# Exam of 26.01.2000 — Solutions

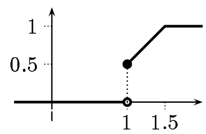
## 1

$$X = \begin{cases} 1 & \text{if } T_1 > 1, \\ 1 + T_2 & \text{if } T_1 < 1. \end{cases}$$

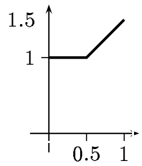
### 1a

$$\mathbb{P}\big(X = 1\big) = \mathbb{P}\big(T_1 > 1\big) = \tfrac{1}{2};$$
$$\mathbb{P}\big(1 < X < x\big) = \mathbb{P}\big(1 + T_2 < x, T_1 < 1\big) = \mathbb{P}\big(T_2 < x - 1\big)\mathbb{P}\big(T_1 < 1\big) =$$
$$= 2(x-1) \cdot \tfrac{1}{2} = x - 1 \quad \text{for } x \in (1, 1.5);$$

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1, \\ x - 0.5 & \text{for } 1 \le x \le 1.5; \\ 1 & \text{for } x \ge 1.5. \end{cases}$$




$$X^*(p) = \begin{cases} 1 & \text{for } 0 < p < 0.5; \\ 0.5 + p & \text{for } 0.5 < p < 1. \end{cases}$$




### 1b

$$x_{1/2} = X^*\big(\tfrac{1}{2}\big) = 1; \quad x_{1/4} = X^*\big(\tfrac{1}{4}\big) = 1; \quad x_{3/4} = X^*\big(\tfrac{3}{4}\big) = 1.25\,.$$

### 1c

The support: $[1, 1.5]$. Discrete, singular, and absolutely continuous components:

$$p_{\mathrm{d}} = \tfrac{1}{2}, \quad p_{\mathrm{s}} = 0, \quad p_{\mathrm{ac}} = \tfrac{1}{2};$$
$$F_{X,\mathrm{d}}(x) = \begin{cases} 0 & \text{for } x < 1, \\ 1 & \text{for } x \ge 1; \end{cases}$$
$$F_{X,\mathrm{ac}}(x) = \begin{cases} 0 & \text{for } x \le 1, \\ 2(x-1) & \text{for } 1 \le x \le 1.5, \\ 1 & \text{for } x \ge 1.5; \end{cases}$$
$$f_{X,\mathrm{ac}}(x) = 2 \cdot \mathbf{1}_{(1,1.5)}(x)\,.$$

## 1d ..........

$$\mathbb{E}X = \int_0^1 X^*(p)\,dp = \int_0^{0.5} dp + \int_{0.5}^1 (0.5+p)\,dp = 0.5 + 0.625 = 1.125\,;$$
$$\mathbb{E}X = p_{\mathrm{d}}\,\mathbb{E}\left( X \mid X \in A_X \right) + p_{\mathrm{ac}}\,\mathbb{E}\left( X \mid X \notin A_X \right) = \tfrac12 \cdot 1 + \tfrac12 \cdot 1.25 = 1.125\,.$$

(Other ways may be used.)

## 1e ..........

$$\mathbb{E}\,(1.5 - X)^{-\alpha} = p_{\mathrm{d}} \int (1.5-x)^{-\alpha}\,dF_{X,\mathrm{d}}(x) + p_{\mathrm{ac}} \int (1.5-x)^{-\alpha} f_{X,\mathrm{ac}}(x)\,dx\,;$$

the discrete part converges for all $\alpha$ (since the atom is not at 1.5); only $(1.5-x)^{-\alpha} f_{X,\mathrm{ac}}(x)$ must be integrable when $x = 1.5 - \varepsilon$, $\varepsilon \to 0+$. So, $\varepsilon^{-\alpha}$ must be integrable, which holds if and only if $\alpha < 1$.

# 3

We have two cases: $T_1 > 1$ and $T_1 < 1$.

## 3a ..........

Case $T_1 > 1$:
$X = 1$, $Y = T_1 + T_2$, $T_1 \in (1,2)$, $T_2 \in (0, 0.5)$; a singular 2-dim distribution, with the support $\{1\} \times [1, 2.5]$ (a segment on the plane).
Case $T_1 < 1$:
$X = 1 + T_2$, $Y = T_1 + T_2$, $T_1 \in (0,1)$, $T_2 \in (0, 0.5)$; an absolutely continuous 2-dim distribution with the support $\{(x,y) : x \in [1, 1.5], y \in [x-1, x]\}$ (a parallelogram).

So, $p_{\mathrm{d}} = 0$, $p_{\mathrm{s}} = 1/2$, $p_{\mathrm{ac}} = 1/2$. The support is the union of the segment and the parallelogram.

## 3b ..........

$$\begin{aligned}
&\mathbb{E}\left( XY \mid T_1 > 1 \right) = \mathbb{E}\left( Y \mid T_1 > 1 \right) = \mathbb{E}\left( T_1 + T_2 \mid T_1 > 1 \right) = \\
&\quad = \mathbb{E}\left( T_1 \mid T_1 > 1 \right) + \mathbb{E}\left( T_2 \mid T_1 > 1 \right) = 1.5 + 0.25 = 1.75\,; \\
&\mathbb{E}\left( XY \mid T_1 < 1 \right) = \mathbb{E}\left( (1+T_2)(T_1+T_2) \mid T_1 < 1 \right) = \\
&\quad = \mathbb{E}\left( T_1 \mid T_1 < 1 \right) + \mathbb{E}\left( T_2 \mid T_1 < 1 \right) + \mathbb{E}\left( T_1 T_2 \mid T_1 < 1 \right) + \mathbb{E}\left( T_2^2 \mid T_1 < 1 \right) = \\
&\quad = 0.5 + 0.25 + 0.5 \cdot 0.25 + \tfrac{1}{12} \approx 0.958\,; \quad (\text{or } \tfrac{23}{24}) \\
&\mathbb{E}(XY) = \mathbb{E}\left( XY \mid T_1 > 1 \right) \mathbb{P}\left( T_1 > 1 \right) + \mathbb{E}\left( XY \mid T_1 < 1 \right) \mathbb{P}\left( T_1 < 1 \right) \approx \\
&\quad \approx 1.75 \cdot 0.5 + 0.958 \cdot 0.5 \approx 1.354\,; \\
&\mathrm{Cov}(X,Y) \approx 1.354 - 1.125 \cdot 1.25 \approx -0.052\,. \quad (\text{or } -\tfrac{5}{96})
\end{aligned}$$

So, $X$ and $Y$ are correlated, therefore, dependent.

### 3c

$P_{X,Y,\mathrm{ac}}$ is $P_{X,Y|T_1<1}$, the uniform distribution on the parallelogram. The area of the parallelogram is $1/2$, so, the density is

$$f_{X,Y,\mathrm{ac}}(x,y) = \begin{cases} 2 & \text{if } x \in [1,1.5],\ y \in [x-1,x], \\ 0 & \text{otherwise.} \end{cases}$$

## 4

### 4a

Yes, $a_n \le a_{n+1}$, since $\max(X_1,\dots,X_n) \le \max(X_1,\dots,X_{n+1})$. The equality $a_n = a_{n+1}$ is possible, since the distribution may be degenerate, $X_1 = \mathrm{const}$.

### 4b

If $X_1$ is bounded from above, that is, $\mathbb{P}\big( X_1 \le c \big) = 1$ for some $c$, then the sequence $(a_n)$ is bounded (by $c$). Conversely, if $(a_n)$ is bounded, then $\lim_n \mathbb{E} \max(X_1,\dots,X_n) < \infty$, which implies (by monotone convergence theorem) that $\lim_n \max(X_1,\dots,X_n) < \infty$ almost sure, that is, $\sup(X_1,X_2,\dots) < \infty$ almost sure. That is possible only for bounded $X_1$. Otherwise, for every $x$ the inequality $X_n > x$ holds infinitely often.

### 4c

$$\frac{a_n}{n} = \frac{1}{n}\mathbb{E}\max(X_1,\dots,X_n) = \frac{1}{n}\bigg( -\int_{-\infty}^0 F^n(x)\,dx + \int_0^\infty (1-F^n(x))\,dx \bigg) .$$

The first term tends to 0; consider the second term. We have

$$1 - F^n(x) = \big(1-F(x)\big)\big(1 + F(x) + F^2(x) + \dots + F^{n-1}(x)\big) \le n\big(1-F(x)\big) .$$

Therefore for every $M$,

$$\frac{1}{n}\int_0^\infty \big(1-F^n(x)\big)\,dx = \underbrace{\frac{1}{n}\int_0^M \big(1-F^n(x)\big)\,dx}_{\to 0} + \frac{1}{n}\int_M^\infty \big(1-F^n(x)\big)\,dx ;$$

$$\frac{1}{n}\int_M^\infty \big(1-F^n(x)\big)\,dx \le \int_M^\infty \big(1-F(x)\big)\,dx \xrightarrow[M\to\infty]{} 0 ,$$

since

$$\int_M^\infty \big(1-F(x)\big)\,dx = \int_0^\infty \big(1-F(x)\big)\,dx - \underbrace{\int_0^M \big(1-F(x)\big)\,dx}_{\to \int_0^\infty (1-F(x))\,dx} .$$

*A more elegant solution found by a student on the exam:*

Denote $Y_n = \max(X_1, \dots, X_n)$, then $F_{Y_n}(\cdot) = F_{X_1}^n(\cdot)$, thus $Y_n^*(p) = X_1^*(\sqrt[n]{p})$ and $a_n = \mathbb{E} Y_n = \int_0^1 Y_n^*(p)\,dp = \int_0^1 X_1^*(\sqrt[n]{p})\,dp = \int_0^1 X_1^*(u) \cdot n u^{n-1}\,du$; so, $\frac{1}{n} a_n = \int_0^1 X_1^*(u) u^{n-1}\,du \to 0$ by the monotone convergence theorem.

## 5

### 5a

$$F_{X|V=1}(x) = F_{U|V=1}(x) = F_U(x) = \int_0^x f_U(u)\,du\,;$$
$$F_{X|V=2}(x) = F_{2U|V=2}(x) = F_{2U}(x) = F_U\Big(\frac{x}{2}\Big) = \int_0^{x/2} f_U(u)\,du = \frac12 \int_0^x f_U\Big(\frac{u}{2}\Big)\,du\,;$$
$$f_{X|V=1}(x) = f_U(x)\,;$$
$$f_{X|V=2}(x) = \frac12 f_U\Big(\frac{x}{2}\Big)\,.$$

### 5b

$$\mathbb{P}\,\big(\,V=1 \,\big|\, X=x\,\big) = \frac{f_{X|V=1}(x)\mathbb{P}\big(\,V=1\,\big)}{f_X(x)} = \frac{f_U(x)\cdot\frac12}{f_U(x)\cdot\frac12 + \frac12 f_U(\frac{x}{2})\cdot\frac12} = \frac{2f_U(x)}{2f_U(x)+f_U(\frac{x}{2})}\,;$$
$$\mathbb{P}\,\big(\,V=2 \,\big|\, X=x\,\big) = \frac{f_U(\frac{x}{2})}{2f_U(x)+f_U(\frac{x}{2})}\,.$$

### 5c

$$\mathbb{E}\,\big(\,Y \,\big|\, X=x\,\big) = \mathbb{E}\,\big(\,\tfrac{3-V}{V}X \,\big|\, X=x\,\big) = x\,\mathbb{E}\,\big(\,\tfrac{3-V}{V} \,\big|\, X=x\,\big) =$$
$$= x\big(2\mathbb{P}\,\big(\,V=1 \,\big|\, X=x\,\big) + \tfrac12\mathbb{P}\,\big(\,V=2 \,\big|\, X=x\,\big)\big) =$$
$$= x\bigg(2\frac{2f_U(x)}{2f_U(x)+f_U(\frac{x}{2})} + \frac12\bigg(1 - \frac{2f_U(x)}{2f_U(x)+f_U(\frac{x}{2})}\bigg)\bigg) = x\bigg(\frac32\frac{2f_U(x)}{2f_U(x)+f_U(\frac{x}{2})} + \frac12\bigg)\,.$$

### 5d

$\mathbb{E}\,\big(\,Y \,\big|\, X\,\big) > X$ means $\frac32 \frac{2f_U(x)}{2f_U(x)+f_U(\frac{x}{2})} + \frac12 > 1$, that is, $6f_U(x) > 2f_U(x) + f_U(\frac{x}{2})$, that is, $4f_U(x) > f_U(\frac{x}{2})$. That must hold for almost all $x$.

Yes, such $f_U$ exist. For example, $f_U(x) = \text{const} \cdot \min(1, 1/x^\alpha)$ for $\alpha \in (1,2)$.

If $X, Y$ are integrable, then $\mathbb{E}\,\big(\,Y \,\big|\, X\,\big) > X$ implies $\mathbb{E}Y > \mathbb{E}X$, since $\mathbb{E}\big(\mathbb{E}\,\big(\,Y \,\big|\, X\,\big)\big) = \mathbb{E}Y$. However, we have $\mathbb{E}(X) = \mathbb{E}(Y)$ (these are identically distributed), so, $\mathbb{E}\,\big(\,Y \,\big|\, X\,\big) > X$ implies that $\mathbb{E}(X) = \infty = \mathbb{E}Y$.