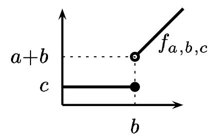
8 Stickiness as a nonclassical noise

8a The sticky factorization

It was shown in 7c that the discrete-time sticky flow has a scaling limit, described by random maps

$$\xi_{s,t} : \Omega \to G_3\,,$$
$$\xi_{s,t} = f_{a(s,t),b(s,t),c(s,t)}\,;$$



and for any $r < s < t$,

$$\xi_{r,s} \text{ and } \xi_{s,t} \text{ are independent,}$$
$$\xi_{s,t} \circ \xi_{r,s} = \xi_{r,t} \text{ almost surely}$$

(recall 7c). Moreover, $\xi_{t_1,t_2}, \xi_{t_2,t_3}, \dots, \xi_{t_{n-1},t_n}$ are independent whenever $t_1 < \dots < t_n$. The distribution of $\xi_{s,t}$ is a probability measure μ_{t-s} on G_3. The three-dimensional random process

$$t \mapsto \big(a(0,t), b(0,t), c(0,t)\big)$$

is continuous (almost surely); in fact, it satisfies the same Hölder condition as the usual Brownian motion. Of course, the same holds for $a(s,t), b(s,t), c(s,t)$ for any *fixed* s. (Note that I did not claim continuity in s.)

Consider the sub-σ-field $\mathcal{F}_{s,t}$ generated by $\xi_{u,v}$ for all $(u,v) \subset (s,t)$; that is, by all $a(u,v)$, $b(u,v)$ and $c(u,v)$.[1] Let $r < s < t$. Clearly, $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$ are independent. Also, $\xi_{r,t} = \xi_{s,t} \circ \xi_{r,s}$ is measurable w.r.t. the sub-σ-field generated by $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$. The same holds for $\xi_{u,v}$. Thus,

$$\mathcal{F}_{r,t} = \mathcal{F}_{r,s} \otimes \mathcal{F} s,t\,,$$

which means that we have a factorization (as defined in 5c); it may be called *the sticky factorization.*

The process $a(\cdot,\cdot)$ in itself generates another factorization, just the Brownian factorization. We have a canonical morphism (recall 5d) from the sticky factorization to the Brownian factorization (according to the canonical homomorphism of semigroups, $G_3 \to G_1$).

The two-dimensional process $\big(a(\cdot,\cdot), b(\cdot,\cdot)\big)$ also generates a factorization, but it is the same Brownian factorization, generated by $a(\cdot,\cdot)$ alone (think, why). In contrast, the process $c(\cdot,\cdot)$ does not fit into the Brownian factorization (recall 7c). The sticky factorization is not equal to its Brownian sub-factorization. The canonical morphism is not an isomorphism. However, it does not mean that the two factorizations are nonisomorphic. Or does it?

8a1 Exercise. Every morphism from a Brownian factorization to a Brownian factorization is an isomorphism.

Prove it.

Hint. Recall (4d3); roughly, $dB^*(t) = \varphi(t)\,dB(t)$, $\varphi(t) = \pm 1$, thus $dB(t) = \varphi(t)\,dB^*(t)$; however, is it a proof?

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[1] Any continuous (or Borel) function of the sample path on (s,t) is an $\mathcal{F}_{s,t}$-measurable random variable (recall 7c, 6b); nothing is missing from $\mathcal{F}_{s,t}$.

8a2 Exercise. The sticky factorization is not isomorphic to the Brownian factorization.

Prove it.

Hint: assume the contrary, and use 8a1.

Maybe, the sticky factorization is isomorphic to the two-dimensional Brownian factorization? Or maybe, to a combination of (necessarily independent) Brownian and Poisson factorizations? The key to answer such questions is given by so-called decomposable processes.

8b Decomposable processes in general

8b1 Definition. Let $(\mathcal{F}_{s,t})_{s<t}$ be a factorization of a probability space $(\Omega, \mathcal{F}, P)$. A family $(X_{s,t})_{s<t}$ of random variables $X_{s,t} : \Omega \to \mathbb{R}$ is called a *decomposable process,* if

$$X_{s,t} \text{ is measurable w.r.t. } \mathcal{F}_{s,t} \,;$$
$$X_{r,s} + X_{s,t} = X_{r,t}$$

whenever $r < s < t$. If in addition

$$\mathbb{E} X_{s,t}^2 < \infty \,, \quad \mathbb{E} X_{s,t} = 0$$

for all $s < t$, we call $(X_{s,t})_{s<t}$ an L_2^0-decomposable process.

We are already acquainted with that notion; see 5c5–5c7, see also 4c, 4d, 4e. Especially, all L_2^0-decomposable processes over the Brownian factorization are of the form $X_{s,t} = \int_s^t \varphi(u) \, dB(u)$, $\varphi \in L_2(\mathbb{R})$.

For every L_2^0-decomposable process (over any factorization),

$$\|X_{r,t}\|^2 = \|X_{r,s}\|^2 + \|X_{s,t}\|^2 \,;$$
$$\mathbb{E} \big(X_{r,t} \,\big|\, \mathcal{F}_{r,s} \big) = X_{r,s} \,;$$
$$X_{r,t} = \mathbb{E} \big(X_{r,t} \,\big|\, \mathcal{F}_{r,s} \big) + \mathbb{E} \big(X_{r,t} \,\big|\, \mathcal{F}_{s,t} \big)$$

whenever $r < s < t$ (think, why).

We introduce the first chaos (for an arbitrary factorization):

$$H_1(s,t) = \{ X_{s,t} : X \text{ is a } L_2^0\text{-decomposable process} \} \,.$$

8b2 Exercise. $X \in H_1(s,t)$ if and only if

$$X = \mathbb{E} \big(X \,\big|\, \mathcal{F}_{s,u} \big) + \mathbb{E} \big(X \,\big|\, \mathcal{F}_{u,t} \big) \quad \text{for all } u \in (s,t) \,.$$

Prove it.

Hint (the 'if' part): for $s < u < v < t$, the random variable

$$\begin{aligned} X_{u,v} &= \mathbb{E} \big(X \,\big|\, \mathcal{F}_{s,v} \big) + \mathbb{E} \big(X \,\big|\, \mathcal{F}_{u,t} \big) - X = \\ &= \mathbb{E} \big(X \,\big|\, \mathcal{F}_{s,v} \big) - \mathbb{E} \big(X \,\big|\, \mathcal{F}_{s,u} \big) = \mathbb{E} \big(X \,\big|\, \mathcal{F}_{u,t} \big) - \mathbb{E} \big(X \,\big|\, \mathcal{F}_{v,t} \big) \end{aligned}$$

is both $\mathcal{F}_{s,v}$-measurable and $\mathcal{F}_{u,t}$-measurable, therefore, $\mathcal{F}_{u,v}$-measurable (treat it as a function on $\Omega_{s,u} \times \Omega_{u,v} \times \Omega_{v,t}$).

8b3 Exercise. Each $H_1(s,t)$ is a (closed linear) subspace of $L_2(\Omega)$, and

$$H_1(r,t) = H_1(r,s) \oplus H_1(s,t) \quad \text{for } r < s < t\,.$$

Prove it.

8b4 Definition. A factorization $(\mathcal{F}_{s,t})$ is *continuous,* if for every $X \in L_2(\Omega)$, the map

$$(s,t) \mapsto \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{s,t} \right)$$

is continuous from $\{(s,t) \in \mathbb{R}^2 : s \le t\}$ to $L_2(\Omega)$.

8b5 Exercise. The Brownian factorization is continuous.
Prove it.
Hint: recall (3c7).

8b6 Exercise. Let $(\mathcal{F}_{s,t})_{s<t}$ be a continuous factorization. Then for every $X \in L_2(\Omega)$ such that $\mathbb{E}X = 0$, the orthogonal projection of X to $H_1(0,1)$ is equal to

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{(k-1)2^{-n},k2^{-n}} \right)$$

(the limit exists in $L_2(\Omega)$).
Prove it.
Hint. For each n we have the orthogonal projection to the subspace

$$\{Y : \forall k \;\; Y = \mathbb{E}\left(Y \,\middle|\, \mathcal{F}_{0,k2^{-n}} \right) + \mathbb{E}\left(Y \,\middle|\, \mathcal{F}_{k2^{-n},1} \right)$$

(recall 8b2).

8b7 Exercise. The following condition is sufficient for a factorization $(\mathcal{F}_{s,t})_{s<t}$ to be continuous:

$$\mathbb{E}\left(X \,\middle|\, \mathcal{F}_{s+\varepsilon,t-\varepsilon} \right) \xrightarrow[\varepsilon\to 0+]{} \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{s,t} \right) \quad \text{for all } s < t\,.$$

Prove it.
Hint (for $s = 0$, $t = \infty$). We have: $\cup_{\varepsilon>0} L_2(\mathcal{F}_{\varepsilon,\infty})$ is dense in $L_2(\mathcal{F}_{0,\infty})$. Assume that $\cap_{\varepsilon>0} L_2(\mathcal{F}_{-\infty,\varepsilon}) \ne L_2(\mathcal{F}_{-\infty,0})$. Take $X \in \cap_{\varepsilon>0} L_2(\mathcal{F}_{-\infty,\varepsilon})$ orthogonal to $L_2(\mathcal{F}_{-\infty,0})$. Then X is orthogonal to YZ for all $Y \in L_2(\mathcal{F}_{-\infty,0})$, $Z \in L_2(\mathcal{F}_{\varepsilon,\infty})$.

8c Decomposable processes over the sticky factorization

Clearly, $\int \varphi(t)\, da(0,t)$ is a decomposable process (over the sticky factorization) for any $\varphi \in L_2(\mathbb{R})$. The question is, what about other decomposable processes. In other words: does the first sticky chaos $H_1(s,t)$ contain something in addition to the first Brownian chaos?

The approach of 4c–4d does not work here, since we have nothing like (3c7) for the sticky case. Also the approach of 4f does not help, since we cannot write something multiplicative, like $\exp(i\lambda\xi_{s,t})$; the obstacle is noncommutativity of the semigroup G_3.

8c1 Exercise. The sticky factorization is continuous.
Prove it.
Hint: use 8b7.

A general form of an element of $H_1(0,1)$ is given by 8b6, where X runs over $L_2(\Omega)$. We may take a dense set of such X, thus getting a set dense in $H_1(0,1)$. In particular, we may take

$$X = \varphi(\xi_{t_0,t_1}, \dots, \xi_{t_{n-1},t_n})$$

for arbitrary n, arbitrary $t_0 < \dots < t_n$ and arbitrary bounded continuous $\varphi : \mathbb{R}^n \to \mathbb{R}$. Moreover, we may take

$$X = \varphi_1(\xi_{t_0,t_1}) \cdot \ldots \cdot \varphi_n(\xi_{t_{n-1},t_n})\,;$$

these are not dense, but their linear combinations are dense. We get

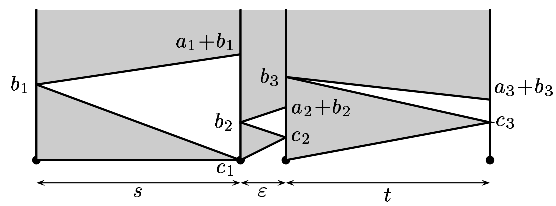
$$\mathbb{E}\left(X \,\middle|\, \mathcal{F}_{t_0,t_1} \right) = \text{const} \cdot \varphi_1(\xi_{t_0,t_1})$$

(think, why). The projection of X to $H_1(t_0,t_1)$ is a linear combination of n terms $\mathbb{E}\,\varphi_k(\xi_{t_{k-1},t_k})$. So, what we really need, is to calculate the limit 8b6 for such a case:

$$X = \varphi(\xi_{0,1})\,, \quad \mathbb{E} X = 0\,, \quad \mathbb{P}\big(-1 \le X \le 1 \big) = 1\,. \tag{8c2}$$

Denoting for convenience $s = (k-1)2^{-n}$, $\varepsilon = 2^{-n}$, $t = 1 - k2^{-n}$, we consider

$$\mathbb{E}\left(\varphi(\xi_{0,1}) \,\middle|\, \mathcal{F}_{s,s+\varepsilon} \right) = \mathbb{E}\left(\varphi(\xi_{s+\varepsilon,1} \circ \xi_{s,s+\varepsilon} \circ \xi_{0,s}) \,\middle|\, \mathcal{F}_{s,s+\varepsilon} \right) =$$
$$= \iint \varphi(f_{a_3,b_3,c_3} \circ f_{a_2,b_2,c_2} \circ f_{a_1,b_1,c_1})\, d\mu_s(a_1,b_1,c_1) d\mu_t(a_3,b_3,c_3) = \alpha(a_2,b_2,c_2)$$



(let us denote it α), where μ_s is the joint distribution of $a(0,s), b(0,s), c(0,s)$; and a_2, b_2, c_2 stand for $a(s,s+\varepsilon), b(s,s+\varepsilon), c(s,s+\varepsilon)$. Clearly,

$$f_{a_3,b_3,c_3} \circ f_{a_2,b_2,c_2} \circ f_{a_1,b_1,c_1} = f_{a_3,b_3,c_3} \circ f_{a_2,b_2,0} \circ f_{a_1,b_1,c_1} \quad \text{unless } b_3 < c_2\,.$$

We get

$$|\alpha(a_2,b_2,c_2) - \alpha(a_2,b_2,0)| \le 2\mathbb{P}\left(b_3 < c_2 \,\middle|\, c_2 \right) = 2\left(2\Phi\Big(\frac{c_2}{\sqrt{t}}\Big) - 1\right) \le 2\sqrt{\frac{2}{\pi}}\frac{c_2}{\sqrt{t}}$$

(recall 6c10); here Φ is the cumulative distribution function of the normal distribution $\mathrm{N}(0,1)$.

Given a_2, b_2, the conditional distribution of c_2 is the same as of $(a_2+b_2-\eta)^+$, $\eta \sim \operatorname{Exp}(1)$ (recall 7c). Thus,

$$\mathbb{E}\Big(|\alpha(a_2,b_2,c_2) - \alpha(a_2,b_2,0)|^2 \,\Big|\, a_2,b_2 \Big) \le \mathbb{E}\Big(\frac{8}{\pi} \frac{\big((a_2+b_2-\eta)^+\big)^2}{t} \,\Big|\, a_2,b_2 \Big) = \\ = \int_0^{a_2+b_2} \frac{8}{\pi} \frac{(a_2+b_2-u)^2}{t} e^{-u}\,du \le \frac{8}{3\pi t}(a_2+b_2)^3 \, ,$$

and

$$\text{(8c3)} \quad \mathbb{E}\,|\alpha(a_2,b_2,c_2) - \alpha(a_2,b_2,0)|^2 \le \frac{8}{3\pi t}\mathbb{E}\,(a_2+b_2)^3 = \\ = \frac{8}{3\pi t}(\sqrt{\varepsilon})^3 \int_0^\infty u^3\,d(2\Phi(u)-1) = \text{const} \cdot \varepsilon^{3/2}/t$$

(with an absolute constant).

Does it mean that $\mathbb{E}\big(\varphi(\xi_{0,1}) \,\big|\, \mathcal{F}_{s,s+\varepsilon} \big)$, being a function of $a(s,s+\varepsilon), b(s,s+\varepsilon), c(s,s+\varepsilon)$, is close to a function of $a(s,s+\varepsilon)$ and $b(s,s+\varepsilon)$ only? Yes, it does. Note that $\mathbb{E}\big(a(0,1) \,\big|\, \mathcal{F}_{s,s+\varepsilon} \big) = a(s,s+\varepsilon)$ is of norm $\sqrt{\varepsilon}$ (in $L_2(\Omega)$). A norm $O(\varepsilon^{3/4})$ is much smaller.

For each $k = 1,\dots,2^n$ we have two random variables

$$X_k = \alpha(a_2,b_2,c_2) = \mathbb{E}\big(X \,\big|\, \mathcal{F}_{(k-1)2^{-n},k2^{-n}} \big) \, , \\ Y_k = \alpha(a_2,b_2,0) \, ;$$

(8c3) gives

$$\|X_k - Y_k\|^2 \le \text{const} \cdot \frac{(2^{-n})^{3/2}}{1-k2^{-n}} \, .$$

Introduce

$$Z_k = \begin{cases} Y_k & \text{if } 1-k2^{-n} \ge 2^{-n/4}, \\ 0 & \text{otherwise}, \end{cases}$$

then

$$\|X_k - Z_k\|^2 \le \begin{cases} \text{const} \cdot 2^{-\frac54 n} & \text{if } 1-k2^{-n} \ge 2^{-n/4}, \\ \|X_k\|^2 & \text{otherwise.} \end{cases}$$

Both X_k and Z_k are $\mathcal{F}_{(k-1)2^{-n},k2^{-n}}$-measurable, thus $X_1 - Z_1, \dots, X_{2^n} - Z_{2^n}$ are independent, and

$$\operatorname{Var}\Big(\sum_{k=1}^{2^n} (X_k - Z_k) \Big) = \sum_{k=1}^{2^n} \operatorname{Var}(X_k - Z_k) \le \sum_{k=1}^{2^n} \|X_k - Z_k\|^2 \le \\ \le \text{const} \cdot 2^{-\frac54 n} \cdot 2^n + \sum_{k:1-k2^{-n} \ge 2^{-n/4}} \|X_k\|^2 \, ;$$

when $n \to \infty$, the first term tends to 0 evidently; the second term also tends to 0, since $\|\mathbb{E}\big(X \,\big|\, \mathcal{F}_{1-\delta,1} \big)\| \to 0$ for $\delta \to 0$ (due to continuity of the sticky factorization).

Let $\mathcal{F}^{\text{Brown}}$ be the sub-σ-field generated by $a(s,t)$ for all $s < t$. Taking into account that Z_k are $\mathcal{F}^{\text{Brown}}$-measurable (think, why), we get

$$\operatorname{dist}\Big(\sum_{k=1}^{2^n} X_k, L_2(\mathcal{F}^{\text{Brown}}) \Big) \to 0 \quad \text{for } n \to \infty \, ,$$

which means that

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \mathbb{E}\,\big(X \,\big|\, \mathcal{F}_{(k-1)2^{-n},k2^{-n}} \big) \in L_2(\mathcal{F}^{\text{Brown}})$$

for every X of the form (8c2). Therefore, for every $X \in L_2(\Omega)$, its projection to the first (sticky) chaos is in fact a function of the Brownian motion $a(0,\cdot)$; thus, it belongs to the first Brownian chaos, and is of the form $\int \psi(t)\, da(0,t)$. So,

(8c4) $$H_1(s,t) \subset L_2(\mathcal{F}^{\text{Brown}}) \, , \quad H_1(s,t) = H_1^{\text{Brown}}(s,t) \, .$$

Decomposable processes do not generate the whole σ-field. The sticky factorization is not isomorphic to the Brownian factorization. Moreover, it is not generated by a process with independent increments (of any dimension).

8d The noise made by a Poisson snake

...it is strongly suggested that the reader draw some pictures.

Jon Warren.[2]

8d1 Exercise. Construct a natural one-parameter group of automorphisms $(\alpha_t)_{t\in\mathbb{R}}$ of the probability space of the sticky flow, thus turning the sticky factorization into a noise. Show that

$$a(s,t)\circ\alpha_r = a(s-r,t-r) \, , \quad b(s,t)\circ\alpha_r = b(s-r,t-r) \, , \quad c(s,t)\circ\alpha_r = c(s-r,t-r) \, .$$

Hint. Similarly to 5e1, use (5d1). In contrast to 5e1, do not use stochastic integrals, use $\xi_{s,t}$ instead.

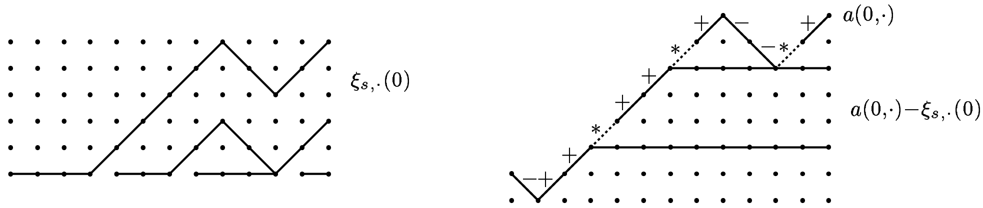
So, we have 'the sticky noise'. Surely, it is not isomorphic to the white noise, since corresponding factorizations are nonisomorphic.

8d2 Exercise. Describe all morphisms from the sticky noise to the white noise. How many morphisms exist?

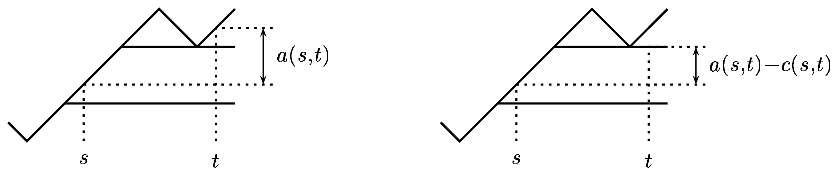
We want to visualize the sticky noise, making it as explicit as possible. To this end, we return for a while to discrete time, namely, to figures (7d1), (7d2). However, instead of

[2] Jonathan Warren, "The noise made by a Poisson snake", unpublished manuscript, Univ. de Pierre et Marie Curie, Paris, November 1998. (See page 1.)

$\xi_{s,\cdot}(x)$ we plot only $\xi_{s,\cdot}(0)$.

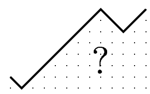
(8d3) 

In discrete time, such a geometric configuration, consisting of a polygonal line and horizontal lines (let us call them 'chords', even though sometimes they finish not on the polygonal line), is just another form of a word in the alphabet $\{-,+,*\}$.[3] In order to speak about scaling limit, however, we have to specify our 'observables'. These are $a(s,t)$ and $a(s,t)-c(s,t)$ shown below.

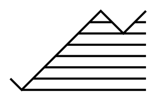
(8d4) 

You see, $a(s,t)$ is just the increment of the polygonal line (irrespective of chords). However, $a(s,t)-c(s,t)$ is the increment of a line that starts on the polygonal line, switches to a chord at the first opportunity (if any), maybe returns to the polygonal line (if the chord terminates on it), maybe switches again, and so on.

We want to describe the probability distribution on the set of such geometric configurations (over a given time interval). No problem with the polygonal line; that is just the simplest random walk. Given a polygonal line, what is the probability distribution of the random set of chords?



The set of all possible chords is easy to describe; each chord starts at a point of the polygonal line, remains (strictly) below it, and finishes when it cannot continue, either hitting the polygonal line, or at the end of the time interval. (Sometimes chords continue one another.)



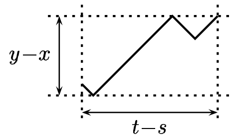
A surprise: the probability distribution on sets of chords is very easy to describe! Namely, each chord belongs to the random set with probability $\sqrt{\varepsilon}$, independently of others (think, why).

The number of all possible chords is also very easy to describe; it is just the number of positive increments (steps upwards) of the polygonal line.

For small ε, typically, the number of positive increments is (relatively) close to $(t-s)/(2\varepsilon)$, where (s,t) is our time interval. Accordingly, the random set of chords typically contains

[3] I do not formulate exact requirements to the geometric configuration; do it yourself. No need to specify the origin on the vertical axis.

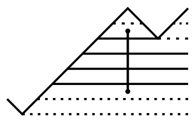
nearly $(t-s)/(2\sqrt{\varepsilon})$ chords. In the scaling limit, we should get an infinite random set of chords (in a bounded domain). Do not panic! The number of chords of length $\ge \delta$ is $O\big(\frac{(t-s)(y-x)}{\delta\sqrt{\varepsilon}}\big)$ (think, why); here (s,t) is our time interval, and (x,y) is the corresponding (spatial) interval visited by the random walk.



Therefore, the random set of chords contains typically $O\big(\frac{(t-s)(y-x)}{\delta}\big)$ chords longer than δ. In the scaling limit, the random set should contain finitely many 'non-short' chords, and infinitely many 'short' chords.

In the scaling limit, the polygonal line turns into a Brownian path, and the set of all possible chords — into the set of all chords of the Brownian path. However, what is the relevant measure on the set of all Brownian chords?

A surprise (once again): the measure on chords has a simple description suitable for the scaling limit.



Consider a vertical segment with endpoints (t,x) and (t,y), $x<y$, and all chords intersecting the segment. In discrete time, the number of such chords is close (for small ε) to $(y-x)/\sqrt{\varepsilon}$, provided that $\min_{[s,t]} B_\varepsilon(\cdot) \le x < y \le \max_{[s,t]} B_\varepsilon(\cdot)$, where B_ε is the random walk (its graph is the polygonal line), and s is the left endpoint of our time interval. Let us give the measure $\sqrt{\varepsilon}$ (rather than 1) to every chord. (Note that $\sqrt{\varepsilon}$ is just the probability of the chord to belong to the random set.) Then the measure of the set of chords intersecting the vertical segment tends (for $\varepsilon \to 0$) to

$$\text{Length}\Big([x,y] \cap \big[\min_{[s,t]} B(\cdot), \max_{[s,t]} B(\cdot)\big)\Big)\,.$$

The set of all chords (in continuous time) is somewhat complicated because of small chords. However, the set of non-small chords, of length $\ge \delta$, is simple.

There exists a finite set of vertical segments such that each (non-small) chord intersects exactly one segment (except maybe for a finite number of chords), and every point of each segment belongs to a chord. Topologically, it is a tree, but we do not need topology, we need only measure. Treated as a measure space, the set of chords is isomorphic to the (disjoint) union of these vertical segments, with the sum of Lebesgue measures on the segments.

For every $\delta>0$, the set of all chords of length $\ge \delta$ is a measure space, with a finite measure. Thus, the set of all chords is a measure space, with a σ-finite measure.[4]

[4] "σ-finite" means that the space is the union of a sequence of sets of finite measure. Lebesgue measure on $\mathbb{R}$ is σ-finite, too.

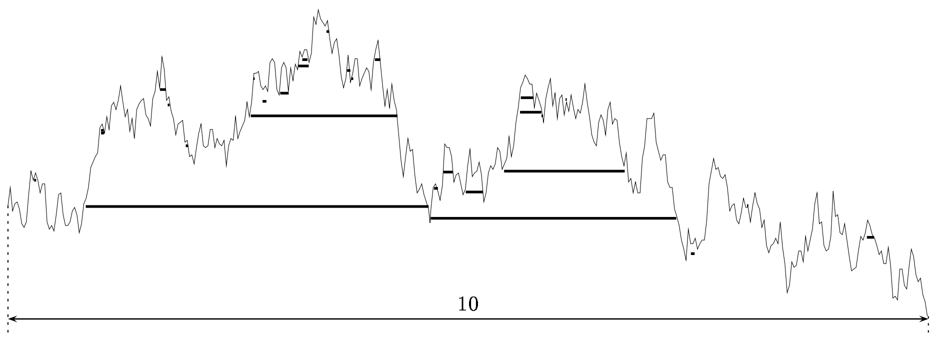
Note that starting points of all chords are special; such a point t satisfies

$$\exists \varepsilon > 0 \quad \min_{[t,t+\varepsilon]} B(\cdot) = B(t) \,.$$

A given (non-random) t satisfies it with probability 0. Therefore (by Fubini theorem), the set of 'starting points' is of Lebesgue measure 0. Another clash between discrete and continuous! In discrete time, nearly a half of points are 'starting points'.

In discrete time, each chord belongs to the random set with probability $\sqrt{\varepsilon}$. The intersection of a vertical segment with the random set contains a random number of points, and the number is distributed binomially, $\operatorname{Binom}\big(\frac{y-x}{\sqrt{\varepsilon}}, \sqrt{\varepsilon}\big)$.[5] In the scaling limit, it should turn into the Poisson distribution, $P(y-x)$.

We return to the idea of a Poisson process over a given measure space. It was already said (in 7d) that the construction of the Poisson process, given in 2b, works over any measure space (with a finite or σ-finite measure). Well, we apply it to the measure space of chords. We get a random set of chords. The set is countable, infinite. For any $\delta > 0$, it contains only a finite number of chords of length $\geq \delta$.



No need to restrict ourselves to a finite time interval. We may consider the whole Brownian path (for $t \in (-\infty, +\infty)$), and its chords (now they are really chords, they start *and finish* on the Brownian path). Still, the set of all chords is a σ-finite measure space, and the corresponding Poisson process selects a random countable subset. The set of selected chords of length $\geq \delta$ is now infinite, but locally finite. Note also that lengths of selected chords are not bounded from above.

Intuitively we feel that the Poisson random set of Brownian chords is the scaling limit of (8d3). What about a proof? Recall our 'observables' $c(s,t)$ shown on figure (8d4). No need to trace a path switching back and forth between the polygonal line and its chords. According to 7d4, in order to know $a(s,t) - c(s,t)$, it suffices to know chords that start after s and do not finish before t. If such chords exist, the lowest of them gives us $a(s,t) - c(s,t)$; otherwise $c(s,t) = 0$. Using this fact, it is easy to prove convergence in distribution (for $\varepsilon \to 0$) for the two-dimensional random variable $\big(a(s,t), c(s,t)\big)$ for any single interval (s,t). In order to prove the scaling limit, joint distributions should be also checked (but I do not explain it further).

[5] Approximately, of course, and assuming that $[x,y] \subset \big[\min_{[s,t]} B, \max_{[s,t]} B\big]$.

8d5 Exercise. Examine the relation between the scaling limit and the result of J. Warren presented in 7c.

Finally, let us interprete our geometric configuration, consisting of a Brownian path and a (countable) set of its chords, as the history of an object changing in time. The state of the object at an instant t is the section of the configuration by the vertical line situated at t. The section contains one point $B(t)$ of the Brownian path, and a sequence $\big(X_k(t)\big)_{k=1,2,\dots}$, $B(t) > X_1(t) > X_2(t) > \dots$, $X_k(t) \to -\infty$ for $k \to \infty$. Imagine that the ray $(-\infty, B(t)]$ is a snake, the point $B(t)$ is the head of the snake, and each $X_k(t)$ is a spot on the snake. The snake is changing in time, but only near its head. Infinitesimally, it either lengthens (increases) or shortens (decreases). When it shortens, it can lose a spot. When it lengthens, it can gain a spot. Otherwise, spots remain unchanged.

Strangely enough, spots are discrete in space but not discrete in time! At every time instant, every bounded part of the snake carries a finite set of spots. However, the set of spots never remains unchanged during a time interval! Short-living spots appear and disappear all the time, without a break. They are the additional (non-white) part of 'the noise made by a Poisson snake'.

8e Time reversal

8e1 Exercise. White noise is time-symmetric in the following sense. There exists an automorphism β of the corresponding probability space such that

$$\Big(\int \varphi(t)\,dB(t) \Big) \circ \beta = \int \varphi(-t)\,dB(t) \quad \text{for all } \varphi \in L_2(\mathbb{R})\,.$$

Prove it. Find $B(t) \circ \beta$. What about the other sign, $-\int \varphi(-t)\,dB(t)$?

Hint. Use (5d1) once again. (You may also recall 5e1 and 8d1.)

8e2 Definition. Let $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s<t}, (\alpha_t)_{t\in\mathbb{R}} \big)$ be a noise. Its *time reversal* is the noise $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{-t,-s})_{s<t}, (\alpha_{-t})_{t\in\mathbb{R}} \big)$. A noise isomorphic to its time reversal will be called time-symmetric.

8e3 Exercise. Check that 8e2 is a correct definition, namely, the new object is indeed a noise.

8e4 Exercise. The white noise is time-symmetric.

Prove it.

Hint: use 8e1.

8e5 Exercise. Generalize 8e1 and 8e4 to an arbitrary process with stationary independent increments.

The sticky noise, is it time-symmetric? We may try the argument of 8e5; after all, we have stationary independent increments in the semigroup G_3. However, G_3 is non-commutative, and the argument fails (think, why). The random geometric configuration ('snake') of 8d is time-symmetric. However, the σ-field $\mathcal{F}_{s,t}$ describes chords ('spots on the snake') whose *left* endpoints belong to (s,t). *Right* endpoints matter for the time reversed noise. The snake makes two different noises. Roughly speaking, appearance of a spot is an event of the 'left oriented' noise, while disappearance of a spot is an event of the 'right oriented' noise. Are the two noises isomorphic?

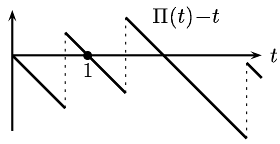
8f Predictable and unpredictable

> This example also illustrates that the time-reversal of a predictable noise need not be predictable...
>
> Jon Warren.[6]

The Brownian motion is continuous; the Poisson process is discontinuous; however, both noises (white and Poisson) are continuous. The Brownian motion as a function $\mathbb{R} \times \Omega \ni (t,\omega) \mapsto B(t,\omega) \in \mathbb{R}$ is such that $B(\cdot,\omega)$ is continuous for almost all ω. The Poisson process $\Pi(t,\omega)$ does not have such a property. However, the two functions $\mathbb{R} \ni t \mapsto B(t,\cdot) \in L_2(\Omega)$ and $\mathbb{R} \ni t \mapsto \Pi(t,\cdot) \in L_2(\Omega)$ both are continuous.[7] Never confuse these two kinds of continuity. (Here is the most elementary example: $(0,1)\times(0,1) \ni (x,y) \mapsto f(x,y) = (\text{if } x < y \text{ then } 1 \text{ else } 0)$.) Continuity in t for almost all ω is called *sample continuity.*

If a factorization $(\mathcal{F}_{s,t})_{s<t}$ is continuous (recall 8b4), then for every $X \in L_2(\Omega)$, the map $[0,\infty) \ni t \mapsto X(t) = \mathbb{E}\big(X \,\big|\, \mathcal{F}_{0,t}\big) \in L_2(\Omega)$ is continuous, but it does not mean that the process $X(\cdot)$ is sample continuous. Here is an example, for the Poisson case: $X = \Pi(1) - 1$; $\mathbb{E}\big(X \,\big|\, \mathcal{F}_{0,t}\big) = \mathbb{E}\big(\Pi(1) - \Pi(t) \,\big|\, \mathcal{F}_{0,t}\big) + \mathbb{E}\big(\Pi(t) \,\big|\, \mathcal{F}_{0,t}\big) - 1 = \mathbb{E}\big(\Pi(1) - \Pi(t)\big) + \Pi(t) - 1 = 1 - t + \Pi(t) - 1$, that is,

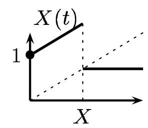
$$X_t = \Pi(t) - t \quad \text{for } t \in [0,1]\,.$$



Another example (still for the Poisson case): let $X : \Omega \to [0,\infty)$ be the time of the first jump of $\Pi(\cdot)$ (after time 0); then $X \sim \mathrm{Exp}(1)$, $\mathbb{E}(X) = 1$. Clearly, for each given t we have $(X > t) \iff (\Pi(t) = 0)$ a.s.

8f1 Exercise.

$$X(t) = \mathbb{E}\big(X \,\big|\, \mathcal{F}_{0,t}\big) = \begin{cases} t+1 & \text{if } \Pi(t) = 0, \\ X & \text{if } \Pi(t) \ge 1 \end{cases} = \begin{cases} t+1 & \text{if } X > t, \\ X & \text{if } X < t. \end{cases}$$



Prove it.

Hint: Markov property ("no memory").

In any case, the process $X_t = \mathbb{E}\big(X \,\big|\, \mathcal{F}_{0,t}\big)$ is a martingale, that is, $\mathbb{E}\big(X_t \,\big|\, \mathcal{F}_{0,s}\big) = X_s$ for $0 \le s \le t$.

[6] Jonathan Warren, "The noise made by a Poisson snake", unpublished manuscript, Univ. de Pierre et Marie Curie, Paris, November 1998. (See the last phrase.)

[7] $\|B(s) - B(t)\| = \sqrt{|s-t|}$; $\|\Pi(s) - \Pi(t)\| = \sqrt{|s-t| + |s-t|^2}$.

Discontinuity of the martingale of 8f1 means that the random time X arrives suddenly; the expected waiting time $\mathbb{E}\left(X - t \,\middle|\, \mathcal{F}_{0,t} \right)$ does not tend to 0 when $t \to X-$.

Can you construct such an example for the Brownian case? Try it, and you'll feel that nothing comes suddenly in the Brownian world.

8f2 Definition. (a) A *filtration* is a one-parameter family $(\mathcal{F}_t)_{t\in\mathbb{R}}$ of sub-σ-fields $\mathcal{F}_t \subset \mathcal{F}$ (on a probability space $(\Omega, \mathcal{F}, P)$) such that $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \le t$.[8]

(b) The filtration of a factorization $(\mathcal{F}_{s,t})_{s<t}$ is $(\mathcal{F}_{0,t})_{t\in[0,\infty)}$.[9]

(c) The filtration of a noise is the filtration of its factorization.

8f3 Definition. (a) A filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ is *predictable,* if for every $X \in L_2(\Omega)$, the martingale $X(t) = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_t \right)$ is sample continuous.[10]

(b) A noise is predictable, if its filtration is predictable.

8f4 Proposition. $\mathbb{P}\left(\sup_t |X(t)| \ge c \right) \le \frac{1}{c^2}\|X\|^2$ whenever $c > 0$, $X(t) = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{0,t} \right)$, $X \in L_2(\Omega)$.[11]

8f5 Exercise. The set of all $X \in L_2(\Omega)$ such that the martingale $X(t) = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_t \right)$ is sample continuous, is a (closed linear) subspace of $L_2(\Omega)$.

Prove it.

Hint. If $\|X_k - X\|^2 \le 2^{-k}$ then $\sum_k \mathbb{P}\left(\max_t |X_k(t) - X(t)| \ge 2^{-k/4} \right) < \infty$; use the first Borel-Cantelli lemma.

8f6 Exercise. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, and $X = \varphi(B(1))$. Then the martingale $X(t) = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_t \right)$ is sample continuous.

Prove it. What about a discontinuous φ?

Hint: $X(t) = \varphi_t\bigl(B(t)\bigr)$, where $\varphi_t : \mathbb{R} \to \mathbb{R}$,

$$\varphi_t(x) = \int \varphi(y)\, d\Phi\Bigl(\frac{y-x}{\sqrt{1-t}} \Bigr) .$$

8f7 Exercise. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a bounded continuous function, $0 < t_1 < \dots < t_n < 1$, and $X = \varphi\bigl(B(t_1), \dots, B(t_n)\bigr)$. Then the martingale $X(t) = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_t \right)$ is sample continuous.

Prove it.

Hint. $X(t_k) = \varphi_k\bigl(B(t_1), \dots, B(t_k)\bigr)$, and $X(t) = \mathbb{E}\left(X(t_k) \,\middle|\, \mathcal{F}_{0,t} \right)$ for $t \in (t_{k-1}, t_k]$; use 8f6.

8f8 Exercise. The Brownian filtration is predictable.

Prove it.

Hint: combine 8f7 and 8f5.

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[8] More generally, t may run over a given subset of $\mathbb{R}$, say, $[0, \infty)$, or $\mathbb{Z}$, $\mathbb{Z}_+$ etc.

[9] Also $(\mathcal{F}_{-\infty,t})_{t\in\mathbb{R}}$ may be used.

[10] Or rather, the restriction of the martingale to (say) rational numbers t is locally uniformly continuous in t for almost all ω. See also Footnote 9 on page 8 (Sect. 1).

[11] Or rather, t runs over a countable subset...

The argument of 8f7 is general enough, it may be used in the noncommutative semigroup G_3 as well, for the sticky Brownian flow, provided that 8f6 remains true. Does it? Instead of $\varphi_t(x) = \int \varphi(y)\, d\Phi\Big(\frac{y-x}{\sqrt{1-t}}\Big)$, we have now

$$\varphi_t(f_{a_1,b_1,c_1}) = \int \varphi(f_{a_2,b_2,c_2} \circ f_{a_1,b_1,c_1})\, d\mu_{1-t}(a_2,b_2,c_2)\,. \tag{8f9}$$

Is it continuous in a_1, b_1, c_1 and t?

8f10 Exercise. For every bounded continuous function $\varphi : G_3 \to \mathbb{R}$ and every $t \in [0,1]$, the function $\varphi_t : G_3 \to \mathbb{R}$ defined by (8f9) is continuous.

Prove it.

Hint. If $f_{a_1^{(k)},b_1^{(k)},c_1^{(k)}} \to f_{a_1,b_1,c_1}$ (in the sense that $a_1^{(k)} \to a_1$, $b_1^{(k)} \to b_1$, $c_1^{(k)} \to c_1$), then $f_{a_2,b_2,c_2} \circ f_{a_1^{(k)},b_1^{(k)},c_1^{(k)}} \to f_{a_2,b_2,c_2} \circ f_{a_1,b_1,c_1}$ for μ_{1-t}-almost all f_{a_2,b_2,c_2}; recall 7b1(h) and the distribution of b_2.

8f11 Exercise. If $t_k \to t$ then $\|\mu_{t_k} - \mu_t\| \to 0$, that is,

$$\sup_{f:G_3\to[-1,+1]} \left| \int f\, d\mu_{t_k} - \int f\, d\mu_t \right| \xrightarrow[k\to\infty]{} 0\,.$$

Prove it.

Hint. First, do it for measures on the two-dimensional semigroup G_2. Then, replace c with $(a+b-\eta)^+$, $\eta \sim \mathrm{Exp}(1)$.

8f12 Exercise. For every bounded continuous function $\varphi : G_3 \to \mathbb{R}$, the function $G_3 \times [0,1] \ni \big(f_{a_1,b_1,c_1}, t\big) \mapsto \varphi_t(f_{a_1,b_1,c_1})$, defined by (8f9), is continuous.

Prove it.

Hint: combine 8f10 and 8f11.

So, the sticky filtration is predictable. That is a wonder; unlike the Poisson process, the Poisson snake is predictable. You see, spots are discrete in space but not in time. Appearance of a long-living spot cannot be detected in real time...

As was said in 8e, the Poisson snake makes two different noises, each being the time reversal of the other. Appearance of a spot is an event of the 'left oriented' noise; disappearance of a spot is an event of the 'right oriented' noise.

Note an important asymmetry: when a spot appears, we do not know, how long will it survive. When a spot disappears, we know its age! True, we did not notice its appearance; however, we know the (past) Brownian path, which allows us to trace back the spot (the chord) retroactively.

For any given $s < t$ consider such an event $A_{s,t}$: "the random set of chords contains a chord[12] that starts at some instant t_{start} and finishes at some t_{finish} such that $s < t_{\text{start}} < t_{\text{start}} + 1 < t_{\text{finish}} < t$". We have

$$A_{s,t} \in \mathcal{F}_{s,t}\,.$$

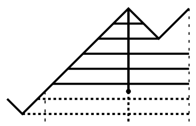
[12] At least one.

8f13 Exercise.

$$\|\mathbb{P}\left(A_{s-\varepsilon,t} \setminus A_{s,t} \,\middle|\, \mathcal{F}_{s,t} \right)\|_{L_\infty(\Omega)} = O(\sqrt{\varepsilon}) \quad \text{for } \varepsilon \to 0+ \,.$$

Prove it. What about $O(\varepsilon)$? What about $\|\mathbb{P}\left(A_{s,t+\varepsilon} \setminus A_{s,t} \,\middle|\, \mathcal{F}_{s,t} \right)\|_{L_\infty(\Omega)}$?

Hint:



8f14 Exercise. The random variable $X = \inf\{t \in (0,\infty) : A_{-t,0}\}$[13] is integrable, and the martingale[14] (X_t), $X_t = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{-t,0} \right)$ is (sample) discontinuous.

Prove it.

Hint. First, $\mathbb{P}\left(X > x_1 + x_2 \right) \le \mathbb{P}\left(X > x_1 \right) \cdot \mathbb{P}\left(X > x_2 \right)$. Second, use 8f13.

So, the noise of stickiness is predictable, but its time reversal is not, which was discovered by Jon Warren (1998).

8g Are you careful enough?

8g1 Exercise. For every bounded continuous function $\varphi : G_3 \to \mathbb{R}$ and every $t \in [0,1]$, the function $\varphi_t : G_3 \to \mathbb{R}$ defined by

$$\varphi_t(f_{a_2,b_2,c_2}) = \int \varphi(f_{a_2,b_2,c_2} \circ f_{a_1,b_1,c_1}) \, d\mu_t(a_1,b_1,c_1)$$

is continuous.

Prove it.

Hint: similar to 8f10; the atom of c_1 at 0 does not invalidate the argument.

8g2 Exercise. For every bounded continuous function $\varphi : G_3 \to \mathbb{R}$, the function $G_3 \times [0,1] \ni \big(f_{a_2,b_2,c_2}, t\big) \mapsto \varphi_t(f_{a_2,b_2,c_2})$ (where φ_t is the same as in 8g1), is continuous.

Prove it.

Hint: similar to 8f12.

8g3 Exercise. Let $\varphi : G_3 \to \mathbb{R}$ be a bounded continuous function, and $X = \varphi(\xi_{-1,0}) = \varphi(f_{a(-1,0),b(-1,0),c(-1,0)})$. Then the martingale[15] (X_t), $X_t = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{-t,0} \right)$, $t \in [0,1]$, is sample continuous.

Prove it.

Hint: recall 8f6, and 8g2.

8g4 Exercise. It follows from 8g3 that the time reversal of the sticky noise is continuous; but 8f states the opposite. Find the error! (Or else, mathematics is inconsistent...)

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[13] That is, $X(\omega) = \inf\{t \in (0,\infty) : \omega \in A_{-t,0}\}$.

[14] Adapted to the filtration $(\mathcal{F}_{-t,0})$.

[15] Adapted to the filtration $(\mathcal{F}_{-t,0})$.