Exam of 22.01.2003 — Solutions

1

1a

Random variables $Y_1, Y_2, \dots$ are not independent. Proof: $Y_2 = 2\min(X_1, X_2) \le 2X_1 = 2Y_1$; $\mathbb{P}\big(Y_2 > 2, Y_1 < 1\big) = 0$, but $\mathbb{P}\big(Y_2 > 2\big) \cdot \mathbb{P}\big(Y_1 < 1\big) > 0$.

Random variables $Y_1, Y_2, \dots$ are identically distributed. Proof: $\mathbb{P}\big(Y_n > y\big) = \mathbb{P}\big(X_1 > \frac{y}{n}, \dots, X_n > \frac{y}{n}\big) = (e^{-y/n})^n = e^{-y}$, thus $Y_n \sim \operatorname{Exp}(1)$.

1b

The joint distribution of Y_n, Y_{2n} does not depend on n. Proof: $Y_{2n} = 2n \min\big(\min(X_1, \dots, X_n), \min(X_{n+1}, \dots, X_{2n})\big) = 2\min(Y_n, Y'_n)$, where $Y'_n = n\min(X_{n+1}, \dots, X_{2n})$. We have $Y_n \sim \operatorname{Exp}(1)$, $Y'_n \sim \operatorname{Exp}(1)$ and these two random variables are independent, that is, $(Y_n, Y'_n) \sim \operatorname{Exp}(1) \otimes \operatorname{Exp}(1)$. The joint distribution of Y_n, Y'_n does not depend on n, therefore the joint distribution of Y_n, $2\min(Y_n, Y'_n)$ does not depend on n.

1c

The distribution of $Y_{2n} - Y_n$ does not depend on n (which follows from (b)), therefore $Y_{2n} - Y_n$ does not converge to 0 in distribution (indeed, $Y_{2n} - Y_n$ is not always 0). It follows that Y_n do not converge in probability (indeed, if $Y_n \to Y$ then $Y_{2n} \to Y$ and so, $Y_{2n} - Y_n \to Y - Y = 0$ in probability, therefore in distribution). It follows that Y_n do not converge in any stronger sense (a.s., in square mean). However, Y_n converge in distribution, since $Y_n \sim \operatorname{Exp}(1) \to \operatorname{Exp}(1)$.

1d

$\mathbb{P}\big(\min(X_1, \dots, X_n) \ne \min(X_1, \dots, X_{n+1})\big) = \frac{1}{n+1}$ by symmetry (indeed, $X_1, \dots, X_{n+1}$ have equal chances to be minimal), thus $\mathbb{P}\big(\frac{Y_n}{n} = \frac{Y_{n+1}}{n+1}\big) = \frac{n}{n+1} \to 1$, that is, $\mathbb{P}\big(Y_{n+1} - Y_n = \frac{1}{n}Y_n\big) \to 1$. We have $\mathbb{P}\big(|Y_{n+1} - Y_n| > \varepsilon\big) \le \mathbb{P}\big(Y_{n+1} - Y_n \ne \frac{1}{n}Y_n\big) + \mathbb{P}\big(\frac{1}{n}Y_n > \varepsilon\big) = \frac{1}{n+1} + e^{-n\varepsilon} \to 0$, which means that $Y_{n+1} - Y_n \to 0$ in probability.

In order to prove convergence in square mean, denote by A_n the event $Y_{n+1} - Y_n \ne \frac{1}{n}Y_n$, then $\mathbb{P}\big(A_n\big) \to 0$. We have $\mathbb{E}\,|Y_{n+1} - Y_n|^2 \le \mathbb{E}\big(|Y_{n+1} - Y_n|^2 \cdot \mathbf{1}_{A_n}\big) + \mathbb{E}\,|\frac{1}{n}Y_n|^2 \le \mathbb{E}\big((2Y_{n+1}^2 + 2Y_n^2) \cdot \mathbf{1}_{A_n}\big) + \frac{1}{n^2}\mathbb{E}Y_n^2 \le 4\int_{1-\mathbb{P}(A_n)}^{1} \big(Y^*(p)\big)^2\,dp + \frac{1}{n^2} \cdot \text{const} \to 0$.

2

2a

$F_{V|X=x}(v) = \mathbb{P}\left(V \le v \mid X = x \right) = \mathbb{P}\left(\min(X,Y) \le v \mid X = x \right) = \mathbb{P}\big(\min(x,Y) \le v\big)$ by independence of X, Y; so,

$$F_{V|X=x}(v) = \begin{cases} 0 & \text{for } v \in (-\infty, 0], \\ 1 - e^{-v} & \text{for } v \in [0, x), \\ 1 & \text{for } v \in [x, \infty). \end{cases}$$

2b

$\mathbb{E}\left(V \mid X = x \right) = \int_0^\infty \big(1 - F_{V|X=x}(v)\big)\, dv = \int_0^x e^{-v}\, dv = 1 - e^{-x}$.

2c

$\mathbb{E}\left(V \mid X \right) = 1 - e^{-X}$; $\mathbb{P}\left(\mathbb{E}\left(V \mid X \right) \le u \right) = \mathbb{P}\left(e^{-X} \ge 1 - u \right) = \mathbb{P}\left(X \le -\ln(1-u) \right) = 1 - e^{-(-\ln(1-u))} = 1 - (1-u) = u$ for $u \in (0,1)$; therefore $\mathbb{E}\left(V \mid X \right) \sim \mathrm{U}(0,1)$.

2d

$\mathbb{E}V = \frac{1}{2}$, since $2V \sim \mathrm{Exp}(1)$ (similarly to 1a); $\mathbb{E}\big(\mathbb{E}\left(V \mid X \right)\big) = \frac{1}{2}$, since $\mathbb{E}\left(V \mid X \right) \sim \mathrm{U}(0,1)$ by (c).

2e

The condition $V \ge v$ decomposes into $X \ge v$ and $Y \ge v$. The conditional distribution of $X - v$ given $X \ge v$ is $\mathrm{Exp}(1)$ by the memoryless property. The same for Y. Thus, the conditional joint distribution of $X - v, Y - v$ given $V \ge v$ is $\mathrm{Exp}(1) \otimes \mathrm{Exp}(1)$; it does not depend on v. Therefore the conditional distribution of $W = (X - v) - (Y - v)$ given $V \ge v$ does not depend on v.

It follows that V, W are independent. Indeed, $F_{W,-V}(w, -v) = \mathbb{P}\big(W \le w, V \ge v\big) = \mathbb{P}\left(W \le w \mid V \ge v \right) \cdot \mathbb{P}\big(V \ge v\big) = \mathbb{P}\big(W \le w\big) \cdot \mathbb{P}\big(V \ge v\big)$, thus W and $(-V)$ are independent.

Distributions of V, W are different, since W is symmetric around 0, but V is always positive.

2f

$$X = \begin{cases} V & \text{if } W < 0, \\ V + W & \text{if } W > 0; \end{cases}$$

$F_{X|V=v}(x) = \mathbb{P}\left(X \le x \,\middle|\, V = v \right) = \mathbb{P}\left(V \le x, W < 0 \,\middle|\, V = v \right) + \mathbb{P}\left(V + W \le x, W > 0 \,\middle|\, V = v \right) = \mathbb{P}\big(v \le x, W < 0 \big) + \mathbb{P}\big(v + W \le x, W > 0 \big)$ by independence of V, W (shown in (e));

$$\mathbb{P}\big(v \le x, W < 0 \big) = \begin{cases} 0 & \text{if } x \in (-\infty, v), \\ 1/2 & \text{if } x \in [v, \infty); \end{cases}$$

$\mathbb{P}\big(v + W \le x, W > 0 \big) = \mathbb{P}\big(0 < W \le x - v \big) = \mathbb{P}\big(0 < X - Y \le x - v \big) = \int_0^\infty dy\, e^{-y} \int_y^{y+x-v} dx_1\, e^{-x_1} = \int_0^\infty dy\, e^{-y} \cdot e^{-y}(1 - e^{-(x-v)}) = \frac{1}{2}(1 - e^{-(x-v)})$ for $x > v$ (and 0 otherwise); so,

$$F_{X|V=v}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, v), \\ 1 - \frac{1}{2} e^{-(x-v)} & \text{for } x \in [v, \infty). \end{cases}$$

3

3a

The probability is equal to 1. Indeed, $\mathbb{P}\big(Y_k \le n \big) \le \sum_{m=k}^{n} \mathbb{P}\big(X_m = X_{m-1} = \dots = X_{m-k+1} = 0 \big) \le n \cdot 10^{-k}$; in particular, $\mathbb{P}\big(Y_k < 9^k \big) \le (9^k - 1) \cdot 10^{-k} \le 0.9^k$; therefore $\sum_k \mathbb{P}\big(Y_k < 9^k \big) < \infty$; by the first Borel-Cantelli lemma, $Y_k \ge 9^k$ eventually.

3b

The probability is equal to 1. Indeed, we have $11^k \ge k^2 \cdot 10^k$ for large k, since the exponential growth of 1.1^k is faster than the power growth of k^2. The sequence $X_1, \dots, X_{11^k}$ contains (at least) $k \cdot 10^k$ *disjoint* sequences of k elements each. They all must differ from $0 \dots 0$ in order to get $Y_k > 11^k$. Using their independence, we have $\mathbb{P}\big(Y_k > 11^k \big) \le (1 - 10^{-k})^{k \cdot 10^k} \le \big(\exp(-10^{-k})\big)^{k \cdot 10^k} = \exp\big(-10^{-k} \cdot k \cdot 10^k\big) = e^{-k}$; therefore $\sum_k \mathbb{P}\big(Y_k > 11^k \big) < \infty$; by the first Borel-Cantelli lemma, $Y_k \le 11^k$ eventually.

3c

Yes, the limit exists almost surely. Indeed, the number '9' in (a) may be replaced with $10 - \varepsilon$ for any $\varepsilon > 0$. Also the number '11' in (b) may be replaced with $10 + \varepsilon$. Eventually, $(10 - \varepsilon)^k \le Y_k \le (10 + \varepsilon)^k$; therefore $\ln(10 - \varepsilon) \le \frac{1}{k} \ln Y_k \le \ln(10 + \varepsilon)$. It means that $\frac{1}{k} \ln Y_k \to \ln 10$ almost surely.