12 Random real zeroes: one derivative

12a Proving Theorem 2b1

For now, X satisfies just assumption A.

12a1 Lemma. Let $u \in \mathbb{R}$. Almost surely, no $t \in \mathbb{R}$ satisfies both $X(t) = u$ and $X'(t) = 0$.

12a2 Exercise. Assume the opposite: $\mathbb{P}\big(\exists t \in \mathbb{R}\ (X(t) = u, X'(t) = 0)\big) > 0$. Then

$$\mathbb{P}\big(\exists t \in [0,1]\ (X(t) = u, X'(t) = 0) \text{ and } \forall t \in [0,1]\ |X''(t)| \le A\big) > 0$$

for some $A < \infty$.

Prove it.

12a3 Exercise.

$$\mathbb{E} \int_0^1 \mathbf{1}_{(u-\varepsilon,u+\varepsilon)}(X(t))\,\mathrm{d}t \ge p \min\Bigg(1, \sqrt{\frac{2\varepsilon}{A}}\Bigg),$$

where $p = \mathbb{P}\big(\exists t \in [0,1]\ (X(t) = u, X'(t) = 0) \text{ and } \forall t \in [0,1]\ |X''(t)| \le A\big)$.

Prove it.

On the other hand, by Lemma 2a4 applied to $\varphi = \mathbf{1}_{(u-\varepsilon,u+\varepsilon)}$,

$$\mathbb{E} \int_0^1 \mathbf{1}_{(u-\varepsilon,u+\varepsilon)}(X(t))\,\mathrm{d}t = O(\varepsilon).$$

Thus, p must vanish, and so, *Lemma 12a1 is proved.* It means that a given number has no chance to be a critical value of $X(\cdot)$. Then, $\{t \in [0,1] : X(t) = u\}$ is a finite set[1] and

$$\xi_v = \sum_{t \in [0,1], X(t) = v} \varphi(X'(t))$$

treated as a function of v for a given $X(\cdot)$ is continuous at u. However, we cannot conclude that $\mathbb{E}\,\xi_v$ is continuous in v unless we have an integrable majorant for these ξ_v.

Now let X satisfy assumption B.

[1] Which also follows from the polynomial form of $X(\cdot)$.

12a4 Exercise. Let $\varphi, \varphi_1, \varphi_2, \dots : \mathbb{R} \to [0, \infty)$ be Borel functions such that either $\varphi_n \downarrow \varphi$ pointwise and φ_1 is bounded, or $\varphi_n \uparrow \varphi$ pointwise. If the equality

$$\mathbb{E} \, \frac{1}{L} \sum_{t \in [0,L], X(t)=0} \psi(X'(t)) = \frac{1}{2\pi} \int \psi(y) |y| \mathrm{e}^{-y^2/2} \, \mathrm{d}y \in [0, \infty]$$

holds for $\psi = \varphi_1, \varphi_2, \dots$ then it holds for $\psi = \varphi$.

Prove it.

Therefore it is sufficient to prove Theorem 2b1 under additional assumptions on φ:

$$\begin{gathered} \varphi : \mathbb{R} \to [0, \infty) \text{ is continuous and bounded}\,, \\ \varphi(\cdot) = 0 \text{ on } [-a, a] \end{gathered} \tag{12a5}$$

for some $a > 0$.

If $\forall t \in [0, L] \quad |X''(t)| \le A$ then points $t \in [0, L]$ such that $X(t) = u$, $|X'(t)| \ge a$ are far apart at least $2a/A$ (think, why), and therefore the number of such points is at most $1 + \frac{AL}{2a}$. It follows that

$$0 \le \underbrace{\frac{1}{L} \sum_{t \in [0,L], X(t)=u} \varphi(X'(t))}_{\xi_u} \le \Big(\frac{1}{L} + \frac{1}{2a} \max_{[0,L]} |X''(\cdot)| \Big) \sup_{\mathbb{R}} \varphi(\cdot) \,,$$

which is an integrable majorant for the random variables ξ_u. Thus, convergence a.s. implies convergence of expectations, and we conclude that

$$\mathbb{E} \, \xi_u \text{ is continuous in } u \,.$$

Now we note that

$$\int_{u-\varepsilon}^{u+\varepsilon} \xi_v \, \mathrm{d}v = \frac{1}{L} \int_0^L |X'(t)| \varphi(X'(t)) \mathbf{1}_{(u-\varepsilon, u+\varepsilon)}(X(t)) \, \mathrm{d}t$$

(basically, $\mathrm{d}v = X'(t) \, \mathrm{d}t$, and $X(\cdot)$ is piecewise monotone). Thus,

$$\begin{aligned} \frac{1}{2\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} \mathbb{E} \, \xi_v \, \mathrm{d}v &= \frac{1}{2\varepsilon L} \int_0^L \Big(\mathbb{E} \, |X'(t)| \varphi(X'(t)) \mathbf{1}_{(u-\varepsilon, u+\varepsilon)}(X(t)) \Big) \, \mathrm{d}t = \\ &= \frac{1}{L} \int_0^L \mathrm{d}t \bigg(\int |y| \varphi(y) \gamma^1(\mathrm{d}y) \bigg) \frac{1}{2\varepsilon} \gamma^1 \big((u - \varepsilon, u + \varepsilon) \big) \,, \end{aligned}$$

since[1] $\big(X(t),X'(t)\big) \sim \gamma^2$. The limit $\varepsilon \to 0$ gives

$$\mathbb{E}\,\xi_u = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-u^2/2} \int |y|\varphi(y)\gamma^1(\mathrm{d}y)$$

for all u. In particular,

$$\mathbb{E}\,\xi_0 = \frac{1}{\sqrt{2\pi}} \int |y|\varphi(y)\gamma^1(\mathrm{d}y)\,,$$

which proves Theorem 2b1 for φ satisfying (12a5), therefore, for all φ.[2]

[1] $0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\,X^2(t) = \mathbb{E}\,2X(t)X'(t)$.

[2] And moreover, $\mathbb{E}\,\frac{1}{L}\sum_{t\in[0,L],X(t)=u}\psi(X'(t)) = \mathrm{e}^{-u^2/2}\frac{1}{2\pi}\int\psi(y)|y|\mathrm{e}^{-y^2/2}\,\mathrm{d}y$.