13 Random real zeroes: two derivatives

13a Restricting the class of functions

Let X satisfy assumption $C_{M,n,L}$.

We have

$$\mathbb{E}\, X'(0)X(t) = \mathbb{E}\Big(\frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=0} X(u) \Big) X(t) = \mathbb{E}\, \frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=0} \big(X(u)X(t) \big) = \\ = \frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=0} \mathbb{E}\, \big(X(u)X(t) \big) \,,$$

since for $u \in (-1,1)$,

$$\Big| \frac{X(u)-X(0)}{u} X(t) \Big| \le \max_{[-1,1]} |X'(\cdot)| \cdot |X(t)| \,,$$

the majorant being integrable. By stationarity,

$$\mathbb{E}\, X'(0)X(t) = \frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=0} \mathbb{E}\, X(0)X(t-u) = -\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\, X(0)X(t) = -\mathbb{E}\, X(0)X'(t) \,.$$

Similarly,

$$\mathbb{E}\, X'(0)X'(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\, X'(0)X(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathbb{E}\, X(0)X(t) = -\mathbb{E}\, X(0)X''(t) \,.$$

In particular,

$$\mathbb{E}\, X(0)X''(0) = -\mathbb{E}\, |X'(0)|^2 = -1 \,,$$

therefore (think, why)

$$\mathbb{E}\, |X''(0)|^2 \ge 1 \,,$$

and $M \ge 1$ (otherwise assumption $C_{M,n,L}$ is never satisfied). Further,

$$\mathbb{E}\, X'(0)X'(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_{k=1}^N |a_k|^2 \cos \lambda_k t = \sum_{k=1}^N |\lambda_k a_k|^2 \cos \lambda_k t$$

as well as

$$\mathbb{E}\, X''(0)X''(t) = \sum_{k=1}^{N} |\lambda_k^2 a_k|^2 \cos \lambda_k t\,.$$

We have $\mathbb{E}\,|X''(0)|^2 = \sigma^2$ for some $\sigma \in [1, \sqrt{M}]$. Taking into account that $\lambda_k^4 \le (1+\lambda_k^2)^2$ and $\sigma \ge 1$ we see that assumption $C_{M,n,L}$ for X implies assumption $A_{n,L}$ for $\frac{1}{\sigma}X''$.

For the two-dimensional process (X', X'') we get

$$\mathbb{E}\,\langle (X'(0), X''(0)), (X'(t), X''(t)) \rangle = \mathbb{E}\,\big(X'(0)X'(t) + X''(0)X''(t) \big) = \\ = \sum_k (\lambda_k^2 + \lambda_k^4) a_k^2 \cos \lambda_k t\,.$$

Taking into account that $\lambda_k^2 + \lambda_k^4 \le (1+\lambda_k^2)^2$ and $\sigma \ge 1$ we see that assumption $C_{M,n,L}$ for X implies assumption $A_{n,L}$ for $(X', \frac{1}{\sigma}X'')$.

Applying 11c1-11c2 to X'' we get

$$\frac{1}{L}\int_0^L |X''(t)|^2 \,\mathrm{d}t \le \frac{C}{n}(X_1^2 + \dots + X_{2N}^2)$$

for some absolute constant C.

13a1 Exercise. Prove that

$$\mathbb{P}\Big(\frac{1}{L}\int_0^L |X''(t)|^2 \,\mathrm{d}t > 2M \Big) \le C_M \mathrm{e}^{-c_M n}$$

for some $c_M > 0$, $C_M < \infty$ (dependent on M only).

In this sense,

$$\frac{1}{L}\int_0^L |X''(t)|^2 \,\mathrm{d}t \le 2M \quad \text{very probably.}$$

Applying Theorem 2a2 (or rather, its two-dimensional generalization) to the two-dimensional process $(X', \frac{1}{\sigma}X'')$ we get for any a.e. continuous $\varphi : \mathbb{R}^2 \to \mathbb{R}$ of linear growth,

$$\frac{1}{L}\int_0^L \varphi\Big(X'(t), \frac{1}{\sigma}X''(t) \Big) \,\mathrm{d}t \in \operatorname{ExpConInt}(n)\,.$$

Using also Lemma 2a1 (or rather, its two-dimensional generalization) we get for every $\varepsilon > 0$,

$$\frac{1}{L}\int_0^L \varphi\Big(X'(t), \frac{1}{\sigma}X''(t) \Big) \,\mathrm{d}t \le \int \varphi \,\mathrm{d}\gamma^2 + \varepsilon \quad \text{very probably.}$$

In particular,

$$\mathbb{E}\,\frac{1}{L}\int_0^L \frac{1}{\sigma}|X''(t)|\mathbf{1}_{[A,\infty)}(|X'(t)|)\,\mathrm{d}t = \biggl(\int |u|\,\gamma^1(\mathrm{d}u)\biggr)\cdot 2\gamma^1([A,\infty)) \le C\mathrm{e}^{-A^2/2}\,,$$

therefore for every $A > 0$ (separately),

$$\frac{1}{L}\int_0^L |X''(t)|\mathbf{1}_{[A,\infty)}(|X'(t)|)\,\mathrm{d}t \le C\sqrt{M}\mathrm{e}^{-A^2/2} \quad \text{very probably,}$$

where C is an absolute constant.[1] Similarly,

$$\frac{1}{L}\int_0^L |X''(t)|\mathbf{1}_{[-a,a]}(X'(t))\,\mathrm{d}t \le C\sqrt{M}a \quad \text{very probably.}$$

Here is a non-probabilistic fact.

13a2 Lemma. Let $f : [0, L] \to \mathbb{R}$ be twice continuously differentiable, and $a > 0$. Then

$$\sum_{t\in[0,L],f(t)=0,0<|f'(t)|\le a} |f'(t)| \le \frac{1}{2}\int_0^L |f''(s)|\mathbf{1}_{[-2a,2a]}(f'(s))\,\mathrm{d}s + 2a\,.$$

Proof. For each $t \in [0, L]$ such that $f(t) = 0$ and $0 < |f'(t)| \le a$ we consider the set $\{s \in [0, L] : 0 < |f'(s)| < 2|f'(t)|\}$ and its connected component I_t containing t. Clearly, I_t is an interval, f is strictly monotone on I_t, and such intervals are pairwise disjoint. If $I_t \subset (0, L)$ then

$$\int_{I_t} |f''(s)|\,\mathrm{d}s \ge 2|f'(t)|$$

(think, why). Taking into account that $|f'(\cdot)| \le 2a$ on I_t we have

$$2\sum |f'(t)| \le \sum \int_{I_t} |f''(s)|\,\mathrm{d}s \le \int_0^L |f''(s)|\mathbf{1}_{[-2a,2a]}(f'(s))\,\mathrm{d}s\,,$$

where the sum is taken over t such that $I_t \subset (0, L)$. Other t (at most two) contribute at most $2a$. □

The random variable

$$\xi_a = \frac{1}{L}\sum_{t\in[0,L],X(t)=0,|X'(t)|\le a} |X'(t)|$$

is a special case of ξ of Theorem 2c1, for $\varphi(x) = |x|\mathbf{1}_{[-a,a]}(x)$.

[1] Every $C > \sqrt{2/\pi}$ fits.

13a3 Exercise. For every $a > 0$ (separately), $\xi_a \le C\sqrt{M}a$ very probably; here C is an absolute constant.

Prove it.

13a4 Exercise. $\mathbb{E}\,\xi_a \le \frac{1}{3\pi}a^3$.

Prove it.

13a5 Exercise. It is sufficient to prove Theorem 2c1 for functions φ that vanish on a neighborhood[1] of 0.

Prove it.

13a6 Lemma. Let $f : [0, L] \to \mathbb{R}$ be twice continuously differentiable, and $A > 2\min_{[0,L]} |f'(\cdot)|$. Then

$$\sum_{t\in[0,L],f(t)=0,|f'(t)|\ge A} |f'(t)| \le 2\int_0^L |f''(s)|\mathbf{1}_{[A/2,\infty)}(|f'(s)|)\,\mathrm{d}s\,.$$

Proof. For each $t \in [0, L]$ such that $f(t) = 0$ and $|f'(t)| \ge A$ we consider the set $\{s \in [0, L] : |f'(s)| > 0.5|f'(t)|\}$ and its connected component I_t containing t. Clearly, I_t is an interval, f is strictly monotone on I_t, and such intervals are pairwise disjoint. It cannot happen that $I_t = [0, L]$, since $|f'(\cdot)| \ge 0.5A$ on I_t. Thus,

$$\int_{I_t} |f''(s)|\,\mathrm{d}s \ge \frac{1}{2}|f'(t)|$$

(think, why). We have

$$\frac{1}{2}\sum |f'(t)| \le \sum \int_{I_t} |f''(s)|\,\mathrm{d}s \le \int_0^L |f''(s)|\mathbf{1}_{[A/2,\infty)}(|f'(s)|)\,\mathrm{d}s\,.$$

□

Taking into account that

$$\min_{[0,L]} |X'(\cdot)| \le \left(\frac{1}{L}\int_0^L |X'(t)|^2\,\mathrm{d}t\right)^{1/2}$$

we see (similarly to 13a1) that

$$\min_{[0,L]} |X'(\cdot)| \le 2 \quad \text{very probably.}$$

[1] The neighborhood depends on φ, of course.

Similarly to 13a3, for every $A > 4$ (separately), the random variable

$$\xi_A = \frac{1}{L} \sum_{t\in[0,L],X(t)=0,|X'(t)|\ge A} |X'(t)|$$

satisfies

$$\xi_A \le C\sqrt{M}\mathrm{e}^{-A^2/8} \quad \text{very probably,}$$

as well as $\mathbb{E}\,\xi_A \le C\sqrt{M}\mathrm{e}^{-A^2/8}$. Here is the conclusion.

13a7 Proposition. It is sufficient to prove Theorem 2c1 for functions φ such that

$$\exists a, A \in (0,\infty) \;\; \forall x \in \mathbb{R} \quad \big(\varphi(x) \ne 0 \quad \Longrightarrow \quad a < |x| < A \big) \,.$$

The condition $\sup(|\varphi(x)|/|x|) < \infty$ becomes just boundedness of φ.

13a8 Exercise. It is sufficient to prove Theorem 2c1 for Lipschitz functions φ (satisfying 13a7).

Prove it.

13b Getting rid of randomness

According to 13a8, we consider a function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying[1]

$$\begin{gathered} \forall x \in \mathbb{R} \quad |\varphi(x)| \le 1 \,, \\ \forall x, y \in \mathbb{R} \quad |\varphi(x) - \varphi(y)| \le |x - y| \,, \\ \forall x \in \mathbb{R} \quad \big(\varphi(x) \ne 0 \quad \Longrightarrow \quad |x| > a \big) \,. \end{gathered}$$

We approximate the random variable of Theorem 2c1,

$$\xi = \frac{1}{L} \sum_{t\in[0,L],X(t)=0} \varphi(X'(t)) \,,$$

by another random variable (for $\varepsilon \to 0+$)

$$\eta_\varepsilon = \frac{1}{2\varepsilon L} \int_0^L \varphi(X'(t))|X'(t)|\mathbf{1}_{(-\varepsilon,\varepsilon)}(X(t)) \,\mathrm{d}t \,.$$

We know that[2]

$$\mathbb{E}\,\xi = \frac{1}{2\pi} \int \varphi(y)|y|\mathrm{e}^{-y^2/2} \,\mathrm{d}y$$

[1] The first two conditions can be enforced multiplying φ by a small number. The condition about $|x| < A$ is not needed.

[2] By Theorem 2b1.

and[1]

$$\begin{aligned}\mathbb{E}\,\eta_\varepsilon &= \frac{1}{2\varepsilon}\iint \varphi(y)|y|\mathbf{1}_{(-\varepsilon,\varepsilon)}(x)\,\gamma^2(\mathrm{d}x\mathrm{d}y) = \\ &= \Big(\int \varphi(y)|y|\,\gamma^1(\mathrm{d}y)\Big)\frac{1}{2\varepsilon}\gamma^1((-\varepsilon,\varepsilon)) \to \frac{1}{\sqrt{2\pi}}\int \varphi(y)|y|\,\gamma^1(\mathrm{d}y)\,,\end{aligned}$$

thus, $|\mathbb{E}\,\eta_\varepsilon - \mathbb{E}\,\xi| \to 0$ (as $\varepsilon \to 0+$). We also know that[2]

$$\eta_\varepsilon \in \mathrm{ExpConInt}(n)$$

for every ε (separately). In order to prove Theorem 2c1 we have to prove that

$$\xi \in \mathrm{ExpConInt}(n)\,;$$

by the approximation lemma (11a9, 11a11) it is sufficient to prove that

$$|\xi - \eta_\varepsilon| \le \varepsilon_0 \quad \text{very probably}$$

if ε is small enough (for a given ε_0).

Here is a non-probabilistic fact, to be proved in Sect. 13c.

13b1 Proposition. Let a twice continuously differentiable function $f : [0, L] \to \mathbb{R}$ and a number $B > 0$ satisfy

$$\frac{1}{L}\int_0^L |f''(t)|^2 \le B^2$$

and $B\varepsilon < \min(1, a^3)$. Then

$$\left|\frac{1}{L}\sum_{t\in[0,L],f(t)=0,f'(t)\ne0} \varphi(f'(t)) - \frac{1}{2\varepsilon L}\int_0^L \varphi(f'(t))|f'(t)|\mathbf{1}_{(-\varepsilon,\varepsilon)}(f(t))\,\mathrm{d}t\right| \le \\ \le C\Big(\frac{\varepsilon^{1/3}B^{4/3}}{a} + \frac{1}{L}\Big)$$

for some absolute constant C.

Note that n does not occur in 13b1, but L does.

13b2 Exercise. $L \ge cn - C$ for some absolute constants c, C.

Prove it.

Given $\varepsilon_0 > 0$, we choose ε such that $C\frac{\varepsilon^{1/3}}{a}(2M)^{2/3} \le \varepsilon_0/2$ and $\varepsilon\sqrt{2M} < \min(1, a^3)$; then, assuming that $C/L \le \varepsilon_0/2$ (which holds for all n large enough) we get $|\xi - \eta_\varepsilon| \le \varepsilon_0$ whenever $\frac{1}{L}\int_0^L |X''(t)|^2\,\mathrm{d}t \le 2M$, which happens very probably. Thus, in order to prove Theorem 2c1 it is sufficient to prove Proposition 13b1.

[1] By the two-dimensional generalization of Lemma 2a1.

[2] By the two-dimensional generalization of Theorem 2a2.

13c Estimating the error

Here we prove the non-probabilistic Proposition 13b1.

We consider the set $\{t \in [0, L] : |f(t)| < \varepsilon, f'(t) \neq 0\}$ and its connected components I such that

$$\sup_I |f'(\cdot)| > a \, .$$

Clearly, I is an interval, f is strictly monotone on I, and of course, such intervals are pairwise disjoint.

13c1 Exercise. The set $\mathcal{I}$ of all these intervals I is finite.

Prove it.

Assume that $\delta \in (0, a)$ is given (it will be chosen later). We say that an interval $I \in \mathcal{I}$ is good, if

$$\sup_I f'(\cdot) - \inf_I f'(\cdot) \le \delta \, ;$$

otherwise I is bad. Denote by $G \subset \mathcal{I}$ the set of all good intervals.

Denoting[1]

$$\xi = \frac{1}{L} \sum_{t \in [0,L], f(t)=0, f'(t) \neq 0} \varphi(f'(t)) \, , \quad \eta_\varepsilon = \frac{1}{2\varepsilon L} \int_0^L \varphi(f'(t)) |f'(t)| \mathbf{1}_{(-\varepsilon,\varepsilon)}(f(t)) \, \mathrm{d}t$$

we have

$$\xi = \sum_{I \in \mathcal{I}} \underbrace{\frac{1}{L} \sum_{t \in I, f(t)=0} \varphi(f'(t))}_{\xi_I} \, , \quad \eta_\varepsilon = \sum_{I \in \mathcal{I}} \underbrace{\frac{1}{2\varepsilon L} \int_I \varphi(f'(t)) |f'(t)| \, \mathrm{d}t}_{\eta_{I,\varepsilon}} \, .$$

Taking into account that $|\varphi(\cdot)| \le 1$ we get

$$\forall I \in \mathcal{I} \quad |\xi_I| \le \frac{1}{L} \, ,$$

since the sum contains no more than one summand; and

$$\forall I \in \mathcal{I} \quad |\eta_{I,\varepsilon}| \le \frac{1}{L} \, ,$$

since $\int_I |f'(t)| \, \mathrm{d}t = |f(t) - f(s)|$ for $I = (s, t)$.

At most two $I \in \mathcal{I}$ may violate $I \subset (0, L)$; their contribution to $|\xi - \eta_\varepsilon|$ cannot exceed $4/L$, that is harmless. From now on we assume that

$$I \subset (0, L)$$

for all considered $I \in \mathcal{I}$.

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[1]We thus redefine ξ and η_ε, which should not be too confusing since the probabilistic context is no more needed.

13c2 Lemma. For every good $I \subset (0, L)$,

$$|\xi_I - \eta_{I,\varepsilon}| \le \frac{\delta}{L} \, .$$

Proof. Let $I = (r, t)$. We have $\min_I |f'(\cdot)| \ge a - \delta > 0$, therefore either $f(r) = -\varepsilon$, $f(t) = \varepsilon$ or $f(r) = \varepsilon$, $f(t) = -\varepsilon$; in every case, $\int_I |f'(u)| \, \mathrm{d}u = 2\varepsilon$. Define $s \in I$ by $f(s) = 0$, then $\xi_I = \frac{1}{L} \varphi(f'(s)) = \frac{1}{2\varepsilon L} \int_I \varphi(f'(s)) |f'(u)| \, \mathrm{d}u$ and

$$|\xi_I - \eta_{I,\varepsilon}| \le \frac{1}{2\varepsilon L} \int_I |\varphi(f'(s)) - \varphi(f'(u))| |f'(u)| \, \mathrm{d}u \le \\ \le \frac{1}{2\varepsilon L} \int_I \underbrace{|f'(s) - f'(u)|}_{\le \delta} |f'(u)| \, \mathrm{d}u = \frac{\delta}{L} \, .$$

□

13c3 Exercise. Prove that

$$\sum |f'(t)| \le \frac{1}{2} \int_0^L |f''(s)| \, \mathrm{d}s \, ,$$

where the sum is taken over t such that $f(t) = 0$ and $f'(t) \ne 0$, except for the least and the greatest of these t.

We have $\frac{1}{L} \int_0^L |f''(t)| \, \mathrm{d}t \le B$, thus,

$$|G| \le \frac{BL}{2a} + 2 \, ,$$

and so,

$$\sum_{I \in G} |\xi_I - \eta_{I,\varepsilon}| \le \Big(\frac{B}{2a} + \frac{2}{L} \Big) \delta \, .$$

13c4 Lemma. For every bad interval $I \subset (0, L)$,

$$\int_I |f''(t)|^2 \, \mathrm{d}t \ge \frac{a\delta^2}{16\varepsilon} \, .$$

Proof. We take $s \in I$ such that $|f'(s)| > a$, note that $\sup_I |f'(\cdot) - f'(s)| > \delta/2$ and take the closest to s point $t \in I$ such that $|f'(t) - f'(s)| = \delta/2$. Assume that $s < t$ (the case $t < s$ is similar). We have $\min_{[s,t]} |f'(\cdot)| \ge a - \frac{\delta}{2} \ge \frac{a}{2}$

and $\int_s^t |f'(u)|\,\mathrm{d}u = |f(t)-f(s)| \le 2\varepsilon$, thus $t-s \le \frac{4\varepsilon}{a}$. Also, $\int_s^t |f''(u)|\,\mathrm{d}u \ge |f'(t)-f'(s)| = \delta/2$. Thus,

$$\frac{\delta}{2} \le \int_s^t |f''(u)|\,\mathrm{d}u \le \biggl(\int_s^t |f''(u)|^2\,\mathrm{d}u \biggr)^{1/2} \biggl(\int_s^t 1^2\,\mathrm{d}u \biggr)^{1/2};$$

$$\int_I |f''(u)|^2\,\mathrm{d}u \ge \int_s^t |f''(u)|^2\,\mathrm{d}u \ge \frac{(\delta/2)^2}{t-s} \ge \frac{a\delta^2}{16\varepsilon}\,.$$

□

Thus, the number of bad intervals $I \subset (0,L)$ does not exceed

$$\frac{16\varepsilon B^2 L}{a\delta^2}\,,$$

and so,

$$\sum_{I\in\mathcal{I}\setminus G, I\subset(0,L)} \bigl(|\xi_I| + |\eta_{I,\varepsilon}|\bigr) \le \frac{16\varepsilon B^2 L}{a\delta^2}\Bigl(\frac{1}{L}+\frac{1}{L}\Bigr) = \frac{32\varepsilon B^2}{a\delta^2}\,;$$

$$\sum_{I\in\mathcal{I}\setminus G} |\xi_I - \eta_{I,\varepsilon}| \le \frac{32\varepsilon B^2}{a\delta^2} + \frac{4}{L}\,;$$

$$|\xi-\eta_\varepsilon| \le \sum_{I\in\mathcal{I}} |\xi_I - \eta_{I,\varepsilon}| \le \Bigl(\frac{B}{2a}+\frac{2}{L}\Bigr)\delta + \frac{32\varepsilon B^2}{a\delta^2} + \frac{4}{L}\,.$$

Finally we choose

$$\delta = (B\varepsilon)^{1/3}\,,$$

note that $\delta < a$ and $\delta < 1$, and get

$$|\xi-\eta_\varepsilon| \le \frac{B}{2a}\delta + \frac{32\varepsilon B^2}{a\delta^2} + \frac{6}{L} \le C\biggl(\frac{\varepsilon^{1/3}B^{4/3}}{a} + \frac{1}{L}\biggr)\,,$$

which completes the proof of Proposition 13b1 and ultimately, Theorem 2c1.

13d Hints to exercises

13a1: the random variable $\xi = \bigl(\frac{1}{L}\int_0^L |X''(t)|^2\,\mathrm{d}t\bigr)^{1/2}$ belongs to $\mathrm{GaussLip}(C/\sqrt{n})$, and $\mathbb{E}\,\xi \le (\mathbb{E}\,\xi^2)^{1/2} \le \sqrt{M}$.

13a8: recall 11d4 and the paragraph after it.

13b2: $\int_{[0,2]}(1+\lambda^2)\,\mu(\mathrm{d}\lambda) \ge \mu([0,2]) \ge 3/4$.

13c3: Hint: similar to 13a2.

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