \dots$ such that sets $\beta_i^{-1}(A) \subset (0,1)$ converge in probability to some $A[\infty] \subset (0,1)$; here $\beta_i : (0,1) \to \Omega[i]$ is such a measure preserving map:

$$\beta_i(x) = \big((-1)^{c_1}, \dots, (-1)^{c_i}\big) \quad \text{when } x - \Big(\frac{c_1}{2} + \dots + \frac{c_i}{2^i}\Big) \in \Big(0, \frac{1}{2^i}\Big)\,,$$

for any $c_1, \dots, c_i \in \{0,1\}$.

You may guess that some limiting procedure produces a ('true', not coarse) probability space out of any given coarse probability space. Indeed, such a procedure (called 'refinement') is described in the next subsection.

2c Good use of joint compactification

Having a coarse probability space $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^{\infty}, \mathcal{A}\big)$ and its refinement $(\Omega, \mathcal{F}, P)$ (to be defined later), we may hope that the Hilbert space $L_2[\infty] = L_2(\Omega, \mathcal{F}, P)$ is in some sense the limit of Hilbert spaces $L_2[i] = L_2\big(\Omega[i], \mathcal{F}[i], P[i]\big)$. That is indeed the case in the framework of joint compactification, as we'll see. A bad use of the framework, tried in 1c, is a joint compactification of given probability spaces. A good use, considered here, is a joint compactification of metric (Hilbert, ...) spaces built over the given probability spaces.

2c1 Definition. A *coarse Polish space* is $\big((S[i], \rho[i])_{i=1}^{\infty}, c\big)$, where each $(S[i], \rho[i])$ is a Polish space (that is, a complete separable metric space[10]), and $c \subset S[1] \times S[2] \times \dots$ is a set of sequences $x = \big(x[1], x[2], \dots\big)$ satisfying the following conditions.

(a) If $x_1, x_2 \in S[1] \times S[2] \times \dots$ are such that $\rho[i]\big(x_1[i], x_2[i]\big) \to 0$ (for $i \to \infty$), then $(x_1 \in c) \iff (x_2 \in c)$.

(b) If $x, x_1, x_2, \dots \in S[1] \times S[2] \times \dots$ are such that $\sup_i \rho[i]\big(x_k[i], x[i]\big) \to 0$ (for $k \to \infty$), then $\big(\forall k \;\; x_k \in c\big) \implies \big(x \in c\big)$.

[10]Many authors define a Polish space as a metrizable topological space admitting a complete separable metric. However, I assume that a metric is given.

(c) $\lim_i \rho[i]\big(x_1[i], x_2[i]\big)$ exists for every $x_1, x_2 \in c$.

(d) There exists a finite or countable subset $c_1 \subset c$ that *generates* c in the sense that the least subset of c satisfying (a), (b) and containing c_1 is the whole c.

2c2 Note. Condition 2c1(d) does not change if 'satisfying (a), (b)' is replaced with 'satisfying (b)'. That is, 2c1(d) is just separability of c in the metric $x_1, x_2 \mapsto \sup_i \rho[i]\big(x_1[i], x_2[i]\big)$.

The refinement of a coarse Polish space $\big((S[i], \rho[i])_{i=1}^\infty, c\big)$ is basically the metric space $\big(c, \tilde\rho\big)$, where

$$\tilde\rho(x_1, x_2) = \lim_i \rho[i]\big(x_1[i], x_2[i]\big) .$$

Though, $\tilde\rho$ is a pseudometric (semimetric), it may vanish for some $x_1 \neq x_2$. The equivalence class, denote it $x[\infty]$, of a sequence $x \in c$ consists of all $x_1 \in c$ such that $\rho[i]\big(x_1[i], x[i]\big) \to 0$. On the set $S[\infty]$ of all equivalence classes we introduce a metric $\rho[\infty]$,

$$\rho[\infty]\big(x_1[\infty], x_2[\infty]\big) = \lim_{i\to\infty} \rho[i]\big(x_1[i], x_2[i]\big) ;$$

thus, $\big(S[\infty], \rho[\infty]\big)$ is a metric space. We write

$$\big(S[\infty], \rho[\infty]\big) = \operatorname{Lim}_{i\to\infty,c}\big(S[i], \rho[i]\big)$$

and call $\big(S[\infty], \rho[\infty]\big)$ the *refinement* of the coarse Polish space $\big((S[i], \rho[i])_{i=1}^\infty, c\big)$. Also, for every $x = (x[1], x[2], \dots) \in c$ we denote its equivalence class $x[\infty] \in S[\infty]$ by

$$x[\infty] = \operatorname{Lim}_{i\to\infty,c} x[i] ,$$

and call it the refinement of x.

2c3 Lemma. For every coarse Polish space, its refinement is a Polish space.

Proof. Separability follows from (d); completeness is to be proven. Let $x_1, x_2, \dots$ be a Cauchy sequence in (S, ρ); we have to find $x \in S$ such that $\rho(x_k, x) \to 0$. We may assume that $\sum_k \rho(x_k, x_{k+1}) < \infty$. Each x_k is an equivalence class; using (a) we choose for each $k = 1, 2, 3, \dots$ a representative $s_k \in S[1] \times S[2] \times \dots$ of x_k such that $\sup_i \rho[i]\big(s_k[i], s_{k+1}[i]\big) \le 2\rho(x_k, x_{k+1})$. Completeness of $\big(S[i], \rho[i]\big)$ ensures existence of $s_\infty[i] = \lim_k s_k[i]$. Condition (b) ensures $s_\infty \in c$. The equivalence class $x \in S$ of s_∞ satisfies $\rho(x_k, x) \le \sup_i \rho[i]\big(s_k[i], s_\infty[i]\big) \to 0$ for $k \to \infty$. □

Let $\big(S[i], \rho[i]\big)_{i=1}^\infty, c\big)$ be a coarse Polish space, and (S, ρ) its refinement. On the disjoint union $\big(S[1] \uplus S[2] \uplus \dots\big) \uplus S$ we introduce a topology, namely, the weakest topology making continuous the following functions $f_s : \big(S[1] \uplus S[2] \uplus \dots\big) \uplus S \to [0, \infty)$ for $s \in c$,

$$f_s(x) = \rho[i]\big(x, s[i]\big) \quad \text{for } x \in S[i] ,$$
$$f_s(x) = \rho(x, s[\infty]) \quad \text{for } x \in S ,$$

and an additional function $f_0 : \big(S[1] \uplus S[2] \uplus \dots\big) \uplus S \to [0, \infty)$, $f_0(x) = 1/i$ for $x \in S[i]$, $f_0(x) = 0$ for $x \in S$. On every $S[i]$ separately (and also on S), the new topology coincides with the old topology, given by $\rho[i]$ (or ρ).

We may choose a sequence (s_k) dense in c; the topology is generated by functions f_{s_k} (and f_0), therefore it is a metrizable topology. Moreover the sequence of functions $\big(\frac{f_{s_k}(\cdot)}{1+f_{s_k}(\cdot)}\big)_{k=1}^{\infty}$ (and f_0) maps the disjoint union into the metrizable compact space $[0,1]^\infty$, and is a homeomorphic embedding. Thus, we have a joint compactification of all $S[i]$ and S; and so, we treat them as subsets of a compact metrizable space K;

$$S[i] \subset K\,, \quad S \subset K\,.$$

2c4 Lemma. Let $s_\infty \in S$, $s_1 \in S[1], s_2 \in S[2], \dots$ Then $s_k \to s_\infty$ in K if and only if $s = (s_1, s_2, \dots) \in c$ and $\operatorname{Lim}_{k\to\infty,c} s_k = s_\infty$.

The proof is left to the reader.

The assumption '$s_\infty \in S$' is essential. Other limiting points (not belonging to S) may exist; corresponding sequences converge in K but do not belong to c. And, of course, sets $S, S[1], S[2], \dots$ are not closed in K, unless they are compact.

2c5 Lemma. A set $c_1 \subset c$ generates c if and only if the set of refinements $\{x[\infty] : x \in c_1\}$ is dense in $S[\infty]$.

The proof is left to the reader.

Given continuous functions $f[i] : S[i] \to \mathbb{R}$, $f[\infty] : S[\infty] \to \mathbb{R}$, we write $f[\infty] = \operatorname{Lim}_{i\to\infty,c} f[i]$ if $f[i](x[i]) \to f[\infty](x[\infty])$ whenever $x[\infty] = \operatorname{Lim}_{i\to\infty,c} x[i]$. If functions $f[i]$ are equicontinuous (say, $|f[i](x) - f[i](y)| \le \rho[i](x,y)$ for all i and $x, y \in S[i]$), then it is enough to check that $f[i](x_k[i]) \to f[\infty](x_k[\infty])$ for some sequence $(x_k)_{k=1}^{\infty}$, $x_k \in c$, such that the sequence $(x_k[\infty])_{k=1}^{\infty}$ is dense in $S[\infty]$.

Given continuous maps $f[i] : S[i] \to S[i]$, $f[\infty] : S \to S$, we write $f[\infty] = \operatorname{Lim}_{i\to\infty,c} f[i]$ if $\operatorname{Lim}_{i\to\infty,c} f[i](x[i]) = f[\infty](x[\infty])$ whenever $x[\infty] = \operatorname{Lim}_{i\to\infty,c} x[i]$. That is, $\operatorname{Lim}\big(f[i](x[i])\big) = \big(\operatorname{Lim} f[i]\big)\big(\operatorname{Lim} x[i]\big)$. If maps $f[i]$ are equicontinuous then, again, convergence may be checked on x_k such that $x_k[\infty]$ are dense.

Given continuous maps $f[i] : S[\infty] \to S[i]$, we may ask, whether $\operatorname{Lim}_{i\to\infty,c} f[i](x) = x$ for all $x \in S[\infty]$, or not. If maps $f[i]$ are equicontinuous then, still, convergence may be checked for a dense subset of $S[\infty]$.

If every $S[i]$ is not only a metric space but also a Hilbert (or Banach) space, and c is linear (that is, closed under linear operations), then the refinement S is also a Hilbert (or Banach) space, and linear operations are continuous on $\big(S[1] \cup S[2] \cup \dots\big) \cup S \subset K$ in the sense that

$$\operatorname{Lim}_{i\to\infty,c}(as_1[i] + bs_2[i]) = a\operatorname{Lim}_{i\to\infty,c} s_1[i] + b\operatorname{Lim}_{i\to\infty,c} s_2[i]$$

for all $s_1, s_2 \in c$.

Consider the case of Hilbert spaces $S[i] = H[i]$, $S = H$. Given linear[11] operators $R[i] : H[i] \to H[i]$, we may ask about $\operatorname{Lim} R[i]$. If it exists, we get

$$\operatorname{Lim}\big(R[i]x[i]\big) = \big(\operatorname{Lim} R[i]\big)\big(\operatorname{Lim} x[i]\big)\,.$$

If $\sup_i \|R[i]\| < \infty$, then $R[i]$ are equicontinuous, and convergence may be checked on a sequence x_k such that vectors $x_k[\infty]$ span H (that is, their linear combinations are dense in

[11] Continuous, of course.

H). For example, one-dimensional orthogonal projections; if $x[\infty] = \operatorname{Lim} x[i]$ then $\operatorname{Proj}_{x[\infty]} = \operatorname{Lim} \operatorname{Proj}_{x[i]}$.

Given linear operators $R[i] : H \to H[i]$, we may ask whether $\operatorname{Lim} R[i](x) = x$ for all $x \in H$, or not. If $\sup_i \|R[i]\| < \infty$ then convergence may be checked on a sequence that spans H. Note that such $R[i]$ always exist; moreover, $\|R[i]\| \le 1$ may be ensured. Indeed, we take x_k such that $x_k[\infty]$ are an orthonormal basis of H. After some correction, $x_k[i]$ become orthogonal (for each i), and $\|x_k(i)\| \le 1$.[12] Now we let $R[i]x_k[\infty] = x_k[i]$.

We return to coarse *probability* spaces.

Let $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A}\big)$ be a coarse probabilty space. For each i the pseudometric $A, B \mapsto P[i](A \triangle B)$ on $\mathcal{F}[i]$ gives us the metric space $\mathrm{MALG}[i] = \mathrm{MALG}\big(\Omega[i], \mathcal{F}[i], P[i]\big)$ of all equivalence classes of measurable sets. It is not only a metric space but also a Boolean algebra, and moreover, a measure algebra (as defined in [3, 17.44]). Treating every coarse event $A \in \mathcal{A}$ as a sequence of $A[1] \in \mathrm{MALG}[1], A[2] \in \mathrm{MALG}[2], \dots$ we get a coarse Polish space $\big((\mathrm{MALG}[i])_{i=1}^\infty, \mathcal{A}\big)$. Its refinement is a Polish space $\mathrm{MALG}[\infty]$. The set $\mathcal{A}$ is closed under Boolean operations (union, intersection, complement). Therefore $\mathrm{MALG}[\infty]$ is not only a metric space but also a Boolean algebra. Using Lemma 2c3 it is easy to check that $\mathrm{MALG}[\infty]$ is a measure algebra. Therefore it is (up to isomorphism) of the form

$$\mathrm{MALG}[\infty] = \mathrm{MALG}(\Omega, \mathcal{F}, P)$$

for some probability space $(\Omega, \mathcal{F}, P)$. In the nonatomic case we may take $(\Omega, \mathcal{F}, P) = (0,1)$ with Lebesgue measure; in general, we may take a shorter interval plus a finite or countable set of atoms. Such a probability space $(\Omega.\mathcal{F}, P)$ (unique up to isomorphism) will be called the refinement of the coarse probability space $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A}\big)$, and we write

$$(\Omega, \mathcal{F}, P) = \operatorname{Lim}_{i\to\infty,\mathcal{A}} \big(\Omega[i], \mathcal{F}[i], P[i]\big)$$

(in practice, sometimes I omit "$i \to \infty$" or "$\mathcal{A}$" (or both) under the "Lim").

Every sequence $A = (A[1], A[2], \dots) \in \mathcal{A}$ has its refinement

$$\operatorname{Lim}_{i\to\infty,\mathcal{A}} A[i] = A[\infty] \in \mathrm{MALG}(\Omega, \mathcal{F}, P)\,.$$

2c6 Lemma. A subset $\mathcal{A}_1$ of a coarse σ-field $\mathcal{A}$ generates $\mathcal{A}$ is and only if the refinement $\mathcal{F}$ of $\mathcal{A}$ is generated (mod 0) by refinements $A[\infty]$ of all $A \in \mathcal{A}_1$.

Proof. We apply Lemma 2c5 to the algebra generated by $\mathcal{A}_1$. □

In order to define $L_2(\mathcal{A})$ as a set of functions on $\Omega[\mathrm{all}]$, we start with indicators $\mathbf{1}_A$ for $A \in \mathcal{A}$, form their linear combinations, and take their completion in the metric

$$\|f\|_{L_2(\mathcal{A})} = \sup_i \|f[i]\|_{L_2[i]}\,,$$

where $L_2[i] = L_2\big(\Omega[i], \mathcal{F}[i], P[i]\big)$; the completion is a Banach (not Hilbert) space $L_2(\mathcal{A})$. Each element f of the completion is evidently identified with a sequence of $f[i] \in L_2[i]$, or a function on $\Omega[\mathrm{all}]$. We have a coarse Polish space $\big((L_2[i])_{i=1}^\infty, L_2(\mathcal{A})\big)$. It has its refinement, $L_2[\infty]$.

[12] Of course, $\|x_k[i]\| \to 1$, but in general we cannot ensure $\|x_k[i]\| = 1$. It may happen that $\dim H[i] < \infty$ but $\dim H = \infty$.

2c7 Lemma. The refinement $L_2[\infty]$ of $\big((L_2[i])_{i=1}^\infty, L_2(\mathcal{A})\big)$ is (canonically isomorphic to) $L_2(\Omega,\mathcal{F},P)$, where $(\Omega,\mathcal{F},P)$ is the refinement of $\big((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^\infty,\mathcal{A}\big)$.

Proof. We define the canonical map $L_2(\mathcal{A}) \to L_2(\Omega,\mathcal{F},P)$ first on indicators by $\mathbf{1}_A \mapsto \mathbf{1}_{A[\infty]}$, and extend it by linearity and continuity to the whole $L_2(\mathcal{A})$. We note that the image of $f \in L_2(\mathcal{A})$ in $L_2(\Omega,\mathcal{F},P)$ depends only on the refinement $f[\infty] \in L_2[\infty]$ of f, and their norms are equal (both are equal to $\lim_i \|f[i]\|$). We have a linear isometric embedding $L_2[\infty] \to L_2(\Omega,\mathcal{F},P)$. Its image is closed (since $L_2[\infty]$ is complete by Lemma 2c3), and contains indicators $\mathbf{1}_B$ for all $B \in \operatorname{MALG}(\Omega,\mathcal{F},P)$; therefore the image is the whole $L_2(\Omega,\mathcal{F},P)$. □

2c8 Note. The same holds for L_p for each $p \in (0,\infty)$, and for the space L_0 of all random variables (equipped with the topology of convergence in probability). Elements of $L_0(\mathcal{A})$ will be called coarsely measurable (w.r.t. $\mathcal{A}$) functions (on $\Omega[\text{all}]$), or *coarse random variables;* elements of $L_2(\mathcal{A})$ — square integrable coarse random variables.

Let f be a coarse random variable, then (usual) random variables $f[i] : \Omega[i] \to \mathbb{R}$ converge in distribution (for $i \to \infty$) to the refinement $f[\infty] : \Omega \to \mathbb{R}$. The distribution of $f[\infty]$ will be called the *limiting distribution* of f.

It may happen that $f \in L_2(\mathcal{A})$ but $(\operatorname{sgn} f) \notin L_2(\mathcal{A})$. An example: $f(\omega) = \frac{(-1)^i}{i}$ for all $\omega \in \Omega[i]$. Here, the limiting distribution is an atom at 0, and the function 'sgn' is discontinuous at 0.

2c9 Lemma. (a) Let $f : \Omega[\text{all}] \to \mathbb{R}$ be a coarse random variable, and $\varphi : \mathbb{R} \to \mathbb{R}$ a continuous function, then $\varphi \circ f : \Omega[\text{all}] \to \mathbb{R}$ is a coarse random variable.

(b) The same as (a) but φ may be discontinuous at points of a set $Z \subset \mathbb{R}$ negligible w.r.t. the limiting distribution of f.

Proof. If f is a linear combination of indicators, then $\varphi \circ f$ is another linear combination of the same indicators. A straightforward approximation gives (a) for uniformly continuous φ. In general, for every ε there exists a compact set $K \subset \mathbb{R} \setminus Z$ of probability $\ge 1-\varepsilon$ w.r.t. the limiting distribution, and also w.r.t. the distribution of $f[i]$ for all i (since all these distributions are a compact set of distributions). The restriction of f to K is uniformly continuous. The limit for $\varepsilon \to 0$ is uniform in i. □

For a given Polish space S we may define a coarse S-valued random variable as a map $f : \Omega[\text{all}] \to S$ such that (usual) random variables $f[i] : \Omega[i] \to S$ converge in distribution (for $i \to \infty$), and $f^{-1}(B) \in \mathcal{A}$ for every $B \subset S$ such that the boundary of B is negligible w.r.t. the limiting distribution of f.

For $S = \mathbb{R}$ the new definition conforms with the old one.

A coarse σ-field generated by a given sequence of sets (coarse events) was defined after Lemma 2b3. Often it is convenient to generate a coarse σ-field by a sequence of functions (coarse random variables). A function $f : \Omega[\text{all}] \to \mathbb{R}$ is coarsely $\mathcal{A}$-measurable if and only if $\mathcal{A}$ contains sets $f^{-1}\big((-\infty,x)\big)$ for all $x \in \mathbb{R}$ except for atoms (if any) of the limiting distribution of f. A dense countable subset of these x is enough. So, a coarse σ-field generated by a finite or countable set of functions f is nothing but the coarse σ-field generated by a countable set of sets of the form $f^{-1}\big((-\infty,x)\big)$. More generally, S-valued (coarse) random variables may be used; they are reduced to the real-valued case by composing with appropriate continuous functions $S \to \mathbb{R}$.

2c10 Lemma. A sequence of functions $f_k : \Omega[\text{all}] \to \mathbb{R}$ generates a coarse σ-field if and only if for every n, n-dimensional random variables $\big(f_1[i], \dots, f_n[i]\big) : \Omega[i] \to \mathbb{R}^n$ converge in distribution (for $i \to \infty$).

2c11 Note. The same holds for an arbitrary Polish space instead of $\mathbb{R}$.

2c12 Note. Comparing 2c10 and (1c9) we see that every joint compactification of $\Omega_1 \uplus \Omega_2 \uplus \dots$ (in the sense of 1c, assuming (1c8)) may be upgraded (or downgraded?) to a coarse probability space. Namely, we take a sequence of functions f_k that generates C and consider the coarse σ-field $\mathcal{A}$ generated by (f_k). Every $f \in C$ is a coarse random variable, since $L_0(\mathcal{A})$ is closed under all operations used in (1c5), or (1c6), or (1c7).[13] Therefore $\mathcal{A}$ does not depend on the choice of (f_k).

[13]Of course, $L_0(\mathcal{A})$ usually contains no sequence dense in the *uniform* topology.