Exam of 27.06.2006 — Solutions

1

1a

For $x \in (-0.5, 0.5)$ the corresponding section $\{y : (x, y) \in B_\varphi\}$ of B_φ is the interval $(\frac{-0.5 - x\cos\varphi}{\sin\varphi}, \frac{0.5 - x\cos\varphi}{\sin\varphi})$ of length $1/\sin\varphi$. The length does not depend on x, therefore the distribution of X is uniform, $X \sim \mathrm{U}(-0.5, 0.5)$.

For a given y the corresponding section $\{x : (x, y) \in B_\varphi\}$ of B_φ is either empty or an interval. The length of the interval depends on y (which is evident on the picture), therefore the distribution of Y is not uniform.

1b

$f_X(x) = \mathbf{1}_{(-0.5,0.5)}(x)$, since $X \sim \mathrm{U}(-0.5, 0.5)$.

$f_{Y|X=x}(y) = \sin\varphi \cdot \mathbf{1}_{(\frac{-0.5 - x\cos\varphi}{\sin\varphi}, \frac{0.5 - x\cos\varphi}{\sin\varphi})}(y)$, since the conditional distribution is uniform on the section.

$f_{X,Y}(x, y) = f_X(x) f_{Y|X=x}(y) = \sin\varphi \cdot \mathbf{1}_{(-0.5,0.5)}(x) \mathbf{1}_{(\frac{-0.5 - x\cos\varphi}{\sin\varphi}, \frac{0.5 - x\cos\varphi}{\sin\varphi})}(y)$. That is, $f_{X,Y}(x, y) = \sin\varphi$ for $(x, y) \in B_\varphi$.

However, $f_{X,Y} = (1/S_\varphi)\mathbf{1}_{B_\varphi}$. Therefore $S_\varphi = 1/\sin\varphi$.

1c

$\mathbb{E}\left(Y \mid X = x \right) = \frac{-x\cos\varphi}{\sin\varphi} = -x\cot\varphi$, the center of the corresponding (conditional) uniform distribution. Thus, $\mathbb{E}\left(Y \mid X \right) = -X\cot\varphi$.

1d

Given $X = x$, $x \in (-0.5, 0.5)$, the conditional distribution of $Y - \mathbb{E}\left(Y \mid X \right) = Y + X\cot\varphi$ is uniform on $(\frac{-0.5}{\sin\varphi}, \frac{0.5}{\sin\varphi})$. The conditional distribution does not depend on x, therefore the random variable is independent of X.

1e

$\mathbb{E}\, X = 0$, the center of the uniform distribution.

$\mathbb{E}\, Y = 0$, since the distribution of Y is symmetric around 0 (according to the central symmetry of B_φ). *Another way* (if you want): $\mathbb{E}\, Y = \mathbb{E}\,(\mathbb{E}\left(Y \mid X \right)) = \mathbb{E}\,(-X\cot\varphi) = 0$.

$\operatorname{Var} X = 1/12$, the variance of the uniform distribution (can be calculated by a simple integration).

By the independence (and using again the variance of a uniform distribution), $\operatorname{Var} Y = \operatorname{Var}(Y + X\cot\varphi) + \operatorname{Var}(-X\cot\varphi) = \frac{1}{12}\frac{1}{\sin^2\varphi} + \frac{1}{12}\cot^2\varphi = \frac{1}{12}\frac{1+\cos^2\varphi}{\sin^2\varphi}$.

2

2a

$B = B_1 \cap B_2 \cap \dots$ where $B_k = \{\omega \in (0,1) : \alpha_{k+1} \le 1 + \alpha_k\}$.

For each k the set B_k is a finite union of intervals (some of the 10^k intervals indexed by $\alpha_1, \dots, \alpha_k$).

Thus, each B_k is a Borel set, and therefore B is a Borel set.

2b

Yes, it can happen that $p < 1$. In particular, it happens for any two independent, identically distributed random variables X, Y, except for degenerate (constant) cases. For example, $X \sim U(0,1)$, $Y \sim U(0,1)$, independent; here $F_X = F_Y$ but $p = 1/2$.

No, it cannot happen that $p = 0$. *Proof.* Assume that $p = 0$, then $X > Y$ a.s. (almost surely), which implies $F_X \le F_Y$ everywhere. [*Warning:* it does not imply $F_X < F_Y$ everywhere.] Thus, $F_X = F_Y$, that is, X and Y are identically distributed.

A contradiction follows easily if we assume in addition that X, Y are integrable. In this case, $\mathbb{E}\, X = \mathbb{E}\, Y$ (since they are identically distributed), but on the other hand, $\mathbb{E}\, X > \mathbb{E}\, Y$ (since $X > Y$ a.s).

The general case may be reduced to the integrable case by a strictly monotone transformation of $\mathbb{R}$ into a bounded interval. We may use (for instance) the arctan function: $\mathbb{E} \arctan X = \mathbb{E} \arctan Y$ (since they are identically distributed), but on the other hand, $\mathbb{E} \arctan X > \mathbb{E} \arctan Y$ (since $\arctan X > \arctan Y$ a.s).

2c

Yes, it can happen that $\limsup A_{n^2} \ne \limsup A_n$. *Example.* Let $A_n = \emptyset$ if n is a square, otherwise $A_n = \Omega$. Then $\limsup A_{n^2} = \emptyset$, but $\limsup A_n = \Omega$. *Another example* (if you want): let A_n be independent events such that $\mathbb{P}\big(A_n \big) = 1/n$, then by Borel-Cantelli Lemma(s), $\limsup A_{n^2}$ is of probability 0, while $\limsup A_n$ is of probability 1.

No, this cannot happen for an increasing sequence. *Proof.* Any increasing sequence (of sets) converges; and any its subsequence converges to the same limit. Thus, $\limsup A_{n^2} = \lim A_{n^2} = \lim A_n = \limsup A_n$.

2d

Yes, it can happen that $f_{X,Y}$ is bounded but f_{X,Y^2} is unbounded. *Example.* Let $X \sim U(0,1)$, $Y \sim U(0,1)$, and X, Y be independent. Then $f_{X,Y}(x,y) = 1$ for $x \in (0,1)$, $y \in (0,1)$. However, $f_{Y^2}(t) = 1/(2\sqrt{t})$ for $t \in (0,1)$, therefore $f_{X,Y^2}(x,t) = 1/(2\sqrt{t})$ for $x \in (0,1)$, $t \in (0,1)$.

Yes, it can happen that $f_{X,Y}$ is unbounded but f_{X,Y^2} is bounded. *Example.* Let $X \sim U(0,1)$ again, and Y be such that $Y > 0$ a.s. and Y^2 is distrubuted uniformly on an unbounded set $B \subset (1, \infty)$ of finite measure. (For instance, $B = (1, 1 + 2^{-1}) \cup (2, 2 +$

$2^{-2}) \cup (3, 3 + 2^{-3}) \cup \dots)$ Then $f_{Y^2}(t) = \text{const}$ for $t \in B$ and $f_Y(y) = \text{const} \cdot 2y$ for y such that $y^2 \in B$. Thus f_Y is unbounded, and $f_{X,Y}(x, y) = f_X(x) f_Y(y)$ is also unbounded.

3

3a

We have $f_{X,Y} = \mathbb{E}\, f_{X,Y|\Phi}$, the expectation of the conditional density. Thus, $f_{X,Y}(x, y) = \frac{2}{\pi} \int_0^{\pi/2} \sin\varphi \cdot \mathbf{1}_{B_\varphi}(x, y)\, \mathrm{d}\varphi$.

If $\sqrt{x^2 + y^2} < 0.5$ then $\mathbf{1}_{B_\varphi}(x, y) = 1$ for all φ (since A_φ results from A_0 by rotation), and we get $f_{X,Y}(x, y) = \frac{2}{\pi} \int_0^{\pi/2} \sin\varphi\, \mathrm{d}\varphi = \frac{2}{\pi}$.

If $\sqrt{x^2 + y^2} > 0.5$ then $\mathbf{1}_{B_\varphi}(x, y) = 0$ for φ of some interval (since A_φ results from A_0 by rotation), and we get $f_{X,Y}(x, y) < \frac{2}{\pi} \int_0^{\pi/2} \sin\varphi\, \mathrm{d}\varphi = \frac{2}{\pi}$.

3b

Every given n belongs to the set with the probability $1/2$ (the integral of the density $2/\pi$ over the disk of the area $\pi/4$). The expected number of these events is equal to the sum of their probabilities, $100 \cdot 0.5 = 50$.

3c

We have an infinite sequence of independent events, each of probability $1/2$. The sum of the probabilities is infinite; by the second Borel-Cantelli Lemma, infinitely many events occur, a.s. That is, the considered set is infinite with probability 1.

3d

We have an infinite sequence of events; the probability of the n-th event (except for $n = 1$) is equal to $2/n^2$ (the integral of the density $2/\pi$ over the disk of the area π/n^2). The sum of the probabilities is finite; by the first Borel-Cantelli Lemma, only finitely many events occur, a.s. That is, the considered set is infinite with probability 0.

3e

The countable set is an infinite sample from a two-dimensional distribution; almost surely, its closure is equal to the support of the distribution. The distribution is a mixture of the uniform distributions on the diamonds B_φ. Its support is equal to the closed unbounded strip $\{(x, y) : -0.5 \le x \le 0.5\}$. (I do not prove these claims.)