
N black balls and N white balls are placed in two urns so that each contains N balls. After each unit of time one ball is selected at random from each urn, and the two balls thus selected are interchanged. Let the number of black balls in the first urn denote the state of the system. Write down the transition matrix of this Markov chain and find the unique stationary distribution. Is the chain reversible in equilibrium?

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The first urn contains i black balls and $N-i$ white balls; the second urn contains i white balls and $N-i$ black balls. The transition $i \to i+1$ occurs if the ball selected from the first urn is white and the ball selected from the second urn is black. Thus,

$$p_{i,i+1} = \frac{N-i}{N} \cdot \frac{N-i}{N} = \frac{(N-i)^2}{N^2} \,.$$

Similarly,

$$p_{i,i-1} = \frac{i^2}{N^2} \,; \quad p_{i,i} = 2\frac{i(N-i)}{N^2} \,.$$

The chain must be reversible in equilibrium, since its graph has no loops. The stationary probabilities μ_i satisfy the detailed balance condition

$$\mu_i p_{i,i+1} = \mu_{i+1} p_{i+1,i} \,.$$

We have

$$\frac{\mu_{i+1}}{\mu_i} = \frac{p_{i,i+1}}{p_{i+1,i}} = \left(\frac{N-i}{i+1} \right)^2 ;$$

$$\frac{\mu_1}{\mu_0} = \left(\frac{N}{1} \right)^2 = \binom{N}{1}^2 ; \quad \frac{\mu_2}{\mu_0} = \left(\frac{N}{1} \cdot \frac{N-1}{2} \right)^2 = \binom{N}{2}^2 ; \dots$$

$$\frac{\mu_i}{\mu_0} = \binom{N}{i}^2 \quad \text{(by induction).}$$

Therefore

$$\mu_i = \frac{\binom{N}{i}^2}{\binom{N}{0}^2 + \binom{N}{1}^2 + \dots + \binom{N}{N}^2} \,.$$

In fact, $\binom{N}{0}^2 + \binom{N}{1}^2 + \dots + \binom{N}{N}^2 = \binom{2N}{N}$, thus, $\mu_i = \frac{\binom{N}{i}^2}{\binom{2N}{N}}$.
