4 Clashes between discrete and continuous

4a White and Poisson noises: not well together

Imagine again the one-dimensional array of random signs, the same as in 1a and 1b. Assume that devices of both types are available. These of Sect. 1,

$$X_{\varepsilon,\varphi} = \sqrt{\varepsilon} \sum_k \varphi(k\varepsilon)\tau(k\varepsilon) \, ,$$

and of Sect. 2,

$$Y_{\varepsilon,\psi} = \sum_k \varphi(k\varepsilon) \frac{1+\tau(k\varepsilon)}{2} \frac{1-\tau\big((k+1)\varepsilon\big)}{2} \dots \frac{1-\tau\big((k+m-1)\varepsilon\big)}{2} \, ;$$

test functions φ, ψ are Riemann integrable and compactly supported; k runs over $\mathbb{Z}$, but only a bounded portion of $\varepsilon\mathbb{Z}$ is relevant; and ε, m are related by $\varepsilon = 2^{-m}$.

In the limit we should get random variables X_φ, Y_ψ on a single probability space; all these X_φ should describe the white noise, as in Sect. 1; all Y_ψ — the Poisson noise, as in Sect. 2.

Have you any idea about their dependence? Especially: whether the Poisson sample path is determined uniquely by the Brownian sample path, or not?

4a1 Exercise.

$$\mathbb{E}\,\big(\tau(0)Y_{\varepsilon,\psi}\big) = 2^{-m}\big(\psi(0) - \psi(-\varepsilon) - \psi(-2\varepsilon) - \dots - \psi(-(m-1)\varepsilon)\big) \, .$$

Prove it.

Hint: open the brackets; nonlinear terms do not contribute.

4a2 Exercise.

$$\mathbb{E}\,\big(X_{\varepsilon,\varphi}Y_{\varepsilon,\psi}\big) = \sqrt{\varepsilon}2^{-m}\bigg(\sum_k \varphi(k\varepsilon)\psi(k\varepsilon) - \sum_k \varphi(k\varepsilon)\psi((k-1)\varepsilon) - \dots - \sum_k \varphi(k\varepsilon)\psi((k-m+1)\varepsilon)\bigg) \, .$$

Prove it.

Hint: use 4a1.

4a3 Exercise.

$$|\mathbb{E}\,\big(X_{\varepsilon,\varphi}Y_{\varepsilon,\psi}\big)| \le \sqrt{\varepsilon} m 2^{-m} \sqrt{\sum_k \varphi^2(k\varepsilon)} \sqrt{\sum_k \psi^2(k\varepsilon)} \, .$$

Prove it.

Hint: use 4a2.

We have $\sqrt{\varepsilon \sum_k \varphi^2(k\varepsilon)} \to \|\varphi\|_2 = \sqrt{\int \varphi^2(x)\,dx}$, and the same for ψ. Thus, $\sqrt{\varepsilon} m 2^{-m} \sqrt{\dots}\sqrt{\dots} = \sqrt{\varepsilon} m \varepsilon \sqrt{\dots}\sqrt{\dots} = (\sqrt{\varepsilon}\log_2 1/\varepsilon)\sqrt{\varepsilon\sum_k \varphi^2(k\varepsilon)}\sqrt{\varepsilon\sum_k \psi^2(k\varepsilon)} \to 0 \cdot \|\varphi\|_2\|\psi\|_2 = 0$. It means that

$$\mathbb{E}\,\big(X_\varphi Y_\psi\big) = 0 \quad \text{for all } \varphi, \psi \tag{4a4}$$

(provided that the limiting model exists, of course). The white noise and the Poisson noise appear to be uncorrelated. Does it mean that they are independent? No, it does not. (It does not contradict even to a functional dependence.) However, we have also $\mathbb{E}\big(X_{\varphi_1}X_{\varphi_2}Y_\psi\big) = \mathbb{E}\big(X_{\varphi_1}X_{\varphi_2}\big)\mathbb{E}\big(Y_\psi\big)$ for all test functions $\varphi_1, \varphi_2, \psi$, which can be proved similarly to (4a4). More generally, $\mathbb{E}\big(X_{\varphi_1}\dots X_{\varphi_n}Y_\psi\big) = \mathbb{E}\big(X_{\varphi_1}\dots X_{\varphi_n}\big)\mathbb{E}\big(Y_\psi\big)$. Still, it does not mean independence; it means rather $\mathbb{E}\big(\,Y_\psi\,\big|\,X\,\big) = \mathbb{E}\,(Y_\psi)$. The conditional expectation of Y_ψ, given the whole Brownian sample path, is equal to the unconditional expectation; it does not depend on the Brownian path. However, something else, say, $\mathbb{E}\big(\,Y_\psi^2\,\big|\,X\,\big)$, could vary. In order to get independence, we need

$$\mathbb{E}\big(X_{\varphi_1}\dots X_{\varphi_m}Y_{\psi_1}\dots Y_{\psi_n}\big) = \mathbb{E}\big(X_{\varphi_1}\dots X_{\varphi_m}\big)\mathbb{E}\big(Y_{\psi_1}\dots Y_{\psi_n}\big)\,,$$

which can be proved by some effort. For now, believe me that it is true, and implies that the two noises are independent. (We'll prove a more general fact in 4f.)

Existence of the limiting model is now trivial; it is just the product of two probability spaces constructed in Sections 1 and 2 respectively.

Usually, probabilists are inclined to say that "Brownian motion is basically an infinitesimal random walk". In other words, the white noise is "basically" the collection $(\tau(k\varepsilon))$ of random signs, ε being infinitesimal. If so, then every function of these random signs is a function of the Brownian sample path. However Y_ψ is not. For adepts of the 'infinitesimal random walk' it is a paradox.

The same paradox is manifested also by models of 1c.

However, the model of 3a does not exhaust possible ideas toward joining the two noises. In order to get them closer to each other, we may try another model of the white noise, say,

$$X_{\varepsilon,\varphi} = \sqrt{\varepsilon}\sum_k \varphi(k\varepsilon)\tau(k\varepsilon)\tau((k+1)\varepsilon)\,,$$

or even

$$X_{\varepsilon,\varphi} = \sqrt{\varepsilon}\sum_k \varphi(k\varepsilon)\tau(k\varepsilon)\tau((k+1)\varepsilon)\dots\tau((k+m-1)\varepsilon)\,, \quad \varepsilon = 2^{-m}\,.$$

On the other hand, we may replace equiprobable random signs τ_k by something else, say, $\mathbb{P}\big(\,\tau_k = 2^m - 1\,\big) = 2^{-m}$, $\mathbb{P}\big(\,\tau_k = -1\,\big) = 1 - 2^{-m}$. Then the Poisson noise emerges immediately. Though, the white noise cannot be constructed from such τ_k (think, why). Well, we could try something intermediate. Also, a continuous random variable could serve better as τ_k. You see, possible models are numerous. It would be nice to have a general theory embracing them all.

4b White and Poisson noises: a one-to-one correspondence?

The white noise is a family of random variables $X_\varphi = \int \varphi(x)\,dB(x)$ defined on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$. The Poisson noise is another family of random variables $Y_\psi = \int \psi(x)\,d\Pi(x)$ defined on another probability space $(\Omega_2, \mathcal{F}_2, P_2)$. Making the Poisson noise a function of the white noise means finding a measure preserving map $\alpha : \Omega_1 \to \Omega_2$. Then, random variables $Y_\psi \circ \alpha$, defined by $(Y_\psi \circ \alpha)(\omega_1) = Y_\psi(\alpha(\omega_1))$ for $\omega_1 \in \Omega_1$, are distributed like Y_ψ,

$$\mathbb{E} f(Y_{\psi_1} \circ \alpha, \dots, Y_{\psi_d} \circ \alpha) = \mathbb{E} f(Y_{\psi_1}, \dots, Y_{\psi_d})$$

(think, why). It means that the family of $Y_\psi \circ \alpha$ is also the Poisson noise; and, being defined on $(\Omega_1, \mathcal{F}_1, P_1)$, it has a joint distribution with the white noise. Moreover, assume that $(\Omega_1, \mathcal{F}_1, P_1)$ is the 'natural' space of the white noise; then every random variable on $(\Omega_1, \mathcal{F}_1, P_1)$ is a function of the white noise (an in particular, the Poisson noise becomes a function of the white noise). Indeed, let $(\Omega_1, \mathcal{F}_1, P_1) = (\mathbb{R}^\infty, \gamma^\infty)$ as in 1b, then every measurable function on Ω_1 is a function of coordinates (on $\mathbb{R}^\infty$), that is, $f(X_{\varphi_1}, X_{\varphi_2}, \dots)$ where $\varphi_1, , \varphi_2, \dots$ are a basis of $L_2(\mathbb{R})$.

How could we construct a measure preserving map $\Omega_1 \to \Omega_2$? First, we can decompose $(\Omega_1, \mathcal{F}_1, P_1)$ into a countable collection of random signs. Starting with a sequence $\zeta_i = X_{\varphi_i} = \int \varphi_i(x)\, dB(x)$ of independent normal $\mathrm{N}(0,1)$ random variables, we transform these into uniform $\mathrm{U}(0,1)$ random variables $U_i = \Phi(\zeta_i)$; their binary digits $\beta_{i,j}$,

$$U_i = \sum_{j=1}^{\infty} \frac{\beta_{i,j}}{2^j}, \quad \beta_{i,j} \in \{0,1\},$$

are *independent* discrete random variables with two equiprobable values $0, 1$. Now we rearrange the two-dimensional array $(\beta_{i,j})_{i,j}$ into a one-dimensional array $(\gamma_k)_k$ and consider

$$U = \sum_{k=1}^{\infty} \frac{\gamma_k}{2^k};$$

U is a random variable on $(\Omega_1, \mathcal{F}_1, P_1)$ distributed uniformly on $(0,1)$ and possessing a wonderful property: the map $\Omega_1 \ni \omega_1 \mapsto U(\omega_1) \in (0,1)$ is one-to-one! It is an *invertible* measure preserving map. We see that the infinite dimension of $(\Omega_1, \mathcal{F}_1, P_1) = (\mathbb{R}^\infty, \gamma^\infty)$ is an illusion; a measure space (in contrast to a topological space) has no dimension; our probability space is isomorphic to $(0,1)$ (with the Lebesgue measure). Therefore, the whole white noise (and Brownian motion) may be defined on $\Omega = (0,1)$.

In fact, every nonatomic probability space is isomorphic to $(0,1)$ with the Lebesgue measure. The space $(\Omega_2, \mathcal{F}_2, P_2)$ of the Poisson noise is nonatomic (as far as the Poisson process lives on the whole $\mathbb{R}$; if it is restricted to a bounded interval, then an atom appears; think, why); it is also isomorphic to $(0,1)$. Combining the two isomorphisms, we get an isomorphism $\alpha : \Omega_1 \to \Omega_2$. (Sets of measure 0 may be neglected throughout.)

So, the Poisson noise can be represented as a function of the white noise. However, this is irrelevant! An important requirement is forgotten: *locality.* For every interval $(a,b) \subset \mathbb{R}$, the Poisson noise on (a,b) should be a function of the white noise on the same interval (a,b) (rather than the whole $\mathbb{R}$).

4c White and Poisson noises: locality is prohibitive

We want to represent the Poisson increment $\Pi(b) - \Pi(a)$ as a function of the white noise on (a, b), which means

$$\text{(4c1)} \qquad \Pi(b) - \Pi(a) = \sum_{n=0}^{\infty} \underset{a<x_1<\dots<x_n<b}{\int \dots \int} \xi_{a,b}(x_1, \dots, x_n)\, dB(x_1) \dots dB(x_n) \, ;$$

here we use the general form (3c7) of a square integrable function of the white noise (a function distributed like $\Pi(b) - \Pi(a)$ must be square integrable); integration is restricted to (a, b) according to the locality requirement. We have to find $\xi_{a,b}$ (for all intervals $(a, b) \subset \mathbb{R}$) such that the process Π defined by (4c1) is (distributed like) the Poisson process. Independence of increments is ensured by locality. We need the Poisson distribution for (4c1) and, more important, additivity:

$$\text{(4c2)} \qquad \Pi(c) - \Pi(a) = \big(\Pi(b) - \Pi(a)\big) + \big(\Pi(c) - \Pi(b)\big)$$

whenever $a < b < c$. Thus, we need

$$\text{(4c3)} \qquad \underset{a<x_1<\dots<x_n<c}{\int \dots \int} \xi_{a,c}(x_1, \dots, x_n)\, dB(x_1) \dots dB(x_n) = \\ = \underset{a<x_1<\dots<x_n<b}{\int \dots \int} \xi_{a,b}(x_1, \dots, x_n)\, dB(x_1) \dots dB(x_n) + \\ + \underset{b<x_1<\dots<x_n<c}{\int \dots \int} \xi_{b,c}(x_1, \dots, x_n)\, dB(x_1) \dots dB(x_n)$$

for each n. That is necessary, because integrals for different n are orthogonal.

For $n = 0$ the stochastic integral is just a number, equal to the value of ξ at the sole point of Δ_0;

$$\xi_{a,c}() = \xi_{a,b}() + \xi_{b,c}() \, .$$

For $n = 1$ we need

$$\int_a^c \xi_{a,c}(x)\, dB(x) = \int_a^b \xi_{a,b}(x)\, dB(x) + \int_b^c \xi_{b,c}(x)\, dB(x) \, ,$$

which means $\xi_{a,c}(x) = \xi_{a,b}(x) + \xi_{b,c}(x)$ (assuming that $\xi_{x,y}$ vanishes outside (x, y)), that is,

$$\xi_{a,c}(x) = \begin{cases} \xi_{a,b}(x) & \text{for } x \in (a, b), \\ \xi_{b,c}(x) & \text{for } x \in (b, c) \end{cases}$$

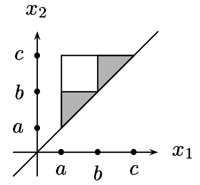
almost everywhere. These functions must be restrictions of a single function,

$$\xi_{a,b}(x) = \begin{cases} \xi(x) & \text{for } x \in (a, b), \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 2$ we need

$$\iint\limits_{a<x_1<x_2<c} \xi_{a,c}(x_1,x_2)\,dB(x_1)dB(x_2) =$$

$$= \iint\limits_{a<x_1<x_2<b} \xi_{a,b}(x_1,x_2)\,dB(x_1)dB(x_2) + \iint\limits_{b<x_1<x_2<c} \xi_{b,c}(x_1,x_2)\,dB(x_1)dB(x_2)\,,$$



which means

$$\xi_{a,c}(x_1,x_2) = \begin{cases} \xi_{a,b}(x_1,x_2) & \text{if } a < x_1 < x_2 < b, \\ \xi_{b,c}(x_1,x_2) & \text{if } b < x_1 < x_2 < c, \\ 0 & \text{if } a < x_1 < b < x_2 < c. \end{cases}$$

However, it cannot hold for all $b \in (a,c)$, unless $\xi_{a,c}(x_1,x_2) = 0$ almost everywhere (think, why). The quadratic term violates locality, therefore it must disappear!

The same for higher terms (cubic, ...); they all must disappear. So, (4c1) becomes

(4c4) $$\Pi(b) - \Pi(a) = \xi_{a,b}() + \int_a^b \xi(x)\,dB(x)\,.$$

Taking the expectation we get $\xi_{a,b}() = \mathbb{E}\big(\Pi(b) - \Pi(a)\big) = b - a$. Taking the variance we get $\int_a^b |\xi(x)|^2\,dx = \operatorname{Var}\big(\Pi(b) - \Pi(a)\big) = b - a$, therefore $\xi(x) = \pm 1$ almost everywhere. So,

$$\Pi(b) - \Pi(a) = b - a + \int_a^b \varphi(x)\,dB(x) \sim \mathrm{N}(b-a, b-a)\,, \qquad \varphi(x) = \pm 1\,,$$

which is absurd; the normal distribution is not at all equal to the Poisson distribution, $\mathrm{Poisson}(b-a)$.

So, the Poisson noise cannot be represented as a *local* function of the white noise. In contrast, the discrete counterpart of the Poisson process can be represented as a local function of the random walk.

4d White noise: locality and linearity

What is a better discrete-time counterpart of the white noise: i.i.d. random signs τ_k, or i.i.d. normal $\mathrm{N}(0,1)$ random variables ζ_k? Both give the white noise in the (natural) scaling limit, of course. However, consider local automorphisms, that is, local one-to-one measure preserving maps to itself. For random signs, these are[1]

(4d1) $$\tau_k^* = c_k \tau_k\,, \qquad c_k = \pm 1\,;$$

[1]Here I treat locality in the most restrictive way: τ_k^* depends on τ_k only. It could depend on a larger number of neighbors.

here (τ_k^*) is another copy of random signs, while c_k are non-random coefficients.

For normal random variables,

$$\zeta_k^* = f_k(\zeta_k)\,, \tag{4d2}$$

where each f_k is an invertible map $\mathbb{R} \to \mathbb{R}$ preserving the $\mathrm{N}(0,1)$ distribution; plenty of such maps exists; the linear case $f_k(\zeta_k) = \pm\zeta_k$ is very special.

For the white noise, a local automorphism may be treated as follows:

$$B^*(b) - B^*(a) = f_{a,b}\big(B|_{[a,b]}\big)\,;$$

here $B(\cdot)$ is a Brownian motion, B^* another (process distributed like) Brownian motion, and $B|_{[a,b]}$ is the restriction of B to $[a,b]$.

It is just the same situation as in 4c; though, the left-hand side of (4c1) is Poissonian (rather than Brownian), but locality and additivity requirements are the same, and similarly to (4c4) we get

$$B^*(b) - B^*(a) = \xi_{a,b}() + \int_a^b \xi(x)\,dB(x)\,.$$

Expectation gives $\xi_{a,b}() = 0$; variance gives $\xi(x) = \pm 1$; so,

$$B^*(b) - B^*(a) = \int_a^b \varphi(x)\,dB(x)\,, \qquad \varphi(x) = \pm 1\,, \tag{4d3}$$

which is similar to (4d1), not (4d2). The transformation $B(\cdot) \to B^*(\cdot)$ was not assumed linear, but appeared to be linear! Informally, $dB^*(x) = \varphi(x)\,dB(x) = \pm dB(x)$. We feel that continuous locality is much more restrictive than discrete locality.

4e White noise: locality and dimension

As noted in 4b, a probability space has no dimension; in particular, $(\mathbb{R}^1, \gamma^1)$ and $(\mathbb{R}^2, \gamma^2)$ are isomorphic; that is, one can find two functions[2] f, g such that if $\zeta \sim \mathrm{N}(0,1)$ then $\big(f(\zeta), g(\zeta)\big) \sim \mathrm{N}(0,1) \otimes \mathrm{N}(0,1)$. Not only $f(\zeta) \sim \mathrm{N}(0,1)$ and $g(\zeta) \sim \mathrm{N}(0,1)$, but also $f(\zeta), g(\zeta)$ are independent.

Can we find a *local* transformation of the (usual, one-dimensional) white noise to the two-dimensional white noise? In other words,

$$\begin{gathered} B^*(b) - B^*(a) = f_{a,b}(B|_{[a,b]})\,, \\ B^{**}(b) - B^{**}(a) = g_{a,b}(B|_{[a,b]})\,, \\ B^*(\cdot), B^{**}(\cdot) \text{ are independent Brownian motions;} \end{gathered} \tag{4e1}$$

is it possible?

As before (see (4d3)), locality implies linearity:

$$B^*(b) - B^*(a) = \int_a^b \varphi_1(x)\,dB(x)\,, \qquad \varphi_1(x) = \pm 1\,,$$

$$B^{**}(b) - B^{**}(a) = \int_a^b \varphi_2(x)\,dB(x)\,, \qquad \varphi_2(x) = \pm 1\,.$$

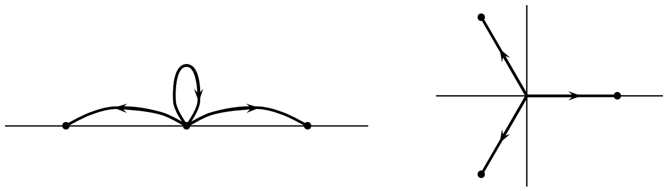
[2]Not continuous, of course; but still, Riemann integrable, if we want.

Independence of $B^*(\cdot), B^{**}(\cdot)$ implies

$$\int_a^b \varphi_1(x)\varphi_2(x)\,dx = 0$$

for all $a < b$ (think, why). It follows that $\varphi_1(x)\varphi_2(x) = 0$ almost everywhere,[3] which is absurd: $\varphi_1(x)\varphi_2(x) = \pm 1$ almost everywhere. So, (4e1) is impossible. The 2-dimensional white noise cannot be represented as a *local* function of the 1-dimensional white noise.

What about discrete counterparts? Of course, we cannot produce two independent random signs out of a single random sign. However, we can approximate 1-dim and 2-dim white noises by random walks with equal number (say, 3) of possible moves:



A 2-dimensional random walk can be a local function of a 1-dimensional random walk, even if we treat locality in the most restrictive way. Otherwise, random signs can be used, too; say,

$$\tau_k^* = \tau_k\,, \qquad \tau_k^{**} = \tau_k\tau_{k+1}\,.$$

4f White and Poisson noises: locality and independence

A single discrete model can produce (in a scaling limit) both noises, white and Poisson; an example was considered in 4a. Each one of the two noises has independent increments, of course. However, locality stipulates much more:

the two-dimensional process $\big(B(x),\Pi(x)\big)_{x\in\mathbb{R}}$ has independent increments.

If test functions φ^-,ψ^- are concentrated on $(-\infty,x)$, while φ^+,ψ^+ are concentrated on $(x,+\infty)$, then

$$\mathbb{E}\,(X_{\varphi^-}Y_{\psi^-}X_{\varphi^+}Y_{\psi^+}) = \mathbb{E}\,(X_{\varphi^-}Y_{\psi^-})\cdot\mathbb{E}\,(X_{\varphi^+}Y_{\psi^+})\,;$$

here, as before, $X_\varphi = \int \varphi\,dB$, $Y_\psi = \int \psi\,d\Pi$. The same for any finite number of test functions,

$$\text{(4f1)}\quad \mathbb{E}\,(X_{\varphi_1^-}\dots X_{\varphi_k^-}Y_{\psi_1^-}\dots Y_{\psi_l^-}X_{\varphi_1^+}\dots X_{\varphi_m^+}Y_{\psi_1^+}\dots Y_{\psi_n^+}) = \\ = \mathbb{E}\,(X_{\varphi_1^-}\dots X_{\varphi_k^-}Y_{\psi_1^-}\dots Y_{\psi_l^-})\cdot\mathbb{E}\,(X_{\varphi_1^+}\dots X_{\varphi_m^+}Y_{\psi_1^+}\dots Y_{\psi_n^+})\,,$$

and any finite number of disjoint intervals (not just $(-\infty,x)$ and $(x,+\infty)$). In the example of 4a the two noises appeared to be independent,

$$\mathbb{E}\,(X_{\varphi_1}\dots X_{\varphi_m}Y_{\psi_1}\dots Y_{\psi_n}) = \mathbb{E}\,(X_{\varphi_1}\dots X_{\varphi_m})\cdot\mathbb{E}\,(Y_{\psi_1}\dots Y_{\psi_n})\,,$$

but we still do not know, whether their independence is forced by locality, or not. We know (recall 4c) that *functional* dependence is impossible. What about some weaker dependence between $B(\cdot)$ and $\Pi(\cdot)$?

[3]Indeed, the product is orthogonal to all step functions.

4f2 Exercise.

$$\operatorname{Cov}\big(B(c)-B(a),\Pi(c)-\Pi(a)\big) = \operatorname{Cov}\big(B(b)-B(a),\Pi(b)-\Pi(a)\big)+\operatorname{Cov}\big(B(c)-B(b),\Pi(c)-\Pi(b)\big)$$

whenever $a < b < c$.

Prove it.

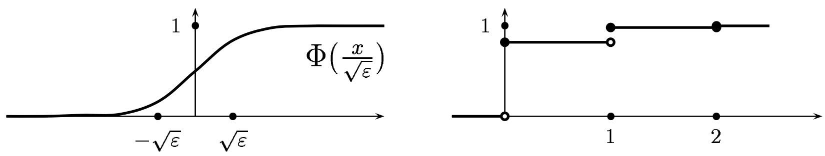
Hint: if you still do not remember what is "Cov", look at footnote 5 on page 21. If you do not remember basic properties of "Cov", just derive them from scratch.

4f3 Exercise. The correlation coefficient[4] $\operatorname{Corr}\big(B(c) - B(a), \Pi(c) - \Pi(a)\big)$ lies between $\operatorname{Corr}\big(B(b) - B(a), \Pi(b) - \Pi(a)\big)$ and $\operatorname{Corr}\big(B(c) - B(b), \Pi(c) - \Pi(b)\big)$.

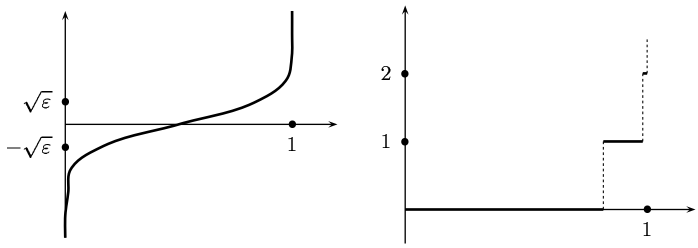
Prove it.

Hint: $\operatorname{Cov}\big(B(y) - B(x), \Pi(y) - \Pi(x)\big) = (y - x) \operatorname{Corr}\big(B(y) - B(x), \Pi(y) - \Pi(x)\big)$; use 4f2.

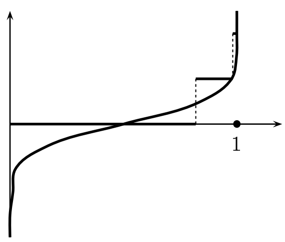
Now we want to maximize $\operatorname{Corr}\big(B(y) - B(x), \Pi(y) - \Pi(x)\big)$ for a small $y - x = \varepsilon$. We have two distributions, $\mathrm{N}(0, \varepsilon)$ and $\mathrm{Poisson}(\varepsilon)$, with their (cumulative) distribution functions



and quantile functions (called also 'inverse distribution functions')



The correlation is maximized by the monotone joining



$$X(\omega) = \sqrt{\varepsilon}\Phi^{-1}(\omega) \quad \text{for } \omega \in (0,1)\,,$$

$$Y(\omega) = \begin{cases} 0 & \text{for } \omega \in (0, e^{-\varepsilon}), \\ 1 & \text{for } \omega \in (e^{-\varepsilon}, (1+\varepsilon)e^{-\varepsilon}), \\ 2 & \text{for } \omega \in ((1+\varepsilon)e^{-\varepsilon}), (1+\varepsilon+\varepsilon^2/2)e^{-\varepsilon}), \\ \dots \end{cases}$$

Then we have

$$\mathbb{E}\,(XY) = 0 + \sqrt{\varepsilon} \int\limits_{e^{-\varepsilon}}^{(1+\varepsilon)e^{-\varepsilon}} \Phi^{-1}(p)\,dp + 2\sqrt{\varepsilon} \int\limits_{(1+\varepsilon)e^{-\varepsilon}}^{(1+\varepsilon+\varepsilon^2/2)e^{-\varepsilon}} \Phi^{-1}(p)\,dp + \dots$$

[4]Recall, $\operatorname{Corr}(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)}\sqrt{\operatorname{Var}(Y)}}$.

However, $\Phi^{-1}(1-\varepsilon) \sim \sqrt{2\ln(1/\varepsilon)}$ for $\varepsilon \to 0$. Thus, for $p \in \big(e^{-\varepsilon}, (1+\varepsilon)e^{-\varepsilon}\big)$ we have $\Phi^{-1}(p) \lesssim \sqrt{2\ln(2/\varepsilon^2)}$, and

$$\int_{e^{-\varepsilon}}^{(1+\varepsilon)e^{-\varepsilon}} \Phi^{-1}(p)\,dp \lesssim \varepsilon\sqrt{2\ln(2/\varepsilon^2)}\,,$$

which is small for small ε. We feel that $\mathbb{E}\,(XY) = o(\sqrt{\varepsilon})$. Here is a way to a proof.

4f4 Exercise. For every $p \in [1,2]$, $\mathbb{E}\,(Y^p) = O(\varepsilon)$ for $\varepsilon \to 0$.
Prove it.
Hint: $Y^p \le Y^2$; $\mathbb{E}\,Y = \varepsilon$, $\operatorname{Var}(Y) = \varepsilon$.

4f5 Exercise. $\mathbb{E}\,|XY| = o(\varepsilon)$ for $\varepsilon \to 0$.
Prove it.
Hint: $\mathbb{E}\,|XY| \le \|X\|_4 \|Y\|_{4/3} = O(\sqrt{\varepsilon}) \cdot O(\varepsilon^{3/4})$.

We see that $|\operatorname{Cov}(B(y)-B(x), \Pi(y)-\Pi(x))| = |\mathbb{E}\,\big((B(y)-B(x))(\Pi(y)-\Pi(x))\big)| = o(y-x)$ for $y-x \to 0+$; therefore

$$\operatorname{Corr}(B(y)-B(x), \Pi(y)-\Pi(x)) \to 0 \quad \text{for } y-x \to 0+\ .$$

Combined with 4f3 it gives us

$$\operatorname{Corr}(B(y)-B(x), \Pi(y)-\Pi(x)) = 0$$

for all $x < y$.

Does it mean that $B(\cdot)$ and $\Pi(\cdot)$ are independent? No, it does not. They are just uncorrelated. Maybe, $(B(y)-B(x))^m$ and $(\Pi(y)-\Pi(x))^n$ are correlated for some m,n?

4f6 Exercise. For any $\lambda, \mu \in \mathbb{R}$ the function

$$f(x,y) = \mathbb{E}\,\exp\big(i\lambda(B(y)-B(x)) + i\mu(\Pi(y)-\Pi(x))\big)$$

satisfies

$$f(x,y)f(y,z) = f(x,z) \quad \text{whenever } x<y<z\,.$$

Prove it.

4f7 Exercise.

$$f(x,y) = \exp\Big(-\frac{1}{2}\varepsilon\lambda^2 + \varepsilon(e^{i\mu}-1)\Big) + o(\varepsilon)$$

when $y-x=\varepsilon \to 0+$.
Prove it.
Hint: $\mathbb{E}\,|e^{i\lambda B(\varepsilon)}-1||e^{i\mu\Pi(\varepsilon)}-1| \le \|e^{i\lambda B(\varepsilon)}-1\|_4 \|e^{i\mu\Pi(\varepsilon)}-1\|_{4/3} \le \|\lambda B(\varepsilon)\|_4 \|\mu\Pi(\varepsilon)\|_{4/3} = o(\varepsilon)$ similarly to 4f5. Also, $\mathbb{E}\,e^{i\lambda B(\varepsilon)} = \exp(-\frac{1}{2}\varepsilon\lambda^2)$ and $\mathbb{E}\,e^{i\mu\Pi(\varepsilon)} = \exp\big(\varepsilon(e^{i\mu}-1)\big)$.

4f8 Exercise.

$$f(x,y) = \exp\Big(-\frac{1}{2}\varepsilon\lambda^2 + \varepsilon(e^{i\mu}-1)\Big)$$

for all $y - x = \varepsilon > 0$; that is,

$$\mathbb{E}\Big(\exp\big(i\lambda(B(y)-B(x))\big) \exp\big(i\mu(\Pi(y)-\Pi(x))\big)\Big) = \\ = \Big(\mathbb{E}\exp\big(i\lambda(B(y)-B(x))\big)\Big)\Big(\mathbb{E}\exp\big(i\mu(\Pi(y)-\Pi(x))\big)\Big)\,.$$

Prove it.

4f9 Exercise.

$$\mathbb{E}\bigg(\exp\bigg(i\lambda\int_a^b \varphi(x)\,dB(x)\bigg) \exp\bigg(i\mu\int_a^b \psi(x)\,d\Pi(x)\bigg)\bigg) = \\ = \bigg(\mathbb{E}\exp\bigg(i\lambda\int_a^b \varphi(x)\,dB(x)\bigg)\bigg)\bigg(\mathbb{E}\exp\bigg(i\mu\int_a^b \psi(x)\,d\Pi(x)\bigg)\bigg)$$

for all $\varphi, \psi \in L_2(\mathbb{R})$.

Prove it.

Hint: first, prove it for *step* functions φ, ψ, using 4f6, 4f8.

4f10 Exercise. For any $\varphi, \psi \in L_2(\mathbb{R})$, random variables $X_\varphi = \int \varphi(x)\,dB(x)$ and $Y_\psi = \int \psi(x)\,d\Pi(x)$ are independent.

Prove it.

Hint: use 4f9; trigonometric polynomials are dense.

Independence of random variables X_φ, Y_ψ is not yet independence of B and Π. We need independence of random vectors $(X_{\varphi_1}, \dots, X_{\varphi_m})$ and $(Y_{\psi_1}, \dots, Y_{\psi_n})$. However, 4f9 is enough for that, since trigonometric polynomials of several variables are also dense.

So, *locally dependent* white noise and Poisson noise are necessarily independent. (The result of 4a is a very special case.)

4g Two white noises: locality and normal correlation

A single discrete model can produce (in a scaling limit) two independent white noises. Some examples were given in 1c; say,

(4g1) $$X_{\varepsilon,\varphi} = \sqrt{\varepsilon}\sum_k \varphi(k\varepsilon)\tau(k\varepsilon) \quad\text{and}\quad Y_{\varepsilon,\varphi} = \sqrt{\varepsilon}\sum_k (-1)^k\varphi(k\varepsilon)\tau(k\varepsilon)\,.$$

Two *normally correlated* white noises can appear; say,[5]

(4g2) $$X_{\varepsilon,\varphi} = \sqrt{\varepsilon}\sum_k \varphi(k\varepsilon)\tau(k\varepsilon) \quad\text{and}\quad Y_{\varepsilon,\varphi} = \sqrt{2\varepsilon}\sum_k \varphi(2k\varepsilon)\tau(2k\varepsilon)\,.$$

[5]Do you see any relation to the fact noted before 1c3? I mean this fact: if the pair $(B_1(\cdot), B_2(\cdot))$ is a 2-dimensional Brownian motion, then the process $B_1(\cdot)\cos\alpha + B_2(\cdot)\sin\alpha$ is another Brownian motion.

4g3 Exercise. For (4g2), calculate the correlation, $\operatorname{Lim}\operatorname{Corr}(X_{\varepsilon,\varphi}, Y_{\varepsilon,\varphi})$.

We want to examine two *locally dependent* white noises, or equivalently, Brownian motions $B_1(\cdot), B_2(\cdot)$. Recall that a local *functional* dependence was examined in 4d; it is always linear, $dB_2(x) = \varphi(x)\,dB_1(x)$. What about some weaker dependence between $B_1(\cdot)$ and $B_2(\cdot)$? Similarly to 4f, locality means that

the two-dimensional process $\big(B_1(x), B_2(x)\big)_{x\in\mathbb{R}}$ has independent increments.

Note that two-dimensional distributions are not assumed to be normal. Similarly to 4f2, the function

$$r(x,y) = \operatorname{Cov}\big(B_1(y) - B_1(x), B_2(y) - B_2(x)\big)$$

is additive:

$$r(x,y) + r(y,z) = r(x,z) \qquad \text{for } x < y < z\,.$$

Also,

$$|r(x,y)| \le y - x \qquad \text{for } x < y$$

(think, why). It follows[6] that

$$r(x,y) = \int_x^y \rho(u)\,du$$

for some $\rho(\cdot) \in L_\infty(\mathbb{R})$, $|\rho(\cdot)| \le 1$. Informally, $\rho(x) = \operatorname{Corr}\big(dB_1(x), dB_2(x)\big)$.

Similarly to 4f6, we introduce the function

$$f(x,y) = \mathbb{E}\exp\Big(i\lambda\big(B_1(y) - B_1(x)\big) + i\mu\big(B_2(y) - B_2(x)\big)\Big)$$

and note that

$$f(x,y)f(y,z) = f(x,z) \qquad \text{whenever } x < y < z\,.$$

4g4 Exercise.

$$f(x,y) = 1 - \frac12\mathbb{E}\Big(\lambda\big(B_1(y) - B_1(x)\big) + \mu\big(B_2(y) - B_2(x)\big)\Big)^2 + o(\varepsilon)$$

when $y - x = \varepsilon \to 0+$.

Prove it.

Thus,

$$f(x,y) = 1 - \frac{\lambda^2}{2}\varepsilon - \frac{\mu^2}{2}\varepsilon - \lambda\mu r(x,y) + o(\varepsilon) = \\ = \exp\bigg(-\int_x^y \Big(\frac{\lambda^2+\mu^2}{2} + \lambda\mu\rho(u)\Big)\,du\bigg) + o(\varepsilon)$$

and, similarly to 4f8, "$o(\varepsilon)$" disappears! We see that $\ln f(x,y)$ is quadratic in λ, μ, which means that the joint distribution of $B_1(y) - B_1(x)$ and $B_2(y) - B_2(x)$ is a two-dimensional normal distribution.

[6] You see, the function $x \mapsto r(0,x)$ is absolutely continuous.

So, *locally dependent* white noises are necessarily jointly normal. They can be represented as linear combinations[7] of *independent* white noises.

Let $B(\cdot)$ be a Brownian motion. Consider the random variable

$$X_n = \big(B(\tfrac{1}{n})\big)^2 + \big(B(\tfrac{2}{n}) - B(\tfrac{1}{n})\big)^2 + \dots + \big(B(1) - B(\tfrac{n-1}{n})\big)^2 \,;$$

we have $\mathbb{E} X_n = 1$ and $\operatorname{Var}(X_n) = 2/n$ (think, why). Due to Central Limit Theorem, $\sqrt{\frac{n}{2}}(X_n - 1)$ is approximately normal, $\mathrm{N}(0,1)$, for large n. Thus, we may introduce a random process

$$X_n(x) = \sqrt{\frac{n}{2}} \sum_{i=1}^{k} \bigg(\Big(B\big(\tfrac{i}{n}\big) - B\big(\tfrac{i-1}{n}\big) \Big)^2 - \tfrac{1}{n} \bigg) \qquad \text{for } \tfrac{k}{n} \le x < \tfrac{k+1}{n} \,.$$

When $n \to \infty$, it converges (in distribution) to a Brownian motion. However, it is a quadratic function of $B(\cdot)$. In the limit, we should get two locally dependent Brownian motions, one being a quadratic function of the other. We know that it cannot happen. Think about the paradox.[8]

Think also about the scaling limit of the pair

$$X_{\varepsilon,\varphi} = \sqrt{\varepsilon} \sum_k \varphi(k\varepsilon)\tau(k\varepsilon) \,, \quad Y_{\varepsilon,\varphi} = \sqrt{\varepsilon} \sum_k \varphi(k\varepsilon)\tau(k\varepsilon)\tau((k+1)\varepsilon) \,.$$

[7] If the correlation coefficient ρ is not constant, then coefficients of these linear combinations are also not constant.

[8] More generally, you may consider

$$\frac{1}{\sqrt{m! n}} \sum_{i=1}^{n} H_m \Big(\sqrt{n} \big(B(\tfrac{i}{n}) - B(\tfrac{i-1}{n}) \big) \Big) \,,$$

where H_m is the Hermite polynomial.