
A random sequence of convex polygons is generated by picking two edges of the current polygon at random, joining their midpoints, and picking one of the two resulting smaller polygons at random to be the next in the sequence. Let $X_n + 3$ be the number of edges of the nth polygon thus constructed. Find $\mathbb{E}(X_n)$ in terms of X_0, and find the stationary distribution of the Markov chain X.

Hint: $\pi(x) - \pi(x+1) = \frac{\pi(x-1)}{x+1}$; $\pi(1) = \pi(0)$; $\pi(2) = \frac{1}{2}\pi(0)$; $\dots$; $\pi(x) = \frac{1}{x!}\mathrm{e}^{-1}$.

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A triangle splits into a triangle and a quadrangle, thus, $\mathbb{P}\left(X_1 + 3 = 3 \mid X_0 + 3 = 3 \right) = \mathbb{P}\left(X_1 + 3 = 4 \mid X_0 + 3 = 3 \right) = 1/2$, that is, $p_{0,0} = p_{0,1} = 1/2$. Similarly, $p_{1,0} = p_{1,1} = p_{1,2} = 1/3$; in general, $p_{i,j} = 1/(i+2)$ for $j = 0, 1, \dots, i, i+1$ (and 0 for other j). The conditional distribution of X_{n+1} given $X_n = i$ is uniform on $\{0, 1, \dots, i+1\}$. The conditional expectation follows: $\mathbb{E}\left(X_{n+1} \mid X_n = i \right) = \frac{0+(i+1)}{2}$, that is, $\mathbb{E}\left(X_{n+1} \mid X_n \right) = (X_n+1)/2$. The unconditional expectation: $\mathbb{E}(X_{n+1}) = \mathbb{E}((X_n+1)/2) = \frac{1}{2}(1 + \mathbb{E}(X_n))$. Denoting $a_n = \mathbb{E}(X_n)$ we have $a_{n+1} = \frac{1}{2}(1 + a_n)$;

$$a_1 = \frac{1}{2} + \frac{1}{2}a_0\,; \quad a_2 = \frac{1}{2} + \frac{1}{2}a_1 = \frac{3}{4} + \frac{1}{4}a_0\,; \quad a_3 = \frac{1}{2} + \frac{1}{2}a_2 = \frac{7}{8} + \frac{1}{8}a_0\,; \dots$$

$$a_n = \frac{2^n - 1}{2^n} + \frac{a_0}{2^n}\,; \quad \mathbb{E}(X_n) = 1 + \frac{\mathbb{E}(X_0) - 1}{2^n}\,.$$

The stationary distribution satisfies $\pi(y) = \sum_x \pi(x) p_{x,y}$, that is,

$$\pi(j) = \sum_{i=j-1}^{\infty} \frac{\pi(i)}{i+2} \quad \text{for } j > 0, \qquad \text{and } \pi(0) = \sum_{i=0}^{\infty} \frac{\pi(i)}{i+2}\,.$$

It follows that

$$\pi(j) - \pi(j+1) = \frac{\pi(j-1)}{j+1} \quad \text{for } j > 0, \qquad \text{and } \pi(0) - \pi(1) = 0\,.$$

Thus, $\pi(1) = \pi(0)$; $\pi(2) = \pi(1) - \frac{1}{2}\pi(0) = \frac{1}{2}\pi(0)$; $\pi(3) = \pi(2) - \frac{1}{3}\pi(1) = \frac{1}{2}\pi(0) - \frac{1}{3}\pi(0) = \frac{1}{6}\pi(0)$; $\dots$; $\pi(j) = \frac{1}{j!}\pi(0)$. However, $1 = \sum_{j=0}^{\infty} \pi(j) = \pi(0) \sum_{j=0}^{\infty} \frac{1}{j!} = \pi(0) \cdot \mathrm{e}$; $\pi(0) = \mathrm{e}^{-1}$; finally, $\pi(j) = \frac{1}{j!}\mathrm{e}^{-1}$ (just the Poisson distribution with parameter 1).

Martingale Markov chains. Let X_n be a Markov chain with state space $\{0, 1, \dots, N\}$ and suppose that X_n is a martingale. (a) Show that 0 and N must be absorbing states. (b) Let $\tau = V_0 \wedge V_N$. Suppose $P_x(\tau < \infty) > 0$ for $1 < x < N$. Show that $P_x(\tau < \infty) = 1$ and $P_x(V_N < V_0) = x/N$.

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(a) On one hand, $\mathbb{E}\left(X_1 \mid X_0 = 0 \right) = 0$; on the other hand, X_1 cannot be negative; therefore $\mathbb{P}\left(X_1 = 0 \mid X_0 = 0 \right) = 1$. Similarly, $\mathbb{P}\left(X_1 = N \mid X_0 = N \right) = 1$.

(b) The states $2, \dots, N-1$ are transient, therefore, visited only a finite number of times, which means that $\tau < \infty$ a.s.

We have $\mathbb{E}_x(X_n) = x$, that is, $\sum_{y=0}^{N} y \cdot p^n(x,y) = x$. However, $p^n(x,y) \to 0$ for all transient states y; thus, $p^n(x,0) + p^n(x,N) \to 1$ and $N \cdot p^n(x,N) \to x$ as $n \to \infty$, which means $N \cdot P_x(V_N < V_0) = x$.
