23 Random real zeroes: no derivatives

23a Random element of $L_2[0,1]$

Continuing Sect. 22d, we consider a Gaussian process

$$\Xi : [0,1] \to G \subset L_2(\Omega, P) \,, \quad \Xi(t) = f_1(t) g_1 + f_2(t) g_2 + \dots \,,$$

where $(g_1, g_2, \dots)$ is an orthonormal basis of G, and $f_k(t) = \langle \Xi(t), g_k \rangle$ are measurable. Necessarily,[1]

$$\forall t \in [0,1] \quad |f_1(t)|^2 + |f_2(t)|^2 + \dots = \|\Xi(t)\|^2 < \infty \,.$$

We upgrade Ξ to the corresponding random element of $L_0[0,1]$ (as explained in Sect. 22d), denoted by $X : \Omega \to L_0[0,1]$. In general, $\int_0^1 \|\Xi(t)\|^2 \, \mathrm{d}t = \sum_k \int |f_k(t)|^2 \, \mathrm{d}t$ need not be finite. From now on we assume that it is:

$$\int_0^1 \|\Xi(t)\|^2 \, \mathrm{d}t < \infty \,;$$

then, by Tonelli's theorem,

$$\mathbb{E} \int_0^1 |X(t)|^2 \, \mathrm{d}t = \int_0^1 \big(\mathbb{E} \, |X(t)|^2 \big) \, \mathrm{d}t = \int_0^1 \|\Xi(t)\|^2 \, \mathrm{d}t < \infty \,,$$

which shows that X is in fact a random element of $L_2[0,1]$. We approximate X by another random element X_n of $L_2[0,1]$,

$$X_n(t) = g_1 f_1(t) + \dots + g_n f_n(t) \,.$$

We may also treat X and X_n as elements of $L_2\big([0,1] \times \Omega\big)$.

23a1 Exercise. $X_n \to X$ in $L_2\big([0,1] \times \Omega\big)$. [2]
Prove it.

[1] This is also sufficient (think, why).
[2] In fact, almost surely the series converges in $L_2(0,1)$.

23a2 Exercise. For every $f \in L_2[0,1]$ the random variables $\langle f, X_n \rangle = \langle f_1, f \rangle g_1 + \dots + \langle f_n, f \rangle g_n$ converge (as $n \to \infty$) in $L_2(\Omega)$ to the random variable $\langle f, X \rangle = \int_0^1 f(t) X(t) \, \mathrm{d}t$.

Prove it.

Thus,

$$\operatorname{Var}\langle f, X \rangle = \sum_k |\langle f, f_k \rangle|^2 \le C \|f\|^2$$

for some $C \le \sum_k \|f_k\|^2 = \int_0^1 \|\Xi(t)\|^2 \, \mathrm{d}t < \infty$.

23a3 Proposition. Let C be such that

$$\forall f \in L_2[0,1] \quad \operatorname{Var}\langle f, X \rangle \le C \|f\|^2 \, .$$

Let $\psi : L_2[0,1] \to \mathbb{R}$ be a Lip(1) function. Then the random variable $\psi(X)$ belongs to GaussLip($\sqrt{C}$).

First, we need the duality argument used already in 11c3.

23a4 Lemma. $\|a_1 f_1 + a_2 f_2 + \dots \|^2 \le C(a_1^2 + a_2^2 + \dots)$ for all $(a_1, a_2, \dots) \in l_2$.

Proof. We introduce a linear operator $S : l_2 \to L_2[0,1]$ by $Sa = \sum a_k f_k$; the series converges in $L_2[0,1]$, since $\sum \|a_k f_k\| = \sum |a_k| \cdot \|f_k\| \le \big(\sum |a_k|^2 \big)^{1/2} \big(\sum \|f_k\|^2 \big)^{1/2} < \infty$. We have $\forall a \in l_2 \;\; \forall f \in L_2[0,1] \;\; \langle f, Sa \rangle = \langle S^* f, a \rangle$, where $S^* : L_2[0,1] \to l_2$, $S^* f = \big(\langle f, f_1 \rangle, \langle f, f_2 \rangle, \dots \big)$.

We note that $\operatorname{Var}\langle f, X \rangle = \|S^* f\|^2$; thus, $\|S^* f\|^2 \le C \|f\|^2$ for all f. Finally,

$$\|Sa\| = \sup_{\|f\| \le 1} \langle f, Sa \rangle = \sup_{\|f\| \le 1} \langle S^* f, a \rangle \le \sup_{\|f\| \le 1} \|S^* f\| \|a\| \le \sqrt{C} \|a\| \, .$$

□

Proof of the proposition. Similarly to the proof of 22d5 we assume that $(\Omega, P) = (\mathbb{R}^\infty, \gamma^\infty)$, g_k are the coordinates, and will prove that $\psi(X)$ is a Lip($\sqrt{C}$) function on $(\mathbb{R}^\infty, \gamma^\infty)$.

We take $n_1 < n_2 < \dots$ such that[1] $\sum_{i=1}^{n_k} f_i g_i \to X$ (as $k \to \infty$) almost everywhere on $[0,1] \times \Omega$.

[1] In fact, $n_k = k$ fit.

Given $a \in l_2$, we introduce $h = a_1 f_1 + a_2 f_2 + \dots \in L_2[0,1]$;[1] $\|h\|^2 \le C\|a\|^2$ by 23a4. For almost all $(t,x) \in [0,1] \times (\mathbb{R}^\infty, \gamma^\infty)$ we have

$$X(x+a,t) - X(x,t) = \lim_k \sum_{i=1}^{n_k} (x_i + a_i) f_i(t) - \lim_k \sum_{i=1}^{n_k} x_i f_i(t) = \\ = \lim_k \sum_{i=1}^{n_k} a_i f_i(t) = h(t) \, .$$

Thus, $X(x+a) - X(x) = h$ for almost all $x \in (\mathbb{R}^\infty, \gamma^\infty)$. Finally,

$$|\psi\big(X(x+a)\big) - \psi\big(X(x)\big)| \le \|X(x+a) - X(x)\| = \|h\| \le \sqrt{C}\|a\| \, .$$

□

Here is a useful formula for the variance:

(23a5) $$\operatorname{Var}\langle f, X\rangle = \int_0^1 \int_0^1 f(s) f(t) \big(\mathbb{E}\, \Xi(s)\Xi(t)\big) \, \mathrm{d}s\mathrm{d}t$$

for every $f \in L_2[0,1]$. Proof:

$$\mathbb{E} \Big(\int f(t) X(t) \, \mathrm{d}t \Big)^2 = \mathbb{E} \iint f(s)X(s)f(t)X(t) \, \mathrm{d}s\mathrm{d}t = \\ = \iint \big(\mathbb{E}\, f(s)X(s)f(t)X(t) \big) \, \mathrm{d}s\mathrm{d}t \, ,$$

since

$$\mathbb{E} \iint |f(s)X(s)f(t)X(t)| \, \mathrm{d}s\mathrm{d}t = \mathbb{E} \Big(\int |f(t)X(t)| \, \mathrm{d}t \Big)^2 \le \\ \le \mathbb{E} \Big(\int |f(t)|^2 \, \mathrm{d}t \Big) \Big(\int |X(t)|^2 \, \mathrm{d}t \Big) = \|f\|^2 \int_0^1 \|\Xi(t)\|^2 \, \mathrm{d}t < \infty \, .$$

23b Using assumption A_n

Let $\Xi : \mathbb{R} \to G \subset L_2(\Omega, P)$ be a mean-square continuous stationary Gaussian random process on $\mathbb{R}$, and μ its spectral measure:

$$\mathbb{E}\, \Xi(0)\Xi(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda t} \, \mu(\mathrm{d}\lambda) = \int_{-\infty}^{+\infty} \cos \lambda t \, \mu(\mathrm{d}\lambda) \, .$$

[1] In fact, the distribution $X[\gamma^\infty]$ of X is a Gaussian measure on $L_2[0,1]$, and h is its admissible shift.

Here is another useful formula for the variance, this time in terms of the spectral measure (recall 11c4):

$$\text{(23b1)} \qquad \operatorname{Var}\langle f, X\rangle = \int \Big| \int_0^1 f(t) \mathrm{e}^{\mathrm{i}\lambda t} \, \mathrm{d}t \Big|^2 \mu(\mathrm{d}\lambda)$$

for every $f \in L_2[0,1]$. Proof:

$$\operatorname{Var}\langle f, X\rangle = \iint f(s) f(t) \Big(\int \mathrm{e}^{\mathrm{i}\lambda(t-s)} \, \mu(\mathrm{d}\lambda) \Big) \, \mathrm{d}s\mathrm{d}t = \\ = \int \mu(\mathrm{d}\lambda) \Big(\int f(s) \overline{\mathrm{e}^{\mathrm{i}\lambda s}} \, \mathrm{d}s \Big) \Big(\int f(t) \mathrm{e}^{\mathrm{i}\lambda t} \, \mathrm{d}t \Big) \, ,$$

since

$$\int \mu(\mathrm{d}\lambda) \iint \big| f(s) f(t) \mathrm{e}^{\mathrm{i}\lambda(t-s)} \big| \, \mathrm{d}s\mathrm{d}t = \mu(\mathbb{R}) \Big(\int |f(t)| \, \mathrm{d}t \Big)^2 < \infty \, .$$

We generalize assumptions A and A_n of Sect. 2 as follows.

Assumption A:

$$\mu(\mathbb{R}) = 1 \, .$$

That is, $X(0) \sim N(0,1)$. Otherwise we may rescale X.

Assumption A_n: assumption A holds, and in addition,[1]

$$\forall \lambda \in [0,\infty) \quad \mu\big([\lambda, \lambda+1]\big) \le \frac{1}{n} \, .$$

The argument of Sect. 11c still applies, recall (11c5): for every $f \in L_2[0,1]$,

$$\int |g|^2 \, \mathrm{d}\mu \le C \Big(\int |g(\lambda)|^2 \, \mathrm{d}\lambda \Big) \sup_\lambda \mu\big([\lambda,\lambda+1]\big) \, ;$$

as before, $g(\lambda) = \int_0^1 \mathrm{e}^{\mathrm{i}\lambda t} f(t) \, \mathrm{d}t$, $\|g\|_2^2 = 2\pi \|f\|_2^2$, and

$$\operatorname{Var}\langle f, X\rangle = \int |g|^2 \, \mathrm{d}\mu \, .$$

Thus, assumption A_n implies (recall 11c3)

$$\operatorname{Var}\langle f, X\rangle \le \frac{C}{n} \|f\|^2 \, ,$$

[1] Alternatively you may take $\lambda \in \mathbb{R}$; it is the same up to a factor 2 absorbed by an absolute constant.

and, by 23a3,

$$\psi(X) \in \operatorname{GaussLip}(C/\sqrt{n})$$

whenever $\psi : L_2[0,1] \to \mathbb{R}$ is a Lip(1) function.

Now all arguments of 11d, 11e apply, and so, Theorems 2a2, 2a3 are generalized as follows.

Let X be a jointly measurable modification of a mean-square continuous stationary Gaussian random process on $\mathbb{R}$, satisfying assumption A_n.

23b2 Proposition. Let a function $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous almost everywhere, and

$$\sup_x \frac{|\varphi(x)|}{1+|x|} < \infty \, .$$

Then the random variable

$$\xi = \int_0^1 \varphi\big(X(t)\big) \, \mathrm{d}t$$

is integrable, $\mathbb{E}\,\xi = \int \varphi \, \mathrm{d}\gamma^1$, and for every $\varepsilon > 0$,

$$\mathbb{P}\big(|\xi - \mathbb{E}\,\xi| \ge \varepsilon \big) \le 2\mathrm{e}^{-c_{\varepsilon,\varphi} n}$$

for some $c_{\varepsilon,\varphi} > 0$ (dependent on ε and φ only, not on n).

23b3 Proposition.

$$\mathbb{P}\big(T(X(\cdot)) \ge \varepsilon \big) \le 2\mathrm{e}^{-c_\varepsilon n}$$

for some $c_\varepsilon > 0$ dependent on ε only.

As before, for $f \in L_1[0,1]$,

$$T(f) = \inf_g \int_0^1 |f(t) - g(t)| \, \mathrm{d}t$$

where the infimum is taken over all measurable $g : (0,1) \to \mathbb{R}$ that send Lebesgue measure to γ^1.

A trivial rescaling of t by arbitrary $L > 0$ turns assumption A_n and Proposition 23b2 into the following.

Assumption $A_{n,L}$: assumption A holds, and in addition,

$$\forall \lambda \in [0,\infty) \quad \mu\Big(\Big[\lambda, \lambda + \frac{1}{L} \Big] \Big) \le \frac{1}{n} \, .$$

23b4 Corollary. Let X satisfy $A_{n,L}$ and φ be as in 23b2. Then the random variable
$$\xi = \frac{1}{L} \int_0^L \varphi\big(X(t)\big)\,\mathrm{d}t$$
is integrable, $\mathbb{E}\,\xi = \int \varphi\,\mathrm{d}\gamma^1$, and for every $\varepsilon > 0$,
$$\mathbb{P}\big(|\xi - \mathbb{E}\,\xi| \ge \varepsilon\big) \le 2\mathrm{e}^{-c_{\varepsilon,\varphi} n}$$
for some $c_{\varepsilon,\varphi} > 0$.

Now, at last, we can deal with a single process, getting rid of assumption $A_{n,L}$.

23b5 Theorem. Let X be a jointly measurable[1] modification of a mean-square continuous stationary Gaussian random process on $\mathbb{R}$ whose spectral measure has a bounded density.[2] Let a function $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous almost everywhere, and
$$\sup_x \frac{|\varphi(x)|}{1+|x|} < \infty\,.$$
Then random variables
$$\xi_L = \frac{1}{L} \int_0^L \varphi\big(X(t)\big)\,\mathrm{d}t \quad \text{for } L \in (0,\infty)$$
are integrable, $\mathbb{E}\,\xi_L = \int \varphi\,\mathrm{d}\gamma^1$, and for every $\varepsilon > 0$,
$$\mathbb{P}\big(|\xi_L - \mathbb{E}\,\xi_L| \ge \varepsilon\big) \le 2\mathrm{e}^{-c_{\varepsilon,\varphi,M} L}$$
for some $c_{\varepsilon,\varphi,M} > 0$ (dependent only on ε, φ and the supremum M of the spectral density, not on L).

23b6 Exercise. Prove Theorem 23b5.

23b7 Exercise. Formulate and prove a single-process counterpart of 23b3.

23c Dimension two, and higher

A two-component (in other words, $\mathbb{R}^2$-valued) Gaussian random process on a set T may be defined as a pair (Ξ_1, Ξ_2) of Gaussian processes $\Xi_1, \Xi_2 : T \to G \subset L_2(\Omega, P)$. Or equivalently, as a Gaussian process $\Xi : T \times \{1,2\} \to$

[1] Sample continuity is of course sufficient (by 22d3).

[2] Equivalently, $\sup_{a<b} \frac{\mu([a,b])}{b-a} < \infty$.

G.[1] Similarly, a two-component random function ξ on T is a pair (ξ_1, ξ_2) of random functions $\xi_1, \xi_2 : \Omega \to \mathbb{R}^T$, or a random function $\xi : \Omega \to \mathbb{R}^{T\times\{1,2\}} = \mathbb{R}^T \times \mathbb{R}^T$. Clearly, (ξ_1, ξ_2) is a modification of (Ξ_1, Ξ_2) if and only if both ξ_1 is a modification of Ξ_1 and ξ_2 is a modification of Ξ_2. Continuity and measurabilty properties are defined evidently.

The covariance function of $\Xi : T\times\{1,2\} \to G$ is $(s,k;t,l) \mapsto \mathbb{E}\,\Xi(s,k)\Xi(t,l) = \mathbb{E}\,\Xi_k(s)\Xi_l(t)$. Stationarity (assuming $T = \mathbb{R}$) is, by definition (recall 21e1),

$$\forall s,t \in \mathbb{R} \;\; \forall k,l \in \{1,2\} \;\; \mathbb{E}\,\Xi_k(s)\Xi_l(t) = \mathbb{E}\,\Xi_k(0)\Xi_l(t-s)\,.$$

For a stationarity $\Xi : \mathbb{R}\times\{1,2\} \to G$ the covariance function $R : \mathbb{R}\times\{1,2\}\times\{1,2\} \to \mathbb{R}$ is, by definition,

$$R(t,k,l) = R_{k,l}(t) = \mathbb{E}\,\Xi_k(0)\Xi_l(t)\,;$$

it determines the process up to isometry. Another function $r : \mathbb{R} \to \mathbb{R}$,

$$r(t) = \mathbb{E}\,\langle \Xi(0), \Xi(t)\rangle = \mathbb{E}\,\bigl(\Xi_1(0)\Xi_1(t) + \Xi_2(0)\Xi_2(t)\bigr) = R_{1,1}(t) + R_{2,2}(t)\,,$$

containing only a partial information about R, will be called the traced covariance function. Normalizing the process to $r(0) = 1$ one may call r the correlation function. However, such normalization is sometimes inconvenient, since the case $\Xi(0) \sim \gamma^2$ leads to $r(0) = 2$.

Clearly, the function r is positive definite. Assuming mean square continuity of Ξ we apply Bochner's theorem and get the traced spectral measure,[2] — a symmetric measure μ on $\mathbb{R}$ such that

$$\mathbb{E}\,\langle \Xi(0), \Xi(t)\rangle = r(t) = \int e^{i\lambda t}\,\mu(d\lambda)\,.$$

In the finite-dimensional case treated in 11f, $r(t) = \sum_k |a_k|^2 \cos\lambda_k t$ (a_k being vectors), thus, $\mu = \sum_k |a_k|^2 (\delta_{\lambda_k} + \delta_{-\lambda_k})/2$.

Similarly to 23a we upgrade a two-component process Ξ to the corresponding random element[3] X of $L_2\bigl([0,1] \to \mathbb{R}^2\bigr)$ and consider

$$\langle f, X\rangle = \langle f_1, X_1\rangle + \langle f_2, X_2\rangle$$

[1] A coordinate-free definition of a E-valued Gaussian process on T, for a finite-dimensional linear space E, may be given as follows: it is a linear map from E^* to G^T.

[2] The full (non-traced) spectral measure may be treated as a matrix-valued measure on $\mathbb{R}$, or equivalently, a 2×2 matrix whose elements are (signed) measures on $\mathbb{R}$. For an E-valued process one gets a "scalar product" on E^* whose values are (signed) measures on $\mathbb{R}$.

[3] Just upgrade Ξ_1 to X_1, Ξ_2 to X_2, and take $X = (X_1, X_2)$.

for $f = (f_1, f_2) \in L_2\big([0,1] \to \mathbb{R}^2\big)$. We cannot calculate $\operatorname{Var}\langle f, X\rangle$ in terms of the *traced* spectral measure μ (like (23b1)), but we can bound it:[1]

$$\operatorname{Var}\langle f, X\rangle \le 2 \int \bigg| \int_0^1 f(t) \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \bigg|^2 \mu(\mathrm{d}\lambda) = \\ = 2 \int \bigg(\bigg| \int_0^1 f_1(t) \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \bigg|^2 + \bigg| \int_0^1 f_2(t) \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \bigg|^2 \bigg) \mu(\mathrm{d}\lambda)\,.$$

Proof:

$$\operatorname{Var}\langle f, X\rangle = \|\langle f, X\rangle\|^2 = \|\langle f_1, X_1\rangle + \langle f_2, X_2\rangle\|^2 \le 2\|\langle f_1, X_1\rangle\|^2 + 2\|\langle f_2, X_2\rangle\|^2 \\ = 2 \int \bigg| \int_0^1 f_1(t) \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \bigg|^2 \mu_{1,1}(\mathrm{d}\lambda) + 2 \int \bigg| \int_0^1 f_2(t) \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \bigg|^2 \mu_{2,2}(\mathrm{d}\lambda)\,,$$

where $\mu_{1,1}$ is the spectral measure for X_1, and $\mu_{2,2}$ — for X_2; it remains to note that $\mu = \mu_{1,1} + \mu_{2,2}$ (think, why).

Assumption A is replaced with

$$\Xi(0) \sim \gamma^2$$

(which implies $\mu(\mathbb{R}) = 2$); assumption A_n still adds

$$\forall \lambda \in [0,\infty) \quad \mu\big([\lambda, \lambda+1]\big) \le \frac{1}{n}$$

where μ is the traced spectral measure. As before we get

$$\forall f \in L_2\big([0,1] \to \mathbb{R}^2\big) \quad \operatorname{Var}\langle f, X\rangle \le \frac{C}{n}\|f\|^2\,; \\ \psi(X) \in \operatorname{GaussLip}(C/\sqrt{n})$$

whenever $\psi : L_2\big([0,1] \to \mathbb{R}^2\big) \to \mathbb{R}$ is a $\operatorname{Lip}(1)$ function. Similarly to 11f, Propositions 23b2 and 23b3 generalize to two-component processes satisfying assumption A_n. Also Theorem 23b5 generalizes to two-component processes whose *traced* spectral measures have bounded densities.

All said about $\mathbb{R}^2$ holds equally well for $\mathbb{R}^d$, $d = 3, 4, \dots$

23d Hints to exercises

23b6: $L = Cn$.

[1] In fact, the coefficient "2" is superfluous (see 11f for the discrete case); however, the stronger inequality is harder to prove.