Exam of 07.02.2001 — Solutions

1

1a

X is a function of the random angle $\varphi \sim \mathrm{U}(0,\pi)$, namely, $X = |\cos\varphi|$ (that is, $X = \sin|\varphi - \frac{\pi}{2}|$). Thus $X \le x$ if and only if $\varphi \in [\arccos x, \pi - \arccos x]$ (that is, $|\varphi - \frac{\pi}{2}| \le \arcsin x$) for $0 < x < 1$. So,

$$F_X(x) = \begin{cases} 0 & \text{for } -\infty < x \le 0, \\ \frac{2}{\pi}\arcsin x & \text{for } 0 \le x \le 1, \\ 1 & \text{for } 1 \le x < \infty; \end{cases}$$

$$f_X(x) = F'_X(x) = \begin{cases} \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise;} \end{cases}$$

$$X^*(p) = \sin\Big(\frac{\pi}{2}p\Big) \quad \text{for } 0 < p < 1\,;$$

$$X^*\Big(\frac{1}{2}\Big) = \sin\Big(\frac{\pi}{4}\Big) = \frac{1}{\sqrt{2}}\,;$$

$$\mathbb{E}\,X = \frac{2}{\pi}\,.$$

There are many ways of calculating the expectation; here is one of them:

$$\mathbb{E}\,X = \int_0^1 X^*(p)\,dp = \int_0^1 \sin\Big(\frac{\pi}{2}p\Big)\,dp = -\frac{2}{\pi}\cos\Big(\frac{\pi}{2}p\Big)\Big|_0^1 = \frac{2}{\pi}\,.$$

It means that a random projection of a straight segment is of length, in the average, $2/\pi$ times the length of the given segment.

1b

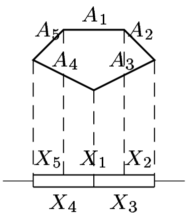
According to the given hint, we have $Y = X_1 + X_2$, where X_1 is a random projection of the vertical edge (no matter, which one), X_2 — of the horizontal edge. Item 1a is applicable both to X_1 and X_2, giving $\mathbb{E}\,X_1 = 2/\pi$ and $\mathbb{E}\,X_2 = 2/\pi$. Therefore

$$\mathbb{E}\,Y = \frac{2}{\pi} + \frac{2}{\pi} = \frac{4}{\pi}\,.$$

(Of course, X_1, X_2 are dependent, but anyway, $\mathbb{E}\,(X_1 + X_2) = \mathbb{E}\,X_1 + \mathbb{E}\,X_2$.)

1c

The projection of the polygon to a random straight line consists of projections of edges, and is twofold. Thus,



$$\mathbb{E}Y = \frac{1}{2}\big(\mathbb{E}X_1 + \dots + \mathbb{E}X_n\big) = \frac{1}{2} \cdot \frac{2}{\pi} \cdot (\text{perimeter}) = \frac{1}{\pi} \cdot (\text{perimeter}) \, .$$

1d

No, the internal perimeter cannot exceed the external perimeter. Indeed, the perimeter is π times the expected projection (which is shown in 1c for polygones, and is true for all convex bodies due to a limiting procedure). However, $\mathbb{E}Y_{\text{internal}} \le \mathbb{E}Y_{\text{external}}$, since $Y_{\text{internal}} \le Y_{\text{external}}$ always.

2

2a

First,

$$\begin{aligned} F_{X|N=n}(x) &= \mathbb{P}\,\big(\max(U_1,\dots,U_N) \le x \;\big|\; N=n\,\big) = \\ &= \mathbb{P}\,\big(\max(U_1,\dots,U_n) \le x \;\big|\; N=n\,\big) = \mathbb{P}\big(\max(U_1,\dots,U_n) \le x\big) = \\ &= \mathbb{P}\big(U_1 \le x\big) \dots \mathbb{P}\big(U_n \le x\big) = x^n \quad \text{for } x \in (0,1)\,; \end{aligned}$$

$$f_{X|N=n}(x) = F'_{X|N=n}(x) = (x^n)' = nx^{n-1} \quad \text{for } x \in (0,1)\,.$$

Second,

$$f_X(x) = \sum_n f_{X|N=n}(x) p_N(n) = \frac{1}{1000} \sum_{n=1}^{1000} nx^{n-1} = \frac{1}{1000}\big(1 + 2x + 3x^2 + \dots + 1000x^{999}\big)\,.$$

Finally,

$$p_{N|X=x}(n) = \frac{f_{X|N=n}(x) p_N(n)}{f_X(x)} = \frac{nx^{n-1}}{1 + 2x + 3x^2 + \dots + 1000x^{999}}\,.$$

2b

First,

$$p_{N|X=x}(n) = \frac{nx^{n-1}}{1 + 2x + \dots + Mx^{M-1}}\,.$$

Second,

$$1 + 2x + 3x^2 + \dots = \frac{d}{dx}(x + x^2 + x^3 + \dots) = \frac{d}{dx}\Big(\frac{x}{1-x}\Big) = \frac{1}{(1-x)^2} \quad (\text{for } |x| < 1)\,.$$

Finally,

$$\lim_{M\to\infty} p_{N|X=x}(n) = \frac{nx^{n-1}}{1/(1-x)^2} = n(1-x)^2 x^{n-1} \quad (\text{for } 0 < x < 1)\,.$$

2c

First, it is a probability distribution, since $n(1-x)^2x^{n-1} \ge 0$, and $\sum_{n=1}^{\infty} n(1-x)^2x^{n-1} = (1-x)^2 \cdot \frac{1}{(1-x)^2} = 1$.

Second (using the given hint),

$$\mathbb{E}\left(N \mid X = x \right) = \sum_{n=1}^{\infty} n \cdot p_{N|X=x}(n) = \sum_{n=1}^{\infty} n^2(1-x)^2x^{n-1} = (1-x)^2 \cdot \frac{1+x}{(1-x)^3} = \frac{1+x}{1-x}\,.$$

2d

$$\mathbb{P}\left(\frac{N}{1000} = y \mid X = 0.999 \right) = \mathbb{P}\left(N = 1000y \mid X = 0.999 \right) =$$
$$= (1000y) \cdot (1 - 0.999)^2 \cdot 0.999^{1000y-1} = \frac{1}{1000} y \Big(1 - \frac{1}{1000}\Big)^{1000y} \cdot \frac{1}{0.999} \approx \frac{1}{1000} y e^{-y}\,,$$

which means that the (conditional) distribution of $N/1000$ is close to the distribution with the density ye^{-y} (for $y > 0$). The latter distribution is Gamma(2).

3

3a

Yes, there is a chance that $S = 2001$. Indeed,

$$\mathbb{P}\left(S = 2001 \right) \ge \mathbb{P}\left(X_1 = 1, \dots, X_{2001} = 1, X_{2002} = 0, X_{2003} = 0, \dots \right) =$$
$$= \prod_{k=1}^{2001} p_k \cdot \prod_{k=2002}^{\infty} (1 - p_k) > 0\,,$$

since $p_k > 0$ and $\sum p_k < \infty$.

3b . . . .

No, both cases are impossible. Indeed, if $\sum p_k < \infty$ then $\mathbb{P}\big(S = 2001\big) > 0$ similarly to 3a, and also $\mathbb{P}\big(S = 3000\big) > 0$ for the same reason. However, if $\sum p_k = \infty$, then $S = \infty$ almost always (by the second Borel-Cantelli lemma), therefore $\mathbb{P}\big(S = 2001\big) = 0$ and $\mathbb{P}\big(S = 3000\big) = 0$.

3c . . . .

Let m_0 out of the numbers p_k are equal to 0, and m_1 out of p_k are equal to 1. Assume that $m_1 < \infty$ (otherwise $S = \infty$ almost always). We have $S = m_0 \cdot 0 + m_1 \cdot 1 + \tilde{S} = \tilde{S} + m_1$, where $\tilde{S}$ is the sum of other X_k (such that $0 < p_k < 1$). Assume for now that there are infinitely many k such that $0 < p_k < 1$.

If $\sum p_k < \infty$ then $\mathbb{P}\big(S = n\big) > 0$ for all $n = 0, 1, 2, \dots$ (similarly to 3a, 3b), therefore, $\mathbb{P}\big(S = n\big) > 0$ for all $n = m_1, m_1 + 1, m_1 + 2, \dots$ (and only these n).

If $\sum p_k = \infty$ then $\tilde{S} = \infty$ almost always (similarly to 3b), thus $S = \infty$ almost always.

Finally, if only a finite number m out of our p_k satisfy $0 < p_k < 1$, then clearly $\mathbb{P}\big(S = n\big) > 0$ for $n = m_1, m_1 + 1, \dots, m_1 + m$, and only these n.

In every case, if $\mathbb{P}\big(S = n\big) > 0$ for two integers n (say, 2001 and 3000), then it holds for all intermediate integers (say, 2345).

4

4a . . . .

$\mathbb{P}\big(X_1 X_2 < 0.05\big)$ is the area of the domain $\{(x_1, x_2) \in (0,1) \times (0,1) : x_1 x_2 < 0.05\}$ consisting of the rectangle $(0, 0.05) \times (0, 1)$ and the subgraph of the function $x_2 = \frac{0.05}{x_1}$ for $x_1 \in (0.05, 1)$. Thus,

$$\mathbb{P}\big(X_1 X_2 < 0.05\big) = 0.05 + \int_{0.05}^{1} \frac{0.05}{x}\, dx = \frac{1 + \ln 20}{20}\,.$$

4b . . . .

$$\mathbb{P}\big(X_1 \dots X_n > 0.05\big) \le \frac{\mathbb{E}\,(X_1 \dots X_n)}{0.05} = \frac{(\mathbb{E}\,X_1) \dots (\mathbb{E}\,X_n)}{1/20} = 20 \cdot 0.5^n < 0.6^n$$

for n large enough.

4c . . . .

First, $(-\ln X_1)$ is distributed Exp(1); indeed,

$$\mathbb{P}\big(-\ln X_1 \le y\big) = \mathbb{P}\big(X_1 \ge e^{-y}\big) = 1 - e^{-y}\,.$$

Therefore, $-\ln(X_1 \dots X_n) = (-\ln X_1) + \dots + (-\ln X_n)$ is distributed $\mathrm{Exp}(1) * \dots * \mathrm{Exp}(1) = \mathrm{Gamma}(1) * \dots * \mathrm{Gamma}(1) = \mathrm{Gamma}(n)$; its density is

$$f_{X_1 \dots X_n}(t) = \frac{1}{(n-1)!} t^{n-1} e^{-t} \, .$$

We have

$$\mathbb{P}\big(X_1 \dots X_n > 0.05 \big) = \mathbb{P}\big(-\ln(X_1 \dots X_n) < -\ln 0.05 \big) = \\ = \int_0^{\ln 20} \frac{1}{(n-1)!} t^{n-1} e^{-t} \, dt \le \frac{1}{(n-1)!} \int_0^{\ln 20} t^{n-1} \, dt \le \frac{(\ln 20)^n}{n!} \, ,$$

which is less than 0.001^n for n large enough.

4d ....

The inequality $e^{-1.1n} < X_1 \dots X_n < e^{-0.9n}$ may be rewritten as

$$0.9n < -\ln(X_1 \dots X_n) < 1.1n \, ,$$

that is,

$$0.9 < \frac{(-\ln X_1) + \dots + (-\ln X_n)}{n} < 1.1 \, .$$

However,

$$\frac{(-\ln X_1) + \dots + (-\ln X_n)}{n} \xrightarrow[n \to \infty]{} 1 \quad \text{almost always}$$

by the strong law of large numbers.