5 Noises as semigroups of probability spaces

...the different possible formulations may give clues about what might happen in other circumstances. In that case they are no longer equivalent...

Richard Feynman.[1]

I hail a semi-group when I see one and I seem to see them everywhere! Friends have observed, however, that there are mathematical objects which are not semi-groups.

Einar Hille.[2]

5a Factorization: discrete case

Recall a very useful notion of a one-parameter semigroup of operators $(U_t)_{t\in[0,\infty)}$, each U_t being an operator (linear, continuous), say, on a Hilbert space. A semigroup must satisfy multiplicativity condition,

(5a1) $$U_{s+t} = U_s U_t \qquad \text{for } s,t \in [0,\infty)\,,$$

and some continuity (in t) condition. A special case is a semigroup of matrices $n \times n$. A more special case is a semigroup of numbers (here $U_t = e^{\lambda t}$, of course).

Anyway, we do not need semigroups of operators. There are other mathematical objects that can be multiplied. Recall the (Cartesian) product of sets, topological spaces, measure spaces, etc. We may define a one-parameter semigroup of spaces (of a given sort). We'll see that noises are naturally related to semigroups of probability spaces.

Recall the discrete model introduced in 1b:

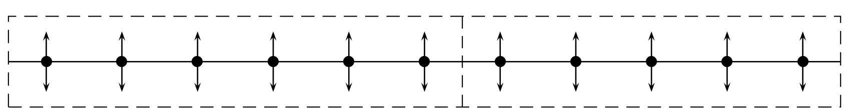
(5a2) $$\Omega_{\varepsilon,M} = \{-1,+1\}^{\varepsilon\mathbb{Z}\cap[-M,M]}\,.$$

Random signs $\tau(k\varepsilon)$ are situated at lattice points $k\varepsilon \in \varepsilon\mathbb{Z} \cap [-M,M]$, and the probability space $\Omega_{\varepsilon,M}$ is the product of many two-point probability spaces $\{-1,+1\}$.[3] If we split the lattice interval in two,

(5a3) $$\varepsilon\mathbb{Z} \cap [-M,M] = \big(\varepsilon\mathbb{Z} \cap [-M,a]\big) \cup \big(\varepsilon\mathbb{Z} \cap (a,M]\big)\,,$$

we get the product of two probability spaces,

(5a4) $$\underbrace{\{-1,+1\}^{\varepsilon\mathbb{Z}\cap[-M,M]}}_{\Omega_{\varepsilon,M}=\Omega_\varepsilon[-M,M]} = \underbrace{\{-1,+1\}^{\varepsilon\mathbb{Z}\cap[-M,a]}}_{\Omega_\varepsilon[-M,a]} \times \underbrace{\{-1,+1\}^{\varepsilon\mathbb{Z}\cap(a,M]}}_{\Omega_\varepsilon(a,M]}\,.$$



That is, each $\omega \in \Omega_\varepsilon[-M,M]$ may be thought of as a pair (ω',ω'') of $\omega' \in \Omega_\varepsilon[-M,a]$ and $\omega'' \in \Omega_\varepsilon(a,M]$.

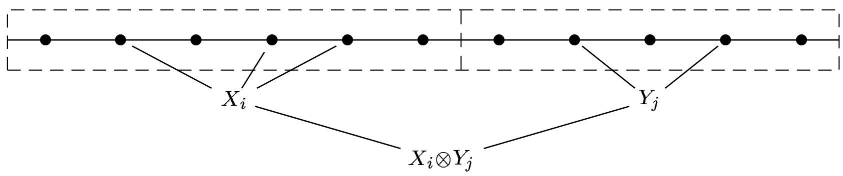
[1] R. Feynman, "The character of physical law", M.I.T. Press 1967; see p. 53 (Chapter 2).

[2] E. Hille, "Functional analysis and semi-groups", Amer. Math. Soc. 1948 (AMS colloquium publications, vol. XXXI); see Foreword.

[3] I omit the σ-field and the probability measure, when they are evident enough.

What happens to the corresponding spaces L_2 of square integrable random variables? A simple and natural orthonormal basis of $L_2\big(\Omega_\varepsilon[-M,M]\big)$ consists of monomials $\tau(k_1\varepsilon)\dots\tau(k_n\varepsilon)$, where $-M/\varepsilon \le k_1 < k_2 < \dots < k_n \le M/\varepsilon$, and $n = 0,1,2,\dots,1+2\,\mathrm{entier}(M/\varepsilon)$. Similar bases exist in $L_2\big(\Omega_\varepsilon[-M,a]\big)$ and $L_2\big(\Omega_\varepsilon(a,M]\big)$. Let $(X_1,\dots,X_k)$ be the basis of $L_2\big(\Omega_\varepsilon[-M,a]\big)$; that is, each $X_i : \Omega_\varepsilon[-M,a] \to \{-1,1\}$ is a monomial; their numbering does not matter; $k = 2^{|\varepsilon\mathbb{Z}\cap[-M,a]|}$, which also does not matter now. Let $(Y_1,\dots,Y_l)$ be the similar basis of $L_2\big(\Omega_\varepsilon(a,M]\big)$. Then the basis of $L_2\big(\Omega_\varepsilon[-M,M]\big)$ is nothing but

$$(X_i \otimes Y_j)_{i=1,\dots,k;j=1,\dots,l}\,;$$



here (similarly to 3c)

(5a5) $$(X\otimes Y)(\omega',\omega'') = X(\omega')Y(\omega'')\,.$$

It means that (see 5b)

(5a6) $$L_2\big(\Omega_\varepsilon[-M,M]\big) = L_2\big(\Omega_\varepsilon[-M,a]\big)\otimes L_2\big(\Omega_\varepsilon(a,M]\big)\,.$$

5b Tensor products and independence

Generally, the *tensor product* $H = H_1\otimes H_2$ of two Hilbert spaces H_1, H_2 may be defined as a Hilbert space H equipped with a function $(x,y)\mapsto x\otimes y \in H$ of two variables $x\in H_1$, $y\in H_2$, such that

(5b1) $$\begin{gathered}\langle x\otimes y, u\otimes v\rangle = \langle x,u\rangle\cdot\langle y,v\rangle \quad \text{for all } x,u\in H_1, y,v\in H_2\,;\\ H \text{ is spanned by vectors } x\otimes y,\ x\in H_1, y\in H_2\,;\end{gathered}$$

the latter means that every vector of H can be approximated by linear combinations of vectors $x\otimes y$. It follows that $x\otimes y$ is bilinear:

$$\begin{aligned}(ax+by)\otimes z &= a(x\otimes z)+b(y\otimes z)\,,\\ x\otimes(ay+bz) &= a(x\otimes y)+b(x\otimes z)\,.\end{aligned}$$

If $(x_i)_i$ is an orthonormal basis of H_1 and $(y_j)_j$ is an orthonormal basis of H_2 then $(x_i\otimes y_j)_{i,j}$ is an orthonormal basis of $H_1\otimes H_2$. Therefore

$$\dim(H_1\otimes H_2) = \dim(H_1)\cdot\dim(H_2)\,,$$

be it finite or infinite.[4]

[4]Do not confuse the tensor product $H_1\otimes H_2$ and the direct sum $H_1\oplus H_2$. You see, $\dim(H_1\oplus H_2) = \dim H_1 + \dim H_2$. Every vector of $H_1\oplus H_2$ is of the form $x\oplus y$, but only some vectors of $H_1\otimes H_2$ (called *factorizable* vectors) are of the form $x\otimes y$. Note also that H_1 and H_2 are subspaces of $H_1\oplus H_2$, but not subspaces of $H_1\otimes H_2$.

Let $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ be two probability spaces, and

$$(\Omega, \mathcal{F}, P) = (\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2)$$

their product.[5] Then

$$L_2(\Omega, \mathcal{F}, P) = L_2(\Omega_1, \mathcal{F}_1, P_1) \otimes L_2(\Omega_2, \mathcal{F}_2, P_2)\,,$$

the operation $(X, Y) \mapsto X \otimes Y$ being defined by (5a5).

For Hilbert spaces in general, each vector $y \in H_2$ of unit norm gives us its own embedding $H_1 \ni x \mapsto x \otimes y \in H_1 \otimes H_2$; no embedding is distinguished. However, Hilbert spaces of the form $L_2(\Omega, \mathcal{F}, P)$ are special; such a space contains a distinguished element $\mathbf{1}_\Omega$ (the constant function, equal to 1 everywhere on Ω). We have the canonical embedding

$$L_2(\Omega_1, \mathcal{F}_1, P_1) \ni X \mapsto X \otimes \mathbf{1}_{\Omega_2} \in L_2(\Omega, \mathcal{F}, P)\,;$$
$$\big(X \otimes \mathbf{1}_{\Omega_2}\big)(\omega_1, \omega_2) = X(\omega_1)\,.$$

Similarly,

$$L_2(\Omega_2, \mathcal{F}_2, P_2) \ni Y \mapsto \mathbf{1}_{\Omega_1} \otimes Y \in L_2(\Omega, \mathcal{F}, P)\,;$$
$$\big(\mathbf{1}_{\Omega_1} \otimes Y\big)(\omega_1, \omega_2) = Y(\omega_2)\,.$$

Note that $X \otimes Y = (X \otimes \mathbf{1})(\mathbf{1} \otimes Y)$; the pointwise multiplication is used (generally, not a good operation on Hilbert spaces).

Basically, $X \otimes \mathbf{1}$ is the same as X; both are $X(\omega_1)$, but X is a function of ω_1 only, while $X \otimes \mathbf{1}$ is a function of two variables ω_1, ω_2 which depends on ω_1 only. In other words, the function $X \otimes \mathbf{1}$ is measurable not only w.r.t. the whole σ-field $\mathcal{F}$, but also w.r.t. its sub-σ-field[6]

$$\tilde{\mathcal{F}}_1 = \{A \times \Omega_2 : A \in \mathcal{F}_1\} \subset \mathcal{F}_1 \otimes \mathcal{F}_2\,.$$

Similarly, functions of the form $\mathbf{1} \otimes Y$ are $\tilde{\mathcal{F}}_2$-measurable,

$$\tilde{\mathcal{F}}_2 = \{\Omega_1 \times B : B \in \mathcal{F}_2\} \subset \mathcal{F}_1 \otimes \mathcal{F}_2\,.$$

These two sub-σ-fields are *independent:*

$$\forall A \in \tilde{\mathcal{F}}_1 \;\; \forall B \in \tilde{\mathcal{F}}_2 \quad P(A \cap B) = P(A)P(B)\,. \tag{5b2}$$

Here is another approach to the same matter. We may start with a single probability space $(\Omega, \mathcal{F}, P)$ (be it a product or not) and two independent[7] sub-σ-fields $\mathcal{F}_1 \subset \mathcal{F}$, $\mathcal{F}_2 \subset \mathcal{F}$. All $\mathcal{F}_1$-measurable square integrable functions on Ω form a Hilbert space $L_2(\Omega, \mathcal{F}_1, P)$ (or $L_2(\Omega, \mathcal{F}_1, P|_{\mathcal{F}_1})$). The same for $\mathcal{F}_2$; we have

$$L_2(\Omega, \mathcal{F}_1, P) \subset L_2(\Omega, \mathcal{F}, P)\,; \quad L_2(\Omega, \mathcal{F}_2, P) \subset L_2(\Omega, \mathcal{F}, P)\,.$$

[5]That is, $\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$; and $\mathcal{F}$ is the σ-field on Ω generated by product sets $A \times B$ for all $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$; and P is the (only) probability measure on $(\Omega, \mathcal{F})$ such that $P(A \times B) = P_1(A)P_2(B)$ for all such A, B. And in addition, all P-negligible sets are added to $\mathcal{F}$ (thus, all subsets of a measurable set of probability 0 are measurable).

[6]Once again, all P-negligible sets should be added. I stop repeating this reservation.

[7]Independence is defined as in (5b2).

Do not think that these are orthogonal subspaces! (Both contain $\mathbf{1}_\Omega$.) Rather, the two subspaces satisfy

$$\forall X \in L_2(\Omega, \mathcal{F}_1, P) \, \forall Y \in L_2(\Omega, \mathcal{F}_2, P) \quad \|XY\| = \|X\| \cdot \|Y\| \, ;$$

here XY is the pointwise product; its norm is taken in $L_2(\Omega, \mathcal{F}, P)$; the norm of X is taken in $L_2(\Omega, \mathcal{F}_1, P)$ (or equivalently in $L_2(\Omega, \mathcal{F}, P)$); the same for Y; finiteness of $\|XY\|$ shows that $XY \in L_2(\Omega, \mathcal{F}, P)$.[8]

5b3 Exercise.

$$\langle XY, UV \rangle = \langle X, U \rangle \langle Y, V \rangle \quad \text{for } X, U \in L_2(\Omega, \mathcal{F}_1, P) \text{ and } Y, V \in L_2(\Omega, \mathcal{F}_2, P) \, .$$

Prove it.

Hint: the expectation of the product of *independent* random variables...

Thus, $L_2(\Omega, \mathcal{F}_1, P) \otimes L_2(\Omega, \mathcal{F}_2, P) \subset L_2(\Omega, \mathcal{F}, P)$; namely, it is the subspace generated by all XY for $X \in L_2(\Omega, \mathcal{F}_1, P)$, $Y \in L_2(\Omega, \mathcal{F}_2, P)$. In fact,

$$L_2(\Omega, \mathcal{F}_1, P) \otimes L_2(\Omega, \mathcal{F}_2, P) = L_2(\Omega, \mathcal{F}_1 \otimes \mathcal{F}_2, P) \, ,$$

where $\mathcal{F}_1 \otimes \mathcal{F}_2$ is the sub-σ-field of $\mathcal{F}$ generated by $A \times B$ for $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$. Note that $\mathcal{F}_1 \otimes \mathcal{F}_2$ is defined only for *independent* $\mathcal{F}_1, \mathcal{F}_2$.

If in addition $\mathcal{F}_1 \otimes \mathcal{F}_2 = \mathcal{F}$ then we get

$$L_2(\Omega, \mathcal{F}_1, P) \otimes L_2(\Omega, \mathcal{F}_2, P) = L_2(\Omega, \mathcal{F}, P)$$

and in fact, $(\Omega, \mathcal{F}, P)$ may be thought of as the product of two probability spaces (such that the second approach conforms to the first approach). These two spaces are so-called quotient spaces,

$$(\Omega, \mathcal{F}, P) \leftrightarrow (\Omega, \mathcal{F}, P)/\mathcal{F}_1 \otimes (\Omega, \mathcal{F}, P)/\mathcal{F}_2 \, ; \tag{5b4}$$

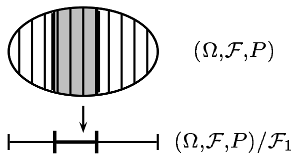
here '$\leftrightarrow$' means a canonical isomorphism. A quotient space $(\Omega, \mathcal{F}, P)/\mathcal{F}_1$ of a probability space $(\Omega, \mathcal{F}, P)$ modulo sub-σ-field $\mathcal{F}_1 \subset \mathcal{F}$ is the set of all equivalence classes of the equivalence relation (partition) corresponding to $\mathcal{F}_1$. We cannot define the equivalence relation simply by $\omega \sim \omega' \iff \forall A \in \mathcal{F}_1 \big(\omega \in \mathcal{A} \iff \omega' \in A \big)$, because of P-negligible sets contained in $\mathcal{F}_1$. Rather, we choose a sequence $A_1, A_2, \dots$ generating the σ-field $\mathcal{F}_1$ and define

$$\omega \sim \omega' \iff \forall n \, \big(\omega \in A_n \iff \omega' \in A_n \big) \, .$$

(A different choice of (A_n) changes the equivalence relation only on a negligible set.) On the quotient set Ω_1 we have a σ-field of all sets A_1 whose inverse images (under the canonical

[8] In general, if $X \in L_2$ and $Y \in L_2$ then $XY \in L_1$ but not L_2. Independence of X and Y is a very special case.

projection $\Omega \to \Omega_1$) belong to $\mathcal{F}$. The probability of such a set is the probability of its inverse image.



So, we have two equivalent pictures of the same situation. One picture is the product of two probability spaces,

$$(\Omega, \mathcal{F}, P) = (\Omega_1, \mathcal{F}_1, P_1) \times (\Omega_2, \mathcal{F}_2, P_2) .$$

The other picture is, two independent sub-σ-fields generating (together) the whole σ-field $\mathcal{F}$ of a given probability space $(\Omega, \mathcal{F}, P)$,

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 .$$

In any case, $L_2(\mathcal{F}) = L_2(\mathcal{F}_1) \otimes L_2(\mathcal{F}_2)$.

5c Factorization: continuous case

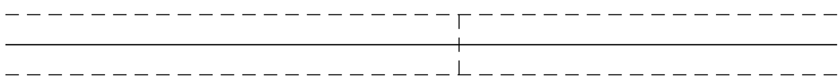
The white noise may be thought of as a (linear isometric) map $L_2(\mathbb{R}) \ni \varphi \mapsto \int \varphi \, dB \in L_2(\Omega)$. A point $a \in \mathbb{R}$ splits $L_2(\mathbb{R})$ in two orthogonal subspaces,

(5c1) $$L_2(-\infty, +\infty) = L_2(-\infty, a) \oplus L_2(a, +\infty) .$$

Let $(\varphi_1, \varphi_3, \varphi_5, \dots)$ be an orthonormal basis of $L_2(-\infty, a)$, and $(\varphi_2, \varphi_4, \varphi_6, \dots)$ an orthonormal basis of $L_2(a, +\infty)$, then $(\varphi_1, \varphi_2, \varphi_3, \dots)$ is an orthonormal basis of $L_2(-\infty, +\infty)$. The probability space $(\Omega, \mathcal{F}, P) = (\mathbb{R}^\infty, \gamma^\infty)$ carrying the white noise is the product of two similar spaces:

(5c2) $$\mathbb{R}^{\{1,2,3,\dots\}} = R^{\{1,3,5,\dots\}} \times \mathbb{R}^{\{2,4,6,\dots\}} , \qquad \bigotimes_{k=1,2,3,\dots} \gamma^1 = \Big(\bigotimes_{k=1,3,5,\dots} \gamma^1 \Big) \otimes \Big(\bigotimes_{k=2,4,6,\dots} \gamma^1 \Big) .$$

In other words, a configuration ω of the white noise over the whole $(-\infty, +\infty)$ may be thought of as a pair (ω', ω'') of a configuration ω' of the white noise over $(-\infty, a)$ and a configuration ω'' of the white noise over $(a, +\infty)$.



Equivalently,

$$\mathcal{F}_{-\infty,+\infty} = \mathcal{F}_{-\infty,a} \otimes \mathcal{F}_{a,+\infty} ;$$

more generally,

(5c3) $$\mathcal{F}_{x,z} = \mathcal{F}_{x,y} \otimes \mathcal{F}_{y,z} \quad \text{for } -\infty \le x \le y \le z \le +\infty ,$$

where $\mathcal{F}_{x,y}$ is the sub-σ-field of $\mathcal{F}$ generated by random variables $\int \varphi \, dB$ for all $\varphi \in L_2(x,y)$.[9] Accordingly,[10]

$$L_2(\mathcal{F}_{x,z}) = L_2(\mathcal{F}_{x,y}) \otimes L_2(\mathcal{F}_{y,z}) \,. \tag{5c4}$$

In terms of the Wiener chaos we have

$$\Big(\sum_{n=0}^\infty \int \dots \int_{x_1<\dots<x_n} \xi(x_1,\dots,x_n) \, dB(x_1) \dots dB(x_n) \Big) \cdot \\ \cdot \Big(\sum_{n=0}^\infty \int \dots \int_{x_1<\dots<x_n} \chi(x_1,\dots,x_n) \, dB(x_1) \dots dB(x_n) \Big) = \\ = \sum_{n=0}^\infty \sum_{k=0}^n \int \dots \int_{x_1<\dots<x_k<a<x_{k+1}<\dots<x_n} \xi(x_1,\dots,x_k)\chi(x_{k+1},\dots,x_n) \, dB(x_1) \dots dB(x_n) \,.$$

A two-parameter family $(\mathcal{F}_{x,y})_{x\le y}$ of sub-σ-fields $\mathcal{F}_{x,y} \subset \mathcal{F}$ satisfying (5c3) and such that $\mathcal{F}_{-\infty,+\infty} = \mathcal{F}$, will be called a *factorization* (of the probability space $(\Omega, \mathcal{F}, P)$ over $\mathbb{R}$).

The two main examples are the white (or Brownian) factorization, and the Poisson factorization.

For the discrete case, we may define a factorization (of $(\Omega, \mathcal{F}, P)$ over $\varepsilon\mathbb{Z}$) as a family $(\mathcal{F}_{x,y})$ where x and y run over $\varepsilon\big(\mathbb{Z}+\frac12\big)$, and $x \le y$; and (5c3) is assumed, of course.

Formally, the two cases look similar. However, there is a deep difference. Every sequence $(\eta_{k\varepsilon})$ of independent random variables generates a factorization over $\varepsilon\mathbb{Z}$, and the factorization carries only a scanty information about these random variables. Knowing only the sub-σ-field generated by a single $\eta_{k\varepsilon}$ we cannot restore the distribution of $\eta_{k\varepsilon}$. For example, we cannot distinguish such cases as $\eta_{k\varepsilon} \sim \mathrm{U}(0,1)$ and $\eta_{k\varepsilon} \sim \mathrm{N}(0,1)$. (Recall also (4d2).) In contrast, the white noise factorization (over $\mathbb{R}$) carries a considerable part of the structure of the white noise. Strangely enough, normal distributions are somehow hidded in the factorization. Several statements of that sort are contained in Sect. 4, somewhat implicitly. Now we'll formulate them explicitly in terms of the white noise factorization $(\mathcal{F}_{x,y})_{x\le y}$ of the probability space $(\Omega, \mathcal{F}, P)$.

5c5 Exercise. Let $(X_{x,y})_{x\le y}$ be a two-parameter family of random variables on $(\Omega, \mathcal{F}, P)$ such that for all $x \le y \le z$

[9] Here $L_2(x,y)$ is treated as a subspace of $L_2(-\infty,+\infty)$; that is, φ is extended by 0 outside (x,y).

[10] We may use the convenient notation

$$L_2(\mathcal{F}_{x,y}) = \operatorname{Exp}\big(L_2(x,y)\big)$$

getting

$$\operatorname{Exp}\big(L_2(x,y) \oplus L_2(y,z)\big) = \operatorname{Exp}\big(L_2(x,y)\big) \otimes \operatorname{Exp}\big(L_2(y,z)\big) \,,$$

which is a special case of the well-known formula

$$\operatorname{Exp}(H_1 \oplus H_2) = \operatorname{Exp}(H_1) \otimes \operatorname{Exp}(H_2) \,.$$

(a) $X_{x,y} \in L_2(\Omega, \mathcal{F}_{x,y}, P)$;
(b) $X_{x,y} + X_{y,z} = X_{x,z}$.
Then each $X_{x,y}$ has a normal distribution.

Prove it.

Hint: recall 4c, 4d.

5c6 Exercise. Consider two families $(X_{x,y})_{x\le y}$ and $(Y_{x,y})_{x\le y}$, each satisfying 5c5(a,b) and in addition,

$$\operatorname{Var}(X_{x,y}) = y - x\,, \quad \operatorname{Var}(Y_{x,y}) = y - x\,,$$
$$\operatorname{Cov}(X_{x,y}, Y_{x,y}) = 0$$

whenever $x \le y$.

Prove that it cannot happen!

Hint: recall 4e.

What about the weaker condition $\operatorname{Var}(X_{x,y}) > 0$, $\operatorname{Var}(Y_{x,y}) > 0$ instead of being equal to $y - x$?

5c7 Exercise. Let $(X_{x,y})_{x\le y}$ and $(Y_{x,y})_{x\le y}$ be two families, each satisfying 5c5(a,b). Then for every $x \le y$, the joint distribution of $X_{x,y}$ and $Y_{x,y}$ is a two-dimensional normal distribution.

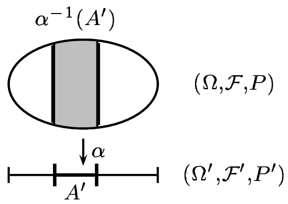
Prove it.

Hint: recall 4g.

5d Morphisms

A *morphism* of a probability space $(\Omega, \mathcal{F}, P)$ into another probability space $(\Omega', \mathcal{F}', P')$ is a measure preserving map

$$\alpha : \Omega \to \Omega'\,,$$
$$\forall A' \in \mathcal{F}' \quad \big(\alpha^{-1}(A') \in \mathcal{F},\, P(\alpha^{-1}(A')) = P'(A')\big)\,.$$



An *isomorphism* (of probability spaces) is an invertible map $\alpha : \Omega \to \Omega'$ such that both α and α^{-1} are morphisms. However (as usual in probability theory), sets of probability 0 are neglected. If an isomorphism exists, the two probability spaces are *isomorphic.* An isomorphism to itself is called *automorphism.*

By the way, if μ is a nonatomic probability measure on the Borel σ-field $\mathcal{B}$ of a Polish[11] space S, then $(S, \mathcal{B}, \mu)$ is isomorphic to $(0,1)$ with Lebesgue measure (recall 4b). If atoms are permitted, then $(S, \mathcal{B}, \mu)$ is isomorphic to $(0, m)$ (with Lebesgue measure) plus a finite or countable set of atoms (of total mass $1 - m$); here $m \in [0, 1]$. Such probability spaces are

[11] That is, a complete separable metric space. For example, $L_2(0,1)$.

known as Lebesgue-Rokhlin spaces. I always restrict myself to Lebesgue-Rokhlin probability spaces (leaving pathologies to measure theory).

Every random variable $X : \Omega \to \mathbb{R}$ is a morphism $(\Omega, \mathcal{F}, P) \to (\mathbb{R}, P_X)$, where P_X is the distribution of X, and also an isomorphism $(\Omega, \mathcal{F}, P)/\mathcal{F}_X \leftrightarrow (\mathbb{R}, P_X)$, where $\mathcal{F}_X$ is the σ-field generated by X. If $\mathcal{F}_X = \mathcal{F}$ then $(\Omega, \mathcal{F}, P)/\mathcal{F}_X = (\Omega, \mathcal{F}, P)$, and X is an isomorphism $(\Omega, \mathcal{F}, P) \leftrightarrow (\mathbb{R}, P_X)$.

If X, X' are identically distributed random variables on different probability spaces $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$ respectively, and $\mathcal{F}_X = \mathcal{F}$, $\mathcal{F}'_{X'} = \mathcal{F}'$, then the two probability spaces are isomorphic,

$$(\Omega, \mathcal{F}, P) \leftrightarrow (\mathbb{R}, P_X) \leftrightarrow (\Omega', \mathcal{F}', P')\,;$$

the isomorphism $\alpha : \Omega \to \Omega'$ satisfies $X = X' \circ \alpha$.

The same holds for d-dimensional random variables. In other words, if random variables $X_1, \dots, X_d : \Omega \to \mathbb{R}$ generate $\mathcal{F}$, random variables $Y_1, \dots, Y_d : \Omega' \to \mathbb{R}$ generate $\mathcal{F}'$, and random vectors $(X_1, \dots, X_d)$, $(Y_1, \dots, Y_d)$ are identically distributed, then the two probability spaces are isomorphic, and moreover, there exists an isomorphism $\alpha : \Omega \to \Omega'$ such that

(5d1) $$\forall k \quad X_k = Y_k \circ \alpha \quad \text{(a.s.)}\,;$$

such α is unique.

The same holds for infinite sequences $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$

It may also happen that $(\Omega, \mathcal{F}, P) = (\Omega', \mathcal{F}', P')$; then α is an automorphism.

If α is a morphism $(\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$ and $\mathcal{E}' \subset \mathcal{F}'$ is a sub-σ-field, then

$$\alpha^{-1}(\mathcal{E}') = \{\alpha^{-1}(A') : A' \in \mathcal{E}'\} \subset \mathcal{F}$$

is also a sub-σ-field.

Let $(\mathcal{F}_{x,y})$ be a factorization of $(\Omega, \mathcal{F}, P)$ over $\mathbb{R}$, and $(\mathcal{F}'_{x,y})$ a factorization of $(\Omega', \mathcal{F}', P')$ over $\mathbb{R}$. We define a *morphism of factorizations* (from $(\mathcal{F}_{x,y})$ to $(\mathcal{F}'_{x,y})$) as a morphism (of probability spaces) $\alpha : (\Omega, \mathcal{F}, P) \to (\Omega', \mathcal{F}', P')$ such that

$$\alpha^{-1}(\mathcal{F}'_{x,y}) = \mathcal{F}_{x,y} \quad \text{whenever } x \le y\,.$$

(The seemingly weaker condition $\alpha^{-1}(\mathcal{F}'_{x,y}) \subset \mathcal{F}_{x,y}$ is in fact equivalent; think, why). An *isomorphism of factorizations* is an isomorphism α of probability spaces such that both α and α^{-1} are morphisms of factorizations. If an isomorphism exists, factorizations are called isomorphic.

5d2 Exercise. The Brownian and Poisson factorizations are non-isomorphic.

Prove it by an elementary argument. Is there any morphism from the Poisson factorization to the Brownian factorization?

Hint. Think about atoms of $\mathcal{F}_{x,y}$.

5d3 Exercise. There is no morphism from the Brownian factorization to the Poisson factorization.

Prove it.

Hint: recall 4c and (5c5).

5d4 Exercise. There is no morphism from the (usual, one-dimensional) Brownian factorization to the two-dimensional Brownian factorization (that is, the factorization generated by the two-dimensional Brownian motion).

Prove it. What about a morphism in the opposite direction?

Hint: recall 4e.

5e Shift invariance

In general, a factorization may behave differently on different intervals. Say, $\mathcal{F}_{0,1}$ can be trivial,[12] while $\mathcal{F}_{1,2}$ nontrivial. Or, we may glue together the Brownian factorization on $(-\infty, 0)$ and the Poisson factorization on $(0, +\infty)$. However, such behavior does not appear for such factorizations as Brownian (or Poisson). Here, $\mathcal{F}_{x,x+a}$ and $\mathcal{F}_{y,y+a}$ are isomorphic. Moreover, we have a natural automorphism α_t of the relevant probability space such that $\alpha_t^{-1}(\mathcal{F}_{x,y}) = \mathcal{F}_{x-t,y-t}$, as we'll see soon.

5e1 Exercise. For every t there exists one and only one automorphism α_t of the white noise probability space $(\Omega, \mathcal{F}, P) = (\mathbb{R}^\infty, \gamma^\infty)$ such that

$$\Big(\int \varphi(x)\, dB(x) \Big) \circ \alpha_t = \int \varphi(x+t)\, dB(x) \quad \text{(a.s.)}$$

for all $\varphi \in L_2(\mathbb{R})$.

Prove it.

Hint. Let $(\varphi_k)_k$ be an orthonormal basis of $L_2(\mathbb{R})$ and $\psi_k(x) = \varphi_k(x+t)$, then $(\psi_k)_k$ also is an orthonormal basis of $L_2(\mathbb{R})$. Apply (5d1) to $X_k = \int \varphi_k\, dB$, $Y_k = \int \psi_k\, dB$.

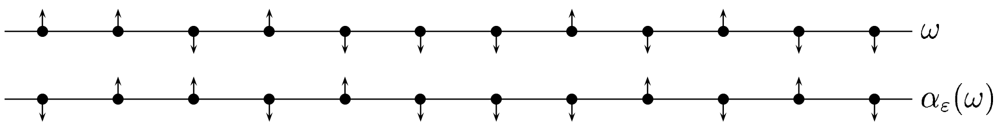
5e2 Exercise. Automorphisms α_t of 5e1 satisfy $\alpha_{s+t} = \alpha_s \circ \alpha_t$ for all $s, t \in \mathbb{R}$.

Prove it.

Hint: they are unique...

The discrete counterpart of α_t acts on the space $\Omega_\varepsilon = \{-1, +1\}^{\varepsilon\mathbb{Z}}$ of two-sided infinite sequences of ± 1; namely,

$$\Big(\sqrt{\varepsilon} \sum_k \varphi(k\varepsilon)\tau(k\varepsilon) \Big) \circ \alpha_{l\varepsilon} = \sqrt{\varepsilon} \sum_k \varphi(k\varepsilon + l\varepsilon)\tau(k\varepsilon) = \sqrt{\varepsilon} \sum_k \varphi(k\varepsilon)\tau(k\varepsilon - l\varepsilon)\,;$$
$$\tau(k\varepsilon) \circ \alpha_{l\varepsilon} = \tau(k\varepsilon - l\varepsilon)\,.$$



5e3 Exercise. Prove that

$$\big(B(y) - B(x) \big) \circ \alpha_t = B(y-t) - B(x-t)\,,$$

but $B(x) \circ \alpha_t \ne B(x-t)$. Find a formula for $B(x) \circ \alpha_t$.

[12] That is, containing sets of probability 0 or 1 only.

5e4 Exercise. Formulate and prove counterparts of 5e1 and 5e2 for the Poisson noise.

5e5 Exercise. $\alpha_t^{-1}(\mathcal{F}_{x,y}) = \mathcal{F}_{x-t,y-t}$ for all $x, y, t \in \mathbb{R}$, $x \le y$. Prove it for the two factorizations, Brownian and Poisson.

5e6 Exercise. Formulate and prove a generalization of 5e1 to

$$\Big(\sum_{n=0}^{\infty} \int \cdots \int_{x_1 < \dots < x_n} \xi(x_1, \dots, x_n) \, dB(x_1) \dots dB(x_n) \Big) \circ \alpha_t \, .$$

5f The definition of a noise

By definition, a *noise* consists of a probability space $(\Omega, \mathcal{F}, P)$, its factorization $(\mathcal{F}_{x,y})_{x \le y}$ over $\mathbb{R}$, and a one-parameter group[13] of automorphisms $(\alpha_t)_{t \in \mathbb{R}}$ such that $\alpha_t^{-1}(\mathcal{F}_{x,y}) = \mathcal{F}_{x-t,y-t}$ for all $x, y, t \in \mathbb{R}$, $x \le y$.

The two main examples are the Brownian (white) noise and the Poisson noise.

An automorphism of the noise is an automorphism $\beta : \Omega \to \Omega$ of probability spaces such that

$$\begin{aligned} &\beta^{-1}(\mathcal{F}_{x,y}) = \mathcal{F}_{x,y} \quad \text{for all } x \le y \, , \\ &\beta \circ \alpha_t = \alpha_t \circ \beta \quad \text{(a.s.)} \quad \text{for all } t \, . \end{aligned}$$

5f1 Exercise. The general form of an automorphism of the Brownian factorization is

$$\begin{gathered} \Big(\int \varphi(x) \, dB(x) \Big) \circ \beta = \int s(x) \varphi(x) \, dB(x) \, , \\ s : \mathbb{R} \to \{-1, +1\} \quad \text{(measurable)} \, . \end{gathered}$$

Prove it.

Hint: recall (4d3), and consider first a step function φ.

5f2 Exercise. The Brownian (white) noise has only two automorphisms: the trivial (identical) automorphism, and the sign change. That is,

$$\Big(\int \varphi(x) \, dB(x) \Big) \circ \beta = s \int \varphi(x) \, dB(x) \, , \quad s \in \{-1, +1\} \, .$$

Prove it.

Hint: use 5f1 and compare $\beta \circ \alpha_t$ with $\alpha_t \circ \beta$.

A sequence of i.i.d. random variables $(\eta_k)_{k \in \mathbb{Z}}$ is (or rather, generates) a discrete counterpart of a noise. Here, factorization automorphisms are of the form $\eta_k^* = f_k(\eta_k)$ (recall (4d1), (4d2)), while noise automorphisms are of the form $\eta_k^* = f(\eta_k)$. If η_k are random signs (± 1), then only two noise automorphisms exist, similarly to 5f2. In contrast, plenty of noise automorphisms exist for $\eta_k \sim \mathrm{N}(0, 1)$.

[13] That is, $\alpha_s \circ \alpha_t = \alpha_{s+t}$ for all $s, t \in \mathbb{R}$.

Let $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_{x,y})$, (α_t) form a Brownian (white) noise, and $\big(B(x)\big)_{x\in\mathbb{R}}$ be a (process distributed like a) Brownian motion, $B(x) : \Omega \to \mathbb{R}$.[14] We say that the Brownian motion is *adapted* to the noise, if

$$
\begin{gathered}
B(y) - B(x) \text{ is } \mathcal{F}_{x,y}\text{-measurable for } x \le y \,, \\
\big(B(y) - B(x)\big) \circ \alpha_t = B(y-t) - B(x-t) \text{ for } x \le y \,, t \in \mathbb{R} \,.
\end{gathered}
$$

5f3 Exercise. There exist exactly two Brownian motions adapted to the white noise. If $\big(B(x)\big)_{x\in\mathbb{R}}$ is one of them, then the other is $\big(-B(x)\big)_{x\in\mathbb{R}}$.

Prove it.

So, we have two quite different ideas of 'white noise'. One idea is described in 1b; it is a map $L_2(\mathbb{R}) \ni \varphi \mapsto \int \varphi \, dB \in L_2(\Omega)$. The other is described here, it sipulates only $(\mathcal{F}_{x,y})$ and (α_t). The latter seems to be a much weaker structure than the former, which is an illusion. They are nearly equivalent! The latter determines the former uniquely up to a (global) sign.

Let $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_{x,y})$, (α_t) form a noise. Consider probability spaces

$$
(\Omega_x, \mathcal{F}_x, P_x) = (\Omega, \mathcal{F}, P)/\mathcal{F}_{0,x}
$$

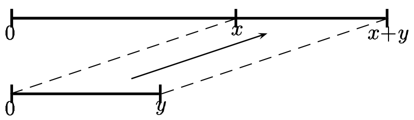
for $x \ge 0$. Informally, a point of Ω_x is a (local) configuration of the noise on $(0, x)$. A shift α_t transforms it into a configuration of the noise on $(t, x+t)$. Formally, we have a commutative diagram:

$$
\begin{array}{ccc}
(\Omega, \mathcal{F}, P) & \xrightarrow{\alpha_t} & (\Omega, \mathcal{F}, P) \\
{\scriptstyle\text{projection}}\big\downarrow & & \big\downarrow{\scriptstyle\text{projection}} \\
(\Omega, \mathcal{F}, P)/\mathcal{F}_{0,x} & \xrightarrow{\text{shift}} & (\Omega, \mathcal{F}, P)/\mathcal{F}_{t,x+t}
\end{array}
$$

(you may give to the 'shift' a more formal name, say, $\alpha_{t,x}$). However (recall (5b4)),

$$
(\Omega, \mathcal{F}, P)/\mathcal{F}_{0,x} \otimes (\Omega, \mathcal{F}, P)/\mathcal{F}_{x,x+y} \leftrightarrow (\Omega, \mathcal{F}, P)/\mathcal{F}_{0,x+y} \,;
$$

that is, a configuration on $(0, x+y)$ may be thought of as a combination of two 'smaller' configurations, on $(0, x)$ and $(x, x+y)$. Given $\omega' \in \Omega/\mathcal{F}_{0,x}$ and $\omega'' \in \Omega/\mathcal{F}_{0,y}$, we may transform the latter into an element of $\Omega/\mathcal{F}_{x,x+y}$ by a shift, and combine the two, getting $\omega \in \Omega/\mathcal{F}_{0,x+y}$ which may be called the concatenation of ω' and ω''.

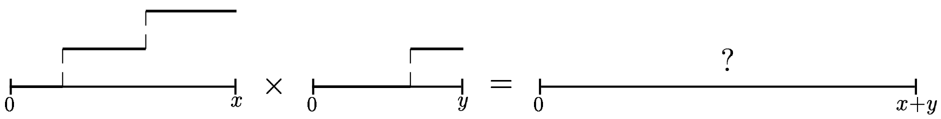


$$
(\Omega_x, \mathcal{F}_x, P_x) \times (\Omega_y, \mathcal{F}_y, P_y) \leftrightarrow (\Omega_{x+y}, \mathcal{F}_{x+y}, P_{x+y}) \,;
$$

the isomorphism denoted by '$\leftrightarrow$' is canonical, provided that α_x is given. The concatenation is associative but not commutative.

[14]For now, no relation is assumed between the noise and the Brownian motion.

5f4 Exercise. For the Brownian noise, describe the concatenation explicitly in terms of the Brownian motion. Also, do it for the Poisson noise.



Hint: recall 5e3.

In general, I do not claim that the disjoint union of all Ω_x is a semigroup, since the concatenation is defined only almost everywhere.

For the corresponding Hilbert spaces we have a concatenation-like operation of tensor multiplication:

$$H_x = L_2(\Omega_x, \mathcal{F}_x, P_x) = L_2(\Omega, \mathcal{F}_{0,x}, P) \, ,$$
$$H_x \otimes H_y \leftrightarrow H_{x+y} \, ;$$

given $u \in H_x$ and $v \in H_y$, we get $u \otimes v \in H_{x+y}$ (though, here '$\otimes$' involves the transition from $H_x \otimes H_y$ to H_{x+y}). That operation turns the disjoint union of all H_x into a semigroup. Such an object is known as a *product system* (of Hilbert spaces).[15]

In particular, for the white noise we have

$$\Big(\exp \int_0^x \varphi \, dB \Big) \otimes \Big(\exp \int_0^y \psi \, dB \Big) = \exp \bigg(\int_0^x \varphi(t) \, dB(t) + \int_x^{x+y} \psi(t-x) \, dB(t) \bigg) \, .$$

[15] I do not define it now.