Exam of 26.09.2006 — Solutions

1

1a

One solution. The set $\{(u_1,u_2) : 2u_1 \le u_2 \le 0.5\}$ is a triangle with vertices $(0,0),(0,0.5),(0.25,0.5)$. The joint distribution of U_1,U_2 is uniform on $(0,1)\times(0,1)$, thus, the probability is equal to the area of the triangle, namely, $1/16$.

Another solution. First, $\mathbb{P}\left(2U_1 \le U_2 \le 0.5 \,\middle|\, U_2 = u_2 \right) = \mathbb{P}\left(U_1 \le 0.5u_2 \right) = 0.5u_2$ (by independence and uniformity) for $u_2 \in (0,0.5)$, and 0 for $u_2 \in (0.5,1)$. Thus, $\mathbb{P}\left(2U_1 \le U_2 \le 0.5 \right) = \mathbb{E}\,\mathbb{P}\left(2U_1 \le U_2 \le 0.5 \,\middle|\, U_2 \right) = \int_0^{0.5} 0.5u_2\,\mathrm{d}u_2 = 1/16$.

1b

For all $v_1,v_2 \in (0,1)$ we have $\mathbb{P}\left(V_1 \le v_1, V_2 \le v_2 \right) = \mathbb{P}\left(\max(U_1^2,U_2^2) \le v_1, \min(U_1,U_2) \le v_2 \max(U_1,U_2) \right) = \mathbb{P}\left(U_1 < U_2, U_2^2 \le v_1, U_1 \le v_2 U_2 \right) + \mathbb{P}\left(U_1 > U_2, U_1^2 \le v_1, U_2 \le v_2 U_1 \right) = 2\mathbb{P}\left(U_1 < U_2, U_2 \le \sqrt{v_1}, U_1 \le v_2 U_2 \right) = 2\mathbb{P}\left(U_1 \le v_2 U_2, U_2 \le \sqrt{v_1} \right)$ (using symmetry); the latter probability is similar to that of Item (a) and can be calculated by either way; it is equal to $0.5\sqrt{v_1}\cdot\sqrt{v_1}v_2 = 0.5v_1v_2$. Thus, $\mathbb{P}\left(V_1 \le v_1, V_2 \le v_2 \right) = v_1v_2$.

1c

Let U_1,U_2 be independent random variables, each distributed $\mathrm{U}(0,1)$. The pair (X_1,X_2) is distributed like the pair $(X^*(U_1),X^*(U_2))$. Therefore the pair $(\max(X_1,X_2),\min(X_1,X_2))$ is distributed like $(\max(X^*(U_1),X^*(U_2)),\min(X^*(U_1),X^*(U_2)))$. The latter is $(X^*(\max(U_1,U_2)),X^*(\min(U_1,U_2)))$, since X^* is increasing. It follows from Item (b) that the pair $(\max(U_1,U_2)),\min(U_1,U_2))$ is distributed like $(\sqrt{V_1},V_2\sqrt{V_1})$. Therefore the pair $(X^*(\max(U_1,U_2)),X^*(\min(U_1,U_2)))$ is distributed like $(X^*(\sqrt{V_1}),X^*(V_2\sqrt{V_1}))$.

2

2a

First, $\mathbb{E}\sum_{k=1}^\infty \frac{\alpha_k}{k(k+1)} = \sum_{k=1}^\infty \mathbb{E}\frac{\alpha_k}{k(k+1)}$ by the monotone convergence theorem (and linearity of expectation). Second, each α_k is distributed uniformly on $\{0,1,\dots,9\}$, thus $\mathbb{E}\,\alpha_k = 4.5$. Third, $\sum_{k=1}^n \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$ (you can discover it easily by inspecting $n=1,2,3$; alternatively, note that $\frac{1}{k(k+1)} = \frac1k - \frac{1}{k+1}$), thus $\sum_{k=1}^\infty \frac{1}{k(k+1)} = 1$. Finally, $\mathbb{E}\sum_{k=1}^\infty \frac{\alpha_k}{k(k+1)} = 4.5$.

2b

No, it cannot happen, because of a contradiction: $\mathbb{P}\left(A \right) = \mathbb{E}\,\mathbb{P}\left(A \,\middle|\, X \right) < \frac12$, but $\mathbb{P}\left(A \right) = \mathbb{E}\,\mathbb{P}\left(A \,\middle|\, Y \right) > \frac12$.

2c

First, $\mathbb{E}\,X = \int_0^1 X^*(p)\,\mathrm{d}p = \int_0^{0.5} X^*(p)\,\mathrm{d}p + \int_{0.5}^1 X^*(p)\,\mathrm{d}p$. Second, $X^*(0.5) = 3$. (More exactly, $X^*(0.5-) \le 3$, $X^*(0.5+) \ge 3$.) Using monotonicity of X^*, $0 \le X^*(p) \le 3$ for $p \in (0, 0.5)$, and $3 \le X^*(p) \le 5$ for $p \in (0.5, 1)$. Therefore $0.5\cdot 0+0.5\cdot 3 \le \mathbb{E}\,X \le 0.5\cdot 3+0.5\cdot 5$, that is, $a \in [1.5, 4]$.

On the other hand, every $a \in [1.5, 4]$ is of the form $0.5b + 0.5c$ where $b \in [0, 3]$ and $c \in [3, 5]$ (for instance, $b = 2 \cdot 0.6(a - 1.5)$ and $c = 2 \cdot 0.4(a - 1.5) + 3$), and we may take $X^*(p) = b$ for $p \in (0, 0.5)$, and $X^*(p) = c$ for $p \in (0.5, 1)$. (The median of X is not unique here, however, 3 is one of the medians.)

Finally, the set is $[1.5, 4]$.

2d

Yes, it follows. We have

$$\mathbb{P}\left(3 + \tfrac{1}{n} < X < 3 + \tfrac{2}{n} \right) = \int_{3+\frac{1}{n}}^{3+\frac{2}{n}} f_X(x)\,\mathrm{d}x \in [\tfrac{1}{n}A_n, \tfrac{1}{n}B_n]\,,$$

where

$$A_n = \inf_{3+\frac{1}{n}<x<3+\frac{2}{n}} f_X(x)\,, \quad B_n = \sup_{3+\frac{1}{n}<x<3+\frac{2}{n}} f_X(x)\,.$$

Both A_n and B_n converge to $f_X(3+)$ as $n \to \infty$; it remains to use the sandwich argument.

3

3a

We take y such that $\mathbb{P}\left(X + \mathrm{e}^X \in [y, y + 1] \right) \ge 0.9$ and x such that $x + \mathrm{e}^x = y$ (such x exists, since the function is bijective). Using also monotonicity of the function we have $\mathbb{P}\left(X \in [x, x+1] \right) = \mathbb{P}\left(X+\mathrm{e}^X \in [x+\mathrm{e}^x, x+1+\mathrm{e}^{x+1}] \right) \ge \mathbb{P}\left(X+\mathrm{e}^X \in [x+\mathrm{e}^x, x+1+\mathrm{e}^x] \right) = \mathbb{P}\left(X + \mathrm{e}^X \in [y, y + 1] \right) \ge 0.9$, thus, X is concentrated.

3b

We take y such that $\mathbb{P}\left(f(X) + g(X) \in [y, y + 1] \right) \ge 0.9$ and consider sets $A = \{x : f(x) + g(x) \in [y, y + 1]\}$, $B = f(A) = \{f(x) : x \in A\}$. For all $x_1, x_2 \in A$ we have $y \le f(x_1) + g(x_1)$ and $f(x_2) + g(x_2) \le y + 1$, thus $(f(x_2) + g(x_2)) - (f(x_1) + g(x_1)) \le 1$, therefore $f(x_2)-f(x_1) \le 1$. It means that $y_2-y_1 \le 1$ for all $y_1, y_2 \in B$, that is, $B \subset [z, z+1]$ for some z (for instance, $z = \inf B$). Finally, $\mathbb{P}\left(z \le f(X) \le z + 1 \right) \ge \mathbb{P}\left(f(X) \in B \right) \ge \mathbb{P}\left(X \in A \right) \ge 0.9$; we see that $f(X)$ is concentrated.

3c ....

Assume the contrary: $\mathbb{P}\big(y \le Y \le y+1 \,\big|\, X \big) < 0.9$ a.s. for every y. Then $\mathbb{P}\big(y \le Y \le y+1 \big) = \mathbb{E}\,\mathbb{P}\big(y \le Y \le y+1 \,\big|\, X \big) < 0.9$ for every y, in contradiction to the concentration of Y.

3d ....

Assume the contrary: $\mathbb{P}\big(x \le X \le x+1 \big) < 0.9$ for every x. By independence, $\mathbb{P}\big(z \le X+Y \le z+1 \,\big|\, Y=y \big) = \mathbb{P}\big(z \le X+y \le z+1 \,\big|\, Y=y \big) = \mathbb{P}\big(z \le X+y \le z+1 \big) = \mathbb{P}\big(z-y \le X \le z-y+1 \big) < 0.9$, which contradicts to Item (c) (and concentration of $X+Y$).

3e ....

First, $X_1 + X_2 = \max(X_1, X_2) + \min(X_1, X_2)$ is distributed like $X^*(\sqrt{V_1}) + X^*(V_2\sqrt{V_1})$ by Item 1(c); here V_1, V_2 are independent $\mathrm{U}(0,1)$. Also, $\max(X_1, X_2)$ is distributed like $X^*(\sqrt{V_1})$. Thus, $X^*(\sqrt{V_1}) + X^*(V_2\sqrt{V_1})$ is concentrated; we have to prove that $X^*(\sqrt{V_1})$ is concentrated. (It does not follow from Item (d), since $X^*(\sqrt{V_1})$ and $X^*(V_2\sqrt{V_1})$ need not be independent!)

Assume the contrary: $X^*(\sqrt{V_1})$ is not concentrated. Conditionally, given $V_2 = v_2$, we may apply Item (b) to the increasing functions $X^*(\sqrt{V_1})$ and $X^*(v_2\sqrt{V_1})$ of the random variabke V_1. We know that $X^*(\sqrt{V_1})$ is not concentrated (the conditioning on V_2 being irrelevant here, by independence). Therefore, the sum $X^*(\sqrt{V_1}) + X^*(v_2\sqrt{V_1})$ cannot be concentrated, in contradiction to Item (c) (and the concentration of $X^*(\sqrt{V_1}) + X^*(V_2\sqrt{V_1})$).