# Exam of 10.10.1999 — Solutions

## 1

### 1a

Both $Y$ and $Z$ are *increasing* functions of $X$, therefore the same relations hold for their quantile functions: $Y^*(p) = (a-b)\mathbf{1}_{(a,\infty)}\big(X^*(p)\big)$, $Z^*(p) = \big(X^*(p)-a\big)_+$. Also,

$$F_Y(y) = \begin{cases} 0 & \text{for } y<0, \\ F_X(a) & \text{for } 0 \le y < a-b, \\ 1 & \text{for } y \ge a-b; \end{cases} \qquad F_Z(z) = \begin{cases} 0 & \text{for } z<0, \\ F_X(a+z) & \text{for } z \ge 0. \end{cases}$$

### 1b

$\mathbb{E}\,Y = (a-b)\mathbb{P}\left( X > a \right) = (a-b)\big(1-F_X(a)\big)$;
$\mathbb{E}\,Z = \int_0^1 \big(X^*(p)-a\big)_+\,dp$;
$\mathbb{E}\,Z = \int_0^\infty \big(1-F_Z(z)\big)\,dz = \int_0^\infty \big(1-F_X(a+z)\big)\,dz = \int_a^\infty \big(1-F_X(x)\big)\,dx$.

### 1c

$\mathbb{E}\,Y = (a-b)e^{-a/c}$; $\frac{d}{da}\mathbb{E}\,Y = e^{-a/c} - (a-b)\frac1c e^{-a/c} = 0$; $1-\frac{a-b}{c} = 0$; $a-b=c$; so, $a=b+c$. For the optimal $a$,
$\mathbb{E}\,Y = ce^{-(b+c)/c}$; $\quad \mathbb{E}\,Z = \int_a^\infty e^{-x/c}\,dx = ce^{-a/c} = ce^{-(b+c)/c}$; $\quad$ we see that $\mathbb{E}\,Y = \mathbb{E}\,Z$.

### 1d

$Y+Z = (a-b)\mathbf{1}_{(a,\infty)}(X) + (X-a)\mathbf{1}_{(a,\infty)}(X) = (X-b)\mathbf{1}_{(a,\infty)}(X)$; for $u>0$ we have $\mathbb{P}\left( Y+Z \le u \right) = \mathbb{P}\left( X \le a \right) + \mathbb{P}\left( a < X \le b+u \right) = \begin{cases} \mathbb{P}\left( X \le a \right) & \text{for } u \le a-b, \\ \mathbb{P}\left( X \le b+u \right) & \text{for } u > a-b; \end{cases}$ $\quad (a-b=c)$

$$F_{Y+Z}(u) = \begin{cases} 0 & \text{for } u<0, \\ 1-e^{-(b+c)/c} & \text{for } 0 \le u < c, \\ 1-e^{-(b+u)/c} & \text{for } u \ge c. \end{cases}$$

The support: $\{0\}\cup[c,\infty)$. Discrete, singular and absolutely continuous components: $p_{\mathrm{d}} = 1-e^{-(b+c)/c}$, $p_{\mathrm{s}} = 0$, $p_{\mathrm{ac}} = e^{-(b+c)/c}$, $F_{Y+Z,\mathrm{d}}(u) = \begin{cases} 0 & \text{for } u<0, \\ 1 & \text{for } u \ge 0; \end{cases}$
$F_{Y+Z,\mathrm{ac}}(u) = \dfrac{e^{-(b+c)/c} - e^{-(b+u)/c}}{e^{-(b+c)/c}} = 1 - e^{-(u-c)/c}$ for $u \ge c$ (and $0$ otherwise).
$f_{Y+Z,\mathrm{ac}}(u) = \frac1c e^{-(u-c)/c}$ for $u>c$ (and $0$ otherwise).

## 1e

Yes, $(Y+Z)^* = Y^* + Z^*$, since $Y$ and $Z$ both are *increasing* functions of $X$.

## 1f

If $X \le b+c$ then $Y=0$ and $Z=0$. If $X > b+c$ then $Y=c$ and $Z = X-(b+c)$. The distribution of $X$ is $\operatorname{Exp}(c)$, and the conditional distribution of $X-(b+c)$ given that $X > b+c$ is also $\operatorname{Exp}(c)$. So, $P_{Y,Z}$ has an atom, and a singular part:

$$P_{Y,Z} = \underbrace{\big(1-e^{-(b+c)/c}\big)}_{p_{\mathrm{d}}} \underbrace{\delta_0 \otimes \delta_0}_{P_{Y,Z,\mathrm{d}}} + \underbrace{e^{-(b+c)/c}}_{p_{\mathrm{s}}} \underbrace{\delta_c \otimes \operatorname{Exp}(c)}_{P_{Y,Z,\mathrm{s}}} .$$

The support is $\{(0,0)\} \cup \{c\} \times [0,\infty)$.

# 3

## 3a

$Y = a_1 \mathbf{1}_{(a_1,\infty)}(X_1) + a_2 \mathbf{1}_{(-\infty,a_1]}(X_1) \mathbf{1}_{(a_2,\infty)}(X_2)$;
$Z_1 = (X_1 - a_1)_+$;
$Z_2 = \mathbf{1}_{(-\infty,a_1]}(X_1) \cdot (X_2 - a_2)_+$.

## 3b

$\mathbb{E}\, Y = a_1 \mathbb{P}\big( X_1 > a_1 \big) + a_2 \mathbb{P}\big( X_1 \le a_1 \big) \mathbb{P}\big( X_2 > a_2 \big) = a_1\big(1-F(a_1)\big) + a_2 F(a_1)\big(1-F(a_2)\big)$; equivalent forms: $\dots = a_1 - (a_1-a_2)F(a_1) - a_2 F(a_1)F(a_2) = a_2\big(1-F(a_1)F(a_2)\big) + (a_1 - a_2)\big(1-F(a_1)\big)$.

## 3c

We need $\max\limits_{a_1,a_2}\big(a_1 e^{-a_1/c} + a_2(1-e^{-a_1/c})e^{-a_2/c}\big)$. Maximize in $a_2$: $\frac{d}{da_2}\big(a_2 e^{-a_2/c}\big) = 0$; $1 - a_2 \cdot \frac1c = 0$; so, $a_2 = c$. Now we need $\max\limits_{a_1}\big(a_1 e^{-a_1/c} + c(1-e^{-a_1/c})e^{-1}\big) = ce^{-1} + \max\limits_{a_1}\big((a_1 - ce^{-1})e^{-a_1/c}\big)$. Differentiation gives $1 - (a_1 - ce^{-1})\frac1c = 0$; $a_1 - ce^{-1} = c$; $a_1 = c(1+e^{-1})$. So,
$a_1 = c(1+e^{-1}), \quad a_2 = c.$

## 3d

$\mathbb{E}\, Y = a_1 e^{-a_1/c} + a_2(1-e^{-a_1/c})e^{-a_2/c} = c(1+e^{-1})e^{-(1+e^{-1})} + c\big(1-e^{-(1+e^{-1})}\big)e^{-1} = c\big(e^{-(1+e^{-1})} + e^{-1}\big) = ce^{-1}(e^{-1/e}+1)$.
$\mathbb{E}\, Z_1 = \mathbb{E}\,(X_1 - a_1)_+ = \int_{a_1}^\infty \big(1-F(x)\big)\,dx = \int_{a_1}^\infty e^{-x/c}\,dx = ce^{-a_1/c} = ce^{-(1+e^{-1})}$;

$\mathbb{E} Z_2 = \mathbb{E}\big(\mathbf{1}_{(-\infty,a_1]}(X_1)\cdot(X_2-a_2)_+\big) = \mathbb{E}\big(\mathbf{1}_{(-\infty,a_1]}(X_1)\big)\cdot\mathbb{E}(X_2-a_2)_+ = F(a_1)\int_{a_2}^{\infty}\big(1-F(x)\big)\,dx = (1-e^{-a_1/c})ce^{-a_2/c} = c\big(1-e^{-(1+e^{-1})}\big)e^{-1}$.

## 3e ..........................................................

No, $(Z_1+Z_2)^* \ne Z_1^* + Z_2^*$. Indeed, $Z_1^*(p) = 0$ for $p < \mathbb{P}\big(Z_1 = 0\big)$; also, $Z_2^*(p) = 0$ for $p < \mathbb{P}\big(Z_2 = 0\big)$; therefore $Z_1^*(p) + Z_2^*(p) = 0$ for $p < \min\big(\mathbb{P}\big(Z_1=0\big), \mathbb{P}\big(Z_2=0\big)\big)$. However, $\mathbb{P}\big(Z_1+Z_2=0\big) < \min\big(\mathbb{P}\big(Z_1=0\big), \mathbb{P}\big(Z_2=0\big)\big)$.

# 4

## 4a ..........................................................

$\mathbb{P}\big(N=n\big) = p(1-p)^{n-1}$ for $n = 1,2,\dots$

$\mathbb{P}\big(Y>y \,\big|\, N=n\big) = \mathbb{P}\big(\min(X_1,\dots,X_N) > y \,\big|\, N=n\big) = \mathbb{P}\big(\min(X_1,\dots,X_n) > y \,\big|\, N=n\big) = \mathbb{P}\big(\min(X_1,\dots,X_n) > y\big) = \mathbb{P}\big(X_1>y,\dots,X_n>y\big) = \mathbb{P}\big(X_1>y\big)\cdot\ldots\cdot\mathbb{P}\big(X_n>y\big) = \big(1-F(y)\big)^n$, therefore $F_{Y|N=n}(y) = 1-\big(1-F(y)\big)^n$.

## 4b ..........................................................

$$F_Y(y) = \mathbb{E} F_{Y|N}(y) = \mathbb{E}\big(1-(1-F(y))^N\big) = 1-\sum_n\big(1-F(y)\big)^n\mathbb{P}\big(N=n\big) =$$
$$= 1-\sum_{n=1}^{\infty}\big(1-F(y)\big)^n p(1-p)^{n-1} = 1-p\big(1-F(y)\big)\cdot\frac{1}{1-(1-F(y))(1-p)} =$$
$$= 1-\frac{p(1-F(y))}{p+F(y)-pF(y)} = \frac{F(y)}{p+F(y)-pF(y)} = \frac{F(y)}{(1-p)F(y)+p};$$
$$\mathbb{P}\big(N=n \,\big|\, Y=y\big) = \frac{f_{Y|N=n}(y)}{f_Y(y)}\mathbb{P}\big(N=n\big) = \frac{F'_{Y|N=n}(y)}{F'_Y(y)}\mathbb{P}\big(N=n\big) =$$
$$= \frac{n(1-F(y))^{n-1}F'(y)}{\frac{(p+F(y)-pF(y))F'(y)-F(y)F'(y)(1-p)}{(p+F(y)-pF(y))^2}}\cdot p(1-p)^{n-1} = \frac{p(1-p)^{n-1}\cdot n(1-F(y))^{n-1}\cdot(p+F(y)-pF(y))^2}{p+F(y)-pF(y)-(1-p)F(y)} =$$
$$= (1-p)^{n-1}n\big(1-F(y)\big)^{n-1}\cdot\big(p+F(y)-pF(y)\big)^2 = n\big((1-p)(1-F(y))\big)^{n-1}((1-p)F(y)+p)^2.$$

## 4c ..........................................................

$\mathbb{E}\big(N \,\big|\, Y=y\big) = \sum_n n\cdot\mathbb{P}\big(N=n \,\big|\, Y=y\big) = \big((1-p)F(y)+p\big)^2\sum_n n^2\big((1-p)(1-F(y))\big)^{n-1}$. The sum may be calculated by differentiating the result of (4b), as follows: $1 = \sum_n \mathbb{P}\big(N=n \,\big|\, Y=y\big) = ((1-p)F(y)+p)^2\sum_n n\big((1-p)(1-F(y))\big)^{n-1}$; $\sum_n n\big((1-p)(1-F(y))\big)^{n-1} = ((1-p)F(y)+p)^{-2}$; $\sum_n n\big((1-p)(1-F(y))\big)^n = \dfrac{(1-p)(1-F(y))}{((1-p)F(y)+p)^2} = \dfrac{1-((1-p)F(y)+p)}{((1-p)F(y)+p)^2} = ((1-p)F(y)+p)^{-2} - ((1-p)F(y)+p)^{-1}$; now differentiate it in $p$: $-(1-F(y))\sum_n n^2\big((1-p)(1-F(y))\big)^{n-1} = -2(1-F(y))((1-p)F(y)+p)^{-3} + (1-F(y))((1-p)F(y)+p)^{-2}$, that is,

$\sum_n n^2 \big((1-p)(1-F(y))\big)^{n-1} = \frac{2}{((1-p)F(y)+p)^3} - \frac{1}{((1-p)F(y)+p)^2}$. Therefore, $\mathbb{E}\,\big(\, N \,\big|\, Y = y \,\big) = ((1-p)F(y)+p)^2 \cdot \sum_n \dots = \frac{2}{(1-p)F(y)+p} - 1$, that is, $\mathbb{E}\,\big(\, N \,\big|\, Y \,\big) = \frac{2}{(1-p)F(Y)+p} - 1$.

### 4d

Combine results of (4b) and (4c): $\mathbb{E}\big(\mathbb{E}\,\big(\, N \,\big|\, Y \,\big)\big) = \mathbb{E}\left(\frac{2}{(1-p)F(Y)=p} - 1\right) = 2\mathbb{E}\left(\frac{1}{(1-p)F(Y)+p}\right) - 1 = 2\int \frac{1}{(1-p)F(y)+p}\, dF_Y(y) - 1 = 2\int \frac{1}{(1-p)F(y)+p}\, d\frac{F(y)}{(1-p)F(y)+p} - 1$. Change the variable: $z = \frac{F(y)}{(1-p)F(y)+p}$; $z(1-p)F(y) + zp - F(y) = 0$; $zp = (1-z+zp)F(y)$; $F(y) = \frac{zp}{1-z+zp}$; $(1-p)F(y) + p = \frac{zp - zp^2 + p - zp + zp^2}{1-z+zp} = \frac{p}{1-z+zp}$. Therefore $\mathbb{E}\big(\mathbb{E}\,\big(\, N \,\big|\, Y \,\big)\big) = 2\int_0^1 \frac{1-z+zp}{p}\, dz - 1 = \frac{2}{p}\left(1 - \frac{1}{2} + \frac{p}{2}\right) - 1 = \frac{1}{p}$. Also $\mathbb{E}N = \frac{1}{p}$, since $N \sim \mathrm{Geom}(p)$. So, $\mathbb{E}\big(\mathbb{E}\,\big(\, N \,\big|\, Y \,\big)\big) = \mathbb{E}N$.

## 5

### 5a

No. Clearly, $\sum_n \mathbb{P}\big(X_n > 5\big) = \infty$ (since terms do not tend to 0, which is necessary ... ); by the second Borel-Cantelli lemma, $X_n > 5$ for infinitely many $n$, almost surely.

### 5b

No. $\sum_n \mathbb{P}\big(|X_n| < \frac{1}{n}\big) = \infty$, since $\mathbb{P}\big(|X_n| < \frac{1}{n}\big) \sim 2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{n}$; the rest is like (5a).

### 5c

Yes. $\sum_n \mathbb{P}\big(|X_n| < \frac{1}{n^2}\big) < \infty$, since $\mathbb{P}\big(|X_n| < \frac{1}{n^2}\big) \sim 2 \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{n^2}$. By the first Borel-Cantelli lemma, $|X_n| < \frac{1}{n^2}$ only for finitely many $n$, almost surely.

### 5d

Yes. $\mathbb{P}\big(\sqrt{X_n^2 + X_{n+1}^2} < \frac{1}{n}\big) \le \mathbb{P}\big(\max(|X_n|, |X_{n+1}|) < \frac{1}{n}\big) = \big(\mathbb{P}\big(|X_1| < \frac{1}{n}\big)\big)^2 \sim \frac{\text{const}}{n^2}$, therefore the series converges; the rest is like (5c). (Note: the events are dependent, which does not invalidate the argument.)

**5e** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Yes. $\mathbb{P}\left(\sqrt{X_n^2 + X_{n+1}^2 + \dots + X_{2n}^2} < 5\right) \le \mathbb{P}\left(\max(|X_n|, |X_{n+1}|, \dots, |X_{2n}|) < 5\right) = \left(\mathbb{P}\left(|X_1| < 5\right)\right)^{n+1}$, therefore the series converges; the rest is like (5d).