0 Lebesgue integration (a summary)

0a Measurability

0a1. A function $f : \mathbb{R} \to \mathbb{R}$ is *measurable* if and only if for every $\varepsilon > 0$ there exist intervals $(a_1, b_1), (a_2, b_2), \dots \subset \mathbb{R}$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\sum_{k=1}^{\infty} (b_k - a_k) \le \varepsilon \,,$$

$$f(x) = g(x) \quad \text{for all } x \in \mathbb{R} \setminus \bigcup_k (a_k, b_k) \,.$$

The same holds for $f : [0,1] \to \mathbb{C}$ etc.

0a2. A function $f : \mathbb{R} \to \mathbb{R}$ vanishes *almost everywhere* if and only if it satisfies 0a1 with $g(\cdot) = 0$ (for all ε).

0a3. A set $A \subset \mathbb{R}$ is a *null set* if and only if its indicator $\mathbb{1}_A(\cdot)$ vanishes almost everywhere. Every subset of a null set is a null set. A finite or countable union of null sets is a null set.

0a4. A function $f : \mathbb{R} \to \mathbb{R}$ is measurable if and only if there exist continuous functions $f_1, f_2, \dots : \mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ almost everywhere (that is, outside a null set).

0a5. If $f, g : \mathbb{R} \to \mathbb{R}$ are measurable and $h : \mathbb{R}^2 \to \mathbb{R}$ is continuous then the function $x \mapsto h(f(x), g(x))$ is measurable. In particular, $f + g$ and fg are measurable.

0a6. If $f_1, f_2, \dots : \mathbb{R} \to \mathbb{R}$ are measurable, $f : \mathbb{R} \to \mathbb{R}$ and $f_n \to f$ almost everywhere, then f is measurable.

0b Integral

0b1. For every measurable function $f : [0,1] \to [-1,1]$ there exists a number $c \in \mathbb{R}$ such that

$$\int_0^1 f_n(x) \, \mathrm{d}x \to c \quad \text{as } n \to \infty$$

for every sequence of continuous functions $f_1, f_2, \dots : [0,1] \to [-1,1]$ such that $f_n \to f$ almost everywhere. The number c is called the *Lebesgue integral* of f and denoted (like the Riemann integral) by $\int_0^1 f(x) \, \mathrm{d}x$ (or just $\int f$).

The same holds for $f : [a,b] \to [c,d]$, a bounded $f : [a,b] \to \mathbb{C}$, etc.

0b2. A measurable function $f : \mathbb{R} \to \mathbb{R}$ is *integrable* if and only if

$$\sup_n \int_{-n}^{n} \min(n, |f(x)|) \, \mathrm{d}x < \infty \, .$$

0b3. For every integrable $f : \mathbb{R} \to \mathbb{R}$ the limit

$$\lim_{a,b,c,d \to +\infty} \int_{-a}^{b} \operatorname{mid}(-c, f(x), d) \, \mathrm{d}x$$

exists. It is called the Lebesgue integral of f and denoted by $\int_{-\infty}^{+\infty} f(x) \, \mathrm{d}x$ (or just $\int f$).

0b4. $\int cf = c \int f$; $\int (f + g) = \int f + \int g$; if $f \le g$ almost everywhere then $\int f \le \int g$.

0b5. It may happen that f and all f_n are integrable and $f_n \to f$ almost everywhere and $\lim_n \int f_n$ exists, but $\int f \ne \lim_n \int f_n$.

0b6. (Dominated convergence theorem) Let f, g and all f_n be integrable functions such that $f_n \to f$ almost everywhere, and $|f_n(\cdot)| \le g(\cdot)$ almost everywhere, for all n. Then $\int f_n \to \int f$.

0b7. (Fatou's lemma) Let $f_n : \mathbb{R} \to [0, \infty)$ be measurable, then

$$\int \Big(\liminf_n f_n(x) \Big) \, \mathrm{d}x \le \liminf_n \int f_n(x) \, \mathrm{d}x \, .$$

Corollary. If $f_n \to f$ almost everywhere and $\lim_n \int f_n$ exists, then $\int f \le \lim_n \int f_n$.

0c The Hilbert space L_2

0c1. Equivalence classes of measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\int |f(x)|^2 \, \mathrm{d}x < \infty$$

become a Hilbert space, denoted by $L_2(\mathbb{R})$, being equipped with the usual linear operations and the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x) \overline{g(x)} \, \mathrm{d}x \, .$$

0c2. Continuous functions with compact supports are dense in $L_2(\mathbb{R})$.

0d The Banach space L_∞

0d1. Equivalence classes of measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\operatorname{ess\,sup} |f(\cdot)| < \infty$$

become a Banach space, denoted by $L_\infty(\mathbb{R})$, being equipped with the usual linear operations and the norm

$$\|f\| = \operatorname{ess\,sup} |f(\cdot)| \, .$$

Recall that $\operatorname{ess\,sup} f(\cdot)$ is the least c such that $f(\cdot) \le c$ almost everywhere.