3 Sensitivity and superconcentration

3a Theory

3a1 Definition. The Gaussian measure γ_t^d on $\mathbb{R}^{2d}$, for $t \in [0,\infty)$, is the joint distribution of two random vectors X and

$$X_t = \mathrm{e}^{-t} X + \sqrt{1-\mathrm{e}^{-2t}} X'$$

where X and X' are independent random vectors distributed γ^d each.

That is, γ_t^d is the image of γ^{2d} under the linear map $(x,x') \mapsto \big(x, \mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}x'\big)$ for $x,x' \in \mathbb{R}^d$. Note that $\gamma_t^d = (\gamma_t^1)^d$.

For small t, X_t is treated as a small perturbation of X. Whether $f(X_t)$ is a small perturbation of $f(X)$ or not, is a matter of sensitivity of a function f.

3a2 Lemma. For every $f \in L_2(\mathbb{R}^d, \gamma^d)$ the function

$$t \mapsto \iint f(x) f(y)\, \gamma_t^d(\mathrm{d}x\mathrm{d}y) \qquad = \mathbb{E}\,\big(f(X)f(X_t)\big)$$

is nonnegative and decreasing on $[0,\infty)$.

The same holds for a vector-function $f : \mathbb{R}^d \to \mathbb{R}^d$ with the scalar product $\langle f(x), f(y)\rangle$.

We consider a random variable of the form (1c1):

$$\xi(X) = \max_{a\in A} \langle X, a\rangle\,,$$

where $X \sim \gamma^d$, and $A \subset \mathbb{R}^d$ is a given finite set, $\forall a \in A \;\; |a| = 1$. We also introduce the A-valued random vector $\alpha(X)$ where $\alpha : \mathbb{R}^d \to A$ is defined (almost everywhere) by

$$\langle X, \alpha(X)\rangle = \max_{a\in A} \langle X, a\rangle = \xi(X)\,.$$

[1]This section is a lightweight introduction into a work of Chatterjee [1].

ASSUMPTION D_n ("SUPERCONCENTRATION"):

$$\operatorname{Var}(\xi) \le \frac{1}{n}\,.$$

Let $(X, X_t) \sim \gamma_t^d$; we compare $\alpha(X)$ and $\alpha(X_t)$ as follows.

ASSUMPTION E_n ("SENSITIVITY"):

$$\mathbb{E}\,\langle \alpha(X), \alpha(X_{1/n}) \rangle \le \frac{1}{n}\,.$$

3a3 Theorem. For every n large enough,
(a) assumption D_{2n^2} implies assumption E_n;
(b) assumption E_{2n} implies assumption D_n.

Thus, sensitivity and superconcentration are equivalent.

The key to the proof of 3a3 is a wonderful equality

$$\operatorname{Var}(\xi) = \int_0^\infty \big(\mathbb{E}\,\langle \alpha(X), \alpha(X_t) \rangle \big) \mathrm{e}^{-t}\,\mathrm{d}t\,. \tag{3a4}$$

3b Application: first-time percolation

Recall the first-time percolation of Sect. 1d:

$$X_L = \sum_k X_{k,l_k} \quad \text{and} \quad \xi_m = \frac{1}{\sqrt{m}} \max_L X_L\,.$$

It appears that

$$\operatorname{Var}(\xi_m) \to 0 \quad \text{as } m \to \infty$$

(superconcentration). Sensitivity follows! Note that $\langle \alpha(X), \alpha(X_t) \rangle$ is the overlap $\frac{1}{m}|L \cap L_t|$ between two optimal paths L and L_t.

References

[1] S. Chatterjee, *Chaos, concentration, and multiple valleys,* arXiv:0810.4221.