8 Unknown distributions, correlated signals

8a The framework

All models considered in sections 1–7 assume that signals are independent and their distributions are common knowledge. It never holds in reality. Often, signals are influenced by some 'common' random variables, which makes them correlated. In other cases, we may treat signals as independent; however, it does not mean that their distributions are *exactly* known to players.

For the first example, return to the 'very simple auction' of Sect. 1, with a small correction: instead of the uniform distribution $\mathrm{U}(0,1)$ consider $\mathrm{U}(0,\theta)$ where θ is a parameter. In reality, θ is not a random variable; it is just a number, say, 0.97. Players know that the number is close to 1. In order to keep the model as simple as possible we describe their uncertainty by the uniform distribution $\mathrm{U}(0.9,1)$ of θ.

Does the true value $\theta = 0.97$ influence actions of players? On one hand, the value, being unknown to players, cannot influence their strategies. On the other hand, it influences signals, and signals influence actions.

As far as we search for an equilibrium, we do not need to distinguish between two frameworks:

- Unknown distributions: in the nature, θ is non-random; however, players, not knowing θ, treat it as being random.
- Correlated signals: θ is chosen (by the nature) at random, and its distribution is common knowledge.

When investigating distributions of actions, still, we may unite the two frameworks.[1] That is, we (just like players) treat θ as being random; however, sometimes we consider conditional probabilities (and expectations) given θ. That is, if we theoreticians (unlike players) know θ, we just condition on it (but only after finding an equilibrium).

Formally, our game is described by (recall (1b2))

$$(\mathcal{S}_1,\mathcal{S}_2;\mathcal{A}_1,\mathcal{A}_2;\Theta,P_\Theta;(P_{S_1|\theta}),(P_{S_2|\theta});\mathbf{\Pi}_1,\mathbf{\Pi}_2)\,;$$

here $\mathcal{S}_1,\mathcal{S}_2$ are signal spaces; $\mathcal{A}_1,\mathcal{A}_2$ are action spaces; Θ is the parameter space; P_Θ is the distribution of the parameter; $(P_{S_1|\theta})$ is a family of distributions of the first signal, parametrized by θ; the same for the second; and $\mathbf{\Pi}_1,\mathbf{\Pi}_2$ are payoff functions. The game is symmetric in the sense that (recall (1b3))

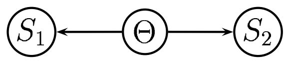
$$\mathcal{S}_1=\mathcal{S}_2\,;\quad \mathcal{A}_1=\mathcal{A}_2\,;\quad P_{S_1|\theta}=P_{S_2|\theta}\text{ for all }\theta\in\Theta\,;\quad \mathbf{\Pi}_1=\mathbf{\Pi}_2\,.$$

[1] Though, the joint distribution of correlated signals in general cannot be represented via a random common parameter θ. In that sense, the 'correlated signals' framework is more general than the 'unknown distributions' framework. However, I believe that in reality it is very typical that correlation is caused by random common parameters.

Our 'not-so-simple auction' is a symmetric game described by (recall (1b4))

$$\begin{gathered}\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S} = \mathbb{R}; \qquad \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A} = [0,\infty)\,;\\ \Theta = \mathbb{R}; \quad P_\Theta = \mathrm{U}(0.9,1)\,;\\ P_{S_1|\theta} = P_{S_2|\theta} = P_{S|\theta} = \mathrm{U}(0,\theta)\,;\\ \mathbf{\Pi}_1 = \mathbf{\Pi}_2 = \mathbf{\Pi} \text{ is the function defined by (3a1), (3d1), (3e1)}\,.\end{gathered}$$

We introduce random variables S_1, S_2, Θ such that Θ is distributed P_Θ, and conditionally, given $\Theta = \theta$, random variables S_1, S_2 are independent, distributed $P_{S|\theta}$ (each). More formally, we have a Markovian random field on a graph,[2]



which means that the conditional distribution of S_2 given S_1 and Θ depends on Θ but not on S_1;

$$\mathbb{P}\left(S_2 \le s_2 \,\middle|\, S_1, \Theta \right) = \mathbb{P}\left(S_2 \le s_2 \,\middle|\, \Theta \right) = F_{S|\Theta}(s_2)\,,$$

or equivalently,

$$\mathbb{E}\left(\varphi(S_2) \,\middle|\, S_1, \Theta \right) = \mathbb{E}\left(\varphi(S_2) \,\middle|\, \Theta \right) = \int \varphi(s_2)\, dF_{S|\Theta}(s_2)$$

for every bounded measurable function $\varphi : \mathcal{S} \to \mathbb{R}$. The same for S_1, given S_2 and Θ. Specifically,

$$\begin{gathered}F_{S|\theta}(s) = \min\Big(1, \frac{s}{\theta}\Big) \quad \text{for } s \in [0,1]\,;\\ \int \varphi(s)\, dF_{S|\theta}(s) = \frac{1}{\theta}\int_0^\theta \varphi(s)\, ds\,.\end{gathered}$$

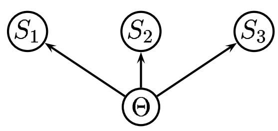
Further, the conditional independence means

$$\mathbb{E}\left(\varphi(S_1)\psi(S_2) \,\middle|\, \Theta \right) = \mathbb{E}\left(\varphi(S_1) \,\middle|\, \Theta \right) \mathbb{E}\left(\varphi(S_2) \,\middle|\, \Theta \right)\,,$$

which follows from the Markov property:

$$\begin{gathered}\mathbb{E}\left(\varphi(S_1)\psi(S_2) \,\middle|\, \Theta \right) = \mathbb{E}\Big(\mathbb{E}\left(\varphi(S_1)\psi(S_2) \,\middle|\, S_1, \Theta \right) \Big|\, \Theta \Big) =\\ \mathbb{E}\Big(\varphi(S_1)\mathbb{E}\left(\psi(S_2) \,\middle|\, S_1, \Theta \right) \Big|\, \Theta \Big) = \mathbb{E}\Big(\varphi(S_1)\mathbb{E}\left(\psi(S_2) \,\middle|\, \Theta \right) \Big|\, \Theta \Big) = \mathbb{E}\left(\psi(S_2) \,\middle|\, \Theta \right) \cdot \mathbb{E}\left(\varphi(S_1) \,\middle|\, \Theta \right)\,.\end{gathered}$$

[2] The graph is a chain, thus we have just a Markov chain (S_1, Θ, S_2). However, for more than two players the graph is not a chain, it is a tree:



Therefore

$$\mathbb{E}\left(\varphi(S_1)\psi(S_2)\chi(\Theta)\right) = \int \left(\int \varphi(s)\, dP_{S|\theta}(s) \right) \left(\int \psi(s)\, dP_{S|\theta}(s) \right) dP_\Theta(\theta)$$

and, more generally,

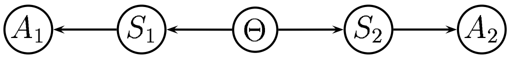
$$\mathbb{E}\left(\varphi(S_1, S_2, \Theta)\right) = \int \left(\iint \varphi(s_1, s_2, \theta)\, dP_{S|\theta}(s_1)\, dP_{S|\theta}(s_2) \right) dP_\Theta(\theta)\,. \tag{8a1}$$

Specifically,

$$\mathbb{E}\left(\varphi(S_1)\psi(S_2)\chi(\Theta)\right) = 10 \int_{0.9}^1 \left(\frac{1}{\theta} \int_0^\theta \varphi(s)\, ds \right) \left(\frac{1}{\theta} \int_0^\theta \psi(s)\, ds \right) \chi(\theta)\, d\theta\,;$$

$$\mathbb{P}\left(S_1 \le s_1, S_2 \le s_2, \Theta \le \theta \right) = 10 \int_{0.9}^1 \min(1, \tfrac{s_1}{\theta}) \min(1, \tfrac{s_2}{\theta})\, d\theta \quad \text{for } \theta \in [0.9, 1]\,.$$

Given strategies μ_1, μ_2, we introduce random variables A_1, A_2 getting a Markovian random field on a larger graph



The Markov property means a number of equalities:

$$\begin{aligned} \mathbb{E}\left(\varphi(A_2) \,\middle|\, A_1, S_1, \Theta, S_2 \right) &= \mathbb{E}\left(\varphi(A_2) \,\middle|\, S_2 \right)\,, \\ \mathbb{E}\left(\varphi(S_2, A_2) \,\middle|\, A_1, S_1, \Theta \right) &= \mathbb{E}\left(\varphi(S_2, A_2) \,\middle|\, \Theta \right)\,, \\ \mathbb{E}\left(\varphi(\Theta, S_2, A_2) \,\middle|\, A_1, S_1 \right) &= \mathbb{E}\left(\varphi(\Theta, S_2, A_2) \,\middle|\, S_1 \right)\,, \end{aligned}$$

etc. It follows that

$$\begin{aligned} &\mathbb{E}\left(\varphi_1(A_1)\psi_1(S_1)\chi(\Theta)\psi_2(S_2)\varphi_2(A_2)\right) = \\ &\quad \int \left(\int \mathbb{E}\left(\varphi_1(A_1) \,\middle|\, S_1 = s_1 \right) \psi_1(s_1)\, dP_{S|\theta}(s_1) \right) \cdot \\ &\qquad \cdot \left(\int \mathbb{E}\left(\varphi_2(A_2) \,\middle|\, S_2 = s_2 \right) \psi_2(s_2)\, dP_{S|\theta}(s_2) \right) \chi(\theta)\, dP_\Theta(\theta)\,; \end{aligned}$$

here $\mathbb{E}\left(\varphi_1(A_1) \,\middle|\, S_1 = s_1 \right)$ is determined by μ_1 (recall 1c); the same for μ_2. You may also write out a counterpart of (8a1).

Similarly to (1d1) we define

$$\mathbf{\Pi}(\mu_1; \mu_2) = \mathbb{E}\, \mathbf{\Pi}(A_1, S_1; A_2, S_2)\,.$$

Now, a best response and an equilibrium are defined in the same way as in 1e.

8b Optimal actions and best response

We want to find the best response μ_1 to a given strategy μ_2. Similarly to 2a we find an upper bound for $\mathbf{\Pi}(\mu_1, \mu_2)$ for every μ_1;

$$\mathbf{\Pi}(\mu_1,\mu_2) = \mathbb{E}\mathbf{\Pi}(A_1,S_1;A_2,S_2) = \mathbb{E}\mathbf{\Pi}(A_1,S_1;A_2) = \mathbb{E}\Big(\mathbb{E}\left(\mathbf{\Pi}(A_1,S_1;A_2)\,\middle|\,A_1,S_1\right)\Big);$$

$$\mathbb{E}\left(\mathbf{\Pi}(A_1,S_1;A_2)\,\middle|\,A_1=a_1,S_1=s_1\right) = \mathbb{E}\left(\mathbf{\Pi}(a_1,s_1;A_2)\,\middle|\,A_1=a_1,S_1=s_1\right) =$$
$$\mathbb{E}\left(\mathbf{\Pi}(a_1,s_1;A_2)\,\middle|\,S_1=s_1\right) = \int \mathbf{\Pi}(a_1,s_1;a_2)\,dP_{A_2|S_1=s_1}(a_2) = \mathbf{\Pi}(a_1,s_1;P_{A_2|S_1=s_1});$$

introducing

$$\mathbf{\Pi}^{\max}(s_1;\mu_2) = \mathbf{\Pi}^{\max}\big(s_1;P_{A_2|S_1=s_1}\big) = \sup_{a_1\in\mathcal{A}} \mathbf{\Pi}(a_1,s_1;P_{A_2|S_1=s_1})$$

we get $\mathbb{E}\left(\mathbf{\Pi}(A_1,S_1;A_2)\,\middle|\,A_1=a_1,S_1=s_1\right) \le \mathbf{\Pi}^{\max}(s_1;\mu_2)$ and

$$\mathbf{\Pi}(\mu_1;\mu_2) \le \mathbb{E}\mathbf{\Pi}^{\max}(S_1;\mu_2) = \int \mathbf{\Pi}^{\max}(s_1;\mu_2)\,dP_{S_1}(s_1) = \mathbf{\Pi}^{\max}(P_{S_1},\mu_2).$$

In contrast to 2a, the signal s_1 plays two roles, according to its two occurrences in such expressions as

$$\mathbf{\Pi}(a_1,s_1;P_{A_2|S_1=s_1}); \quad \mathbf{\Pi}^{\max}(s_1;P_{A_2|S_1=s_1}).$$

In order to make it more clear (or more vague?) we may separate the two roles by splitting the signal in two, s_1^{int} and s_1^{ext}; the 'internal' signal s_1^{int} informs the first player about his valuation; the 'external' signal s_1^{ext}, being correlated with s_2, informs the first player (to some extent) about s_2. That is, we consider

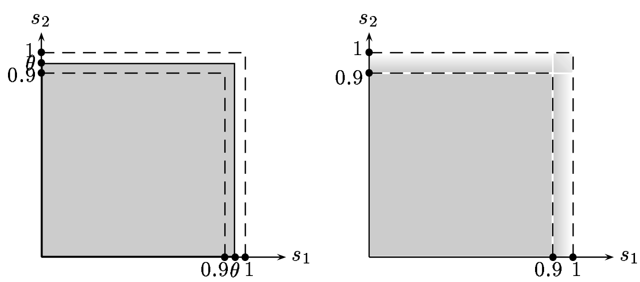
$$\mathbf{\Pi}(a_1,s_1^{\text{int}};P_{A_2|S_1=s_1^{\text{ext}}});$$

only the case $s_1^{\text{int}} = s_1^{\text{ext}} = s_1$ is relevant; however, sometimes it is instructive, to split the effect of s_1 into effects of s_1^{int} and s_1^{ext}.

If s_1^{ext} is kept constant while s_1^{int} runs, then we have the situation studied in Sect. 2: a best response to a given distribution $P_{A_2|S_1=s_1^{\text{ext}}}$. We know that it is an increasing pure strategy except (maybe) for never-winning actions (recall 2c6). Let $a_2^{\min}$ be the least point of the support of $P_{A_2|S_1=s_1^{\text{ext}}}$ (it can be proven that $a_2^{\min}$ does not depend on s_1^{ext}); for $s_1 > a_2^{\min}$ we have the optimal action $a_1 = \varphi(s_1^{\text{int}}, s_1^{\text{ext}})$, an increasing function of s_1^{int} for a constant s_1^{ext}.

What happens when s_1^{ext} varies?

Return to our example, the 'not-so-simple auction'. The joint distribution P_{S_1,S_2} is the mixture of $\mathrm{U}(0,\theta)\otimes\mathrm{U}(0,\theta)$ for $\theta \sim \mathrm{U}(0.9,1)$.



The one-dimensional density $f_{S_1} = f_{S_2} = f_S$ is

$$f_S(s) = 10 \int_{0.9}^{1} \frac{1}{\theta} \mathbf{1}_{(0,\theta)}(s)\, d\theta = 10 \int_{0.9}^{1} \frac{1}{\theta} \mathbf{1}_{(s,\infty)}(\theta)\, d\theta =$$

$$= 10 \int_{\max(s,0.9)}^{1} \frac{d\theta}{\theta} = -10 \ln \max(s, 0.9) = 10 \min\bigl(\ln(1/s), \ln(10/9)\bigr)\,.$$

The two-dimensional density f_{S_1,S_2} is

$$f_{S_1,S_2}(s_1, s_2) = 10 \int_{0.9}^{1} \frac{1}{\theta} \mathbf{1}_{(0,\theta)}(s_1) \frac{1}{\theta} \mathbf{1}_{(0,\theta)}(s_2)\, d\theta =$$

$$10 \int_{\max(s_1,s_2,0.9)}^{1} \frac{d\theta}{\theta^2} = \frac{10}{\max(s_1, s_2, 0.9)} - 10 \quad \text{for } s_1, s_2 \in (0,1)\,.$$

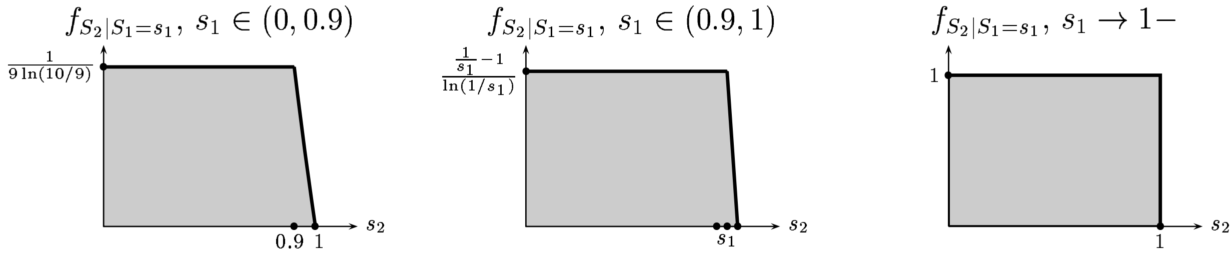
The conditional density $f_{S_2|S_1}$ is

$$f_{S_2|S_1=s_1}(s_2) = \frac{f_{S_1,S_2}(s_1, s_2)}{f_{S_1}(s_1)} = \frac{\bigl(1/\max(s_1, s_2, 0.9)\bigr) - 1}{\min(\ln(1/s_1), \ln(10/9))}\,;$$

when $s_1 \in (0, 0.9)$, the conditional density does not depend on s_1,

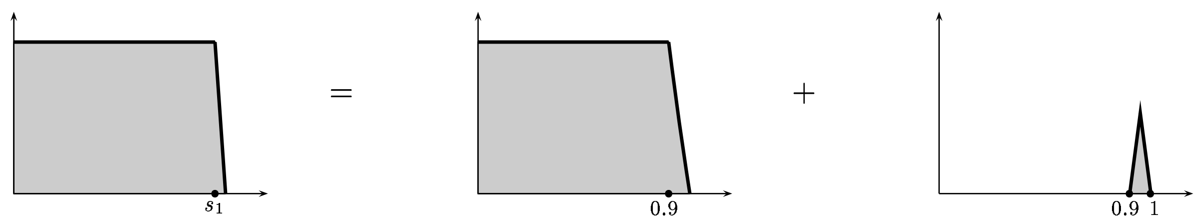
$$f_{S_2|S_1=s_1}(s_2) = \frac{\bigl(1/\max(s_2, 0.9)\bigr) - 1}{\ln(10/9)} \quad \text{for } s_1 \in (0, 0.9)\,,$$

which is, of course, a special property of our example.



The case $s_1^{\text{ext}} \in (0, 0.9)$ is easy; all these values may be replaced with one of them, say, 0.9, thus $\varphi(s_1^{\text{int}}, s_1^{\text{ext}}) = \varphi(s_1^{\text{int}}, 0.9)$ and $\varphi(s_1, s_1) = \varphi(s_1, 0.9)$, an increasing function on $(0, 0.9)$.

The case $s_1^{\text{ext}} \in (0.9, 1)$ is not easy. The conditional distribution $P_{S_2|S_1=s_1}$ may be represented as a mixture of $P_{S_2|S_1=0.9}$ and another distribution, concentrated on $(0.9, 1)$.

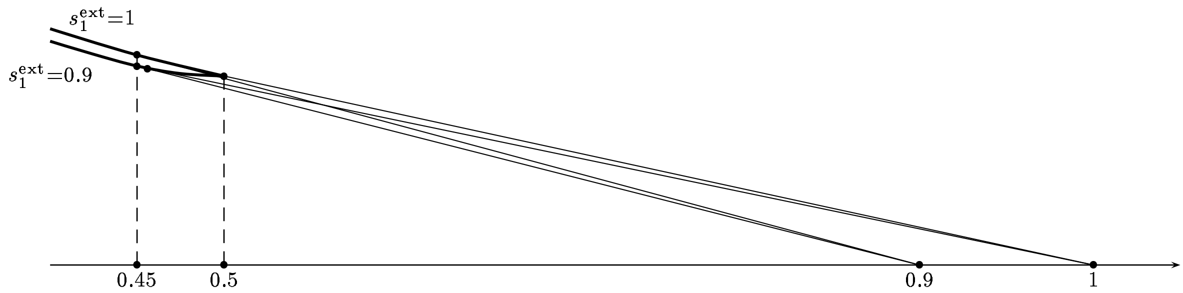


Accordingly, $P_{A_2|S_1=s_1}$ is a mixture of $P_{A_2|S_1=0.9}$ and another distribution, actions of signals on $(0.9, 1)$. How does it influence the best response? It depends on the strategy μ_2.

Assume for a while that μ_2 is just $A_2 = \frac12 S_2$. Of course, it need not be true; however, we may hope that it is nearly true, since our 'not-so-simple auction' is close to the 'very simple auction' of Sect. 1. We have $\mathbb{P}\left(A_2 \le a \,\middle|\, S_1 = s_1 \right) = \mathbb{P}\left(S_2 \le 2a \,\middle|\, S_1 = s_1 \right)$; for $a \in [0, 0.45]$ it is simple:[3]

$$F_{A_2|S_1=s_1}(a) = \operatorname{const}(s_1) \cdot a\,;$$
$$\max_{a\in[0,0.45]} (s_1^{\text{int}} - a) F_{A_2|S_1=s_1^{\text{ext}}}(a) = \operatorname{const}(s_1) \cdot \max_{a\in[0,0.45]} a(s_1^{\text{int}} - a)\,;$$

the optimum is reached at $a_1 = \frac12 s_1^{\text{int}}$, if $s_1^{\text{int}} \in [0, 0.9]$; note that s_1^{ext} does not matter here. Till now, the solution is the same as in Sect. 1. What happens for larger a_1? Here, s_1^{ext} matters.



If $s_1^{\text{ext}} = 1$ then the formula $\frac12 s_1^{\text{int}}$ holds till $s_1^{\text{int}} = 1$, and so, the optimal action for $s_1^{\text{int}} = 1$ is 0.5. However, if $s_1^{\text{ext}} = 0.9$ then the optimal action for $s_1^{\text{int}} = 1$ is $0.456\dots$ You see, in that case, more aggressive bidding of the competitor makes the best response more aggressive.

Generally, for any two distribution functions W_1 and W_2 such that functions $a \mapsto 1/W_1(a)$ and $a \mapsto 1/W_2(a)$ are strictly convex, the following two conditions are equivalent.

- $\varphi_1(s) \le \varphi_2(s)$ for all s; here $\varphi_1(s)$ is the action optimal for s against W_1, and φ_2 — against W_2.
- The function $a \mapsto W_2(a)/W_1(a)$ increases.

Here is a sufficient (but not necessary) condition.

- The function $a \mapsto W_2'(a)/W_1'(a)$ increases.

8b1. Exercise. Assuming $A_2 = \frac12 S_2$ show that

(a) The optimal action $\varphi(s_1^{\text{int}}, s_1^{\text{ext}})$ increases in s_1^{ext} for a fixed s_1^{int};

(b) $\varphi(s_1, s_1)$ increases in s_1.

Hint: (a) consider the quotient of densities; (b) $\varphi(s_1^{\text{int}}, s_1^{\text{ext}})$ increases in each argument separately.

8b2. Exercise. Generalize 8b1 for an arbitrary increasing pure strategy μ_2.

Hint. Increase of W_2/W_1 is invariant under increasing transformations of the argument.

8b3. Exercise. Generalize 8b2 for an arbitrary joint distribution of signals S_1, S_2 having a two-dimensional density $f_{S_1,S_2} = f$ such that[4]

$$\begin{vmatrix} f(x,u) & f(x,v) \\ f(y,u) & f(y,v) \end{vmatrix} \ge 0 \qquad \text{whenever } x \le y\,, u \le v\,.$$

[3]Here $\operatorname{const}(s_1) = 2\left(\frac{1}{s_1} - 1\right)/\left(\ln(1/s_1)\right)$, but we do not need it.

[4]Recall that $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| = ad - bc$.

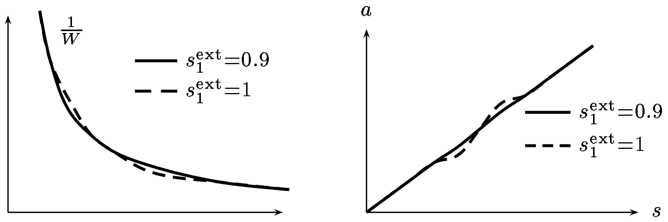
Hint: $\dfrac{f_{S_2|S_1=y}(u)}{f_{S_2|S_1=x}(u)} \le \dfrac{f_{S_2|S_1=y}(v)}{f_{S_2|S_1=x}(v)}$.

Random variables S_1, S_2 satisfying the assumption of 8b3 are called *affiliated.* The inequality is equivalent to

$$\frac{\partial^2}{\partial x \partial y} \ln f(x,y) \ge 0 \quad \text{for all } x, y$$

provided that f is smooth.

For independent signals we know that the best response is monotone (by 2c6). For correlated signals the situation is much less clear, even if signals are affiliated. If μ_2 is monotone (and $1/W_1$ is convex) then μ_1 is monotone. So what? Maybe, both are non-monotone? If μ_2 maps signals on $(0.9, 1)$ to actions somewhere in the middle of $(0, 0.5)$ then the ratio of distribution functions for (say) $s_1^{\text{ext}} = 0.9$ and $s_1^{\text{ext}} = 1$ is non-monotone.



The increase of s_1^{ext} makes some signals more aggressive, but some other signals — less aggressive. Can it lead to a non-monotone equilibrium? I do not know.[5] On the other hand, results of 8b1–8b3 inspire a hope that a monotone equilibrium exists (due to affiliation).

8c Monotone equilibria (informal)

We consider a continuous, strictly increasing function φ such that $\varphi(s) < s$ for all s, and the corresponding strategy $A = \varphi(S)$; we want to find a condition for the strategy to support a symmetric equilibrium (of our 'not-so-simple auction' game).

For almost every $s \in (0,1)$, the action $\varphi(s)$ must be optimal for s against $P_{A_2|S_1=s} = P_{\varphi(S_2)|S_1=s}$. The optimality means[6]

$$(s-a)F_{\varphi(S_2)|S_1=s}(a) \le \big(s - \varphi(s)\big) F_{\varphi(S_2)|S_1=s}\big(\varphi(s)\big) \quad \text{for all } a\,. \tag{8c1}$$

It is necessary that the inequality holds for $a = \varphi(t)$:

$$\big(s - \varphi(t)\big) F_{S_2|S_1=s}(t) \le \big(s - \varphi(s)\big) F_{S_2|S_1=s}(s) \quad \text{for all } t\,, \tag{8c2}$$

since $F_{A_2|S_1=s}\big(\varphi(t)\big) = \mathbb{P}\big(\, \varphi(S_2) \le \varphi(t) \,\big|\, S_1 = s \,\big) = \mathbb{P}\big(\, S_2 \le t \,\big|\, S_1 = s \,\big) = F_{S_2|S_1=s}(t)$. We assume that $\varphi(t) < s$ (otherwise the inequality holds for a trivial reason). We may write the inequality in the form

$$\frac{F_s(t)}{F_s(s)} \le \frac{s - \varphi(s)}{s - \varphi(t)}\,,$$

[5]Convexity of $1/W_k$ is also a problem (especially for n players).

[6]Note that we consider a *private value* auction; for a more general case see 8e.

where $F_s(t) = F_{S_2|S_1=s}(t)$. That is,[7]

(8c3)
$$\ln F_s(t) - \ln F_s(s) \le \ln\bigl(s - \varphi(s)\bigr) - \ln\bigl(s - \varphi(t)\bigr) .$$

Assuming that functions $F_s(\cdot)$ and $\varphi(\cdot)$ are smooth (namely, have continuous first derivatives) we try $t \to s-$, $t \to s+$ and conclude that the equality

$$\frac{d}{dt}\Big|_{t=s} \ln F_s(t) = -\frac{d}{dt}\Big|_{t=s} \ln\bigl(s - \varphi(t)\bigr)$$

is necessary. That is,[8]

(8c4)
$$\frac{F_s'(s)}{F_s(s)} = \frac{\varphi'(s)}{s - \varphi(s)} ;$$

it must hold for almost all $s \in (0,1)$, therefore (by continuity), for all $s \in [0,1]$. For a given distribution of signals, it is a differential equation for φ; it determines φ uniquely up to a constant. On the other hand, for a given φ, does it determine the corresponding distribution of signals? Surely, not; $F_s'(s)/F_s(s)$ tells us nothing about $F_s(t)$ outside (a neighborhood of) the diagonal $t = s$.

It is instructive to think, what happens for *independent* signals. Here, $F_s(t) = F(t)$, and so, $\frac{F'(s)}{F(s)} = \frac{\varphi'(s)}{s-\varphi(s)}$. And, of course, we know the solution (recall 3d3); no need to solve the differential equation. Now, returning to correlated signals and their $F_s(t)$, we may construct another distribution function $\tilde F(\cdot)$ such that

(8c5)
$$\frac{F_s'(s)}{F_s(s)} = \frac{\tilde F'(s)}{\tilde F(s)} \quad \text{for all } s ,$$

namely,

(8c6)
$$\tilde F(s) = \exp\left(\int \frac{F_s'(s)}{F_s(s)} \, ds \right) .$$

We may consider the first price auction with independent signals distributed $\tilde F$ (call it 'associated' with the original auction), and its equilibrium strategy $A = \varphi(S)$. Hopefully, the same φ gives us an equilibrium strategy for correlated signals, too!

Specifically, we have the conditional density (recall 8b)

$$f_s(t) = F_s'(t) = \frac{(1/\max(s,t,0.9)) - 1}{\min(\ln(1/s), \ln(10/9))} ;$$
$$f_s(s) = \frac{1}{9\ln(10/9)} \approx 1.055 \quad \text{for } s \in [0, 0.9] ,$$
$$f_s(s) = \frac{\frac{1}{s} - 1}{\ln(1/s)} \quad \text{for } s \in [0.9, 1] .$$

[7]Do not confuse the symbol $F_s(\cdot)$ introduced here (temporarily, for convenience) with the symbol $F_S(\cdot)$ (always stands for the cumulative distribution function of S).

[8]Note that $F_s'(s)$ is not the same as $\frac{d}{ds}F_s(s)$, and $F_s'(s)/F_s(s)$ is not the same as $(\ln F_s(s))'$.

We could calculate the corresponding conditional distribution function $F_s(\cdot)$ by integration, and then $F_s'(s)/F_s(s) = f_s(s)/F_s(s)$. However, the answer is evident! Just recall that $f_s(\cdot)$ is constant on $(0, s]$; we have

$$f_s(t) = \text{const} \quad \text{for } t \in (0, s)\,;$$
$$F_s(s) = \int_0^s f_s(t)\,dt = \text{const} \cdot s\,;$$
$$\frac{F_s'(s)}{F_s(s)} = \frac{f_s(s)}{F_s(s)} = \frac{\text{const}}{\text{const} \cdot s} = \frac{1}{s} \quad \text{for } s \in (0,1)\,.$$

We construct the function $\tilde{F}(\cdot)$:

$$\tilde{F}(s) = \exp\left(\int \frac{1}{s}\,ds \right) = \text{const} \cdot s\,;$$

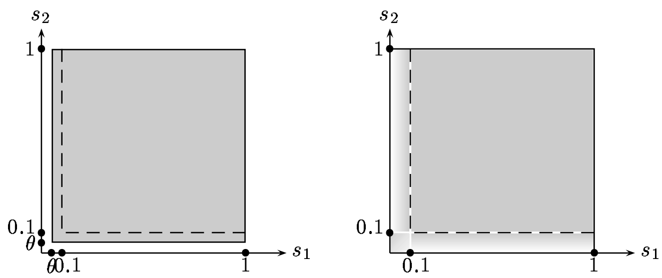
in order to get a distribution function we choose const $= 1$, and so, $\tilde{F}$ describes the uniform distribution $\mathrm{U}(0,1)$. The corresponding equilibrium is known to us long ago:

$$\varphi(s) = \frac{1}{2}s\,; \qquad A = \frac{1}{2}S\,.$$

We did not prove yet that it really is an equilibrium for our 'not-so-simple auction' with correlated signals, but hopefully it is.

Why signal dependence does not change the strategy function φ? Due to a special property of our case; given $S_1 = s_1$, the conditional distribution of S_2 is uniform within $(0, s_1)$ (but not uniform in the whole). The first player, knowing his signal s_1, optimizes his action against S_2 assuming that $S_2 < s_1$ (since otherwise he does not win, as far as the equilibrium is monotone and symmetric).[9] In that sense, the optimization is one-sided.

In order to make the story more dramatical, consider another case: let the joint distribution P_{S_1,S_2} be the mixture of $\mathrm{U}(\theta,1) \otimes \mathrm{U}(\theta,1)$ for $\theta \sim \mathrm{U}(0, 0.1)$.



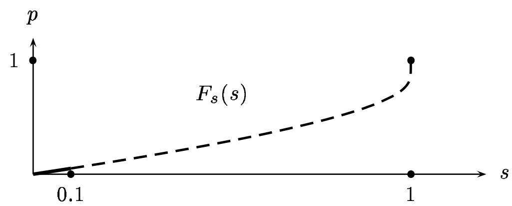
[9] The argument is not quite convincing, since the first player should check all alternative actions, including $a > \varphi(s_1)$, and here he should not restrict himself to $S_2 < s_1$. For now we only check actions infinitesimally close to $\varphi(s_1)$; that is, we perform only local optimization. Global optimization will be treated later.

We have

$$f_S(s) = 10 \min\Bigl(\ln \frac{1}{1-s}, \ln \frac{10}{9}\Bigr)\,;$$
$$f_{S_1,S_2}(s_1,s_2) = \frac{10}{1-\min(s_1,s_2,0.1)} - 10\,;$$
$$f_s(t) = f_{S_2|S_1=s}(t) = \frac{1}{\min(\ln \frac{1}{1-s}, \ln \frac{10}{9})}\biggl(\frac{1}{1-\min(s,t,0.1)} - 1\biggr)\,;$$
$$f_s(s) = \frac{1}{\ln \frac{1}{1-s}} \cdot \biggl(\frac{1}{1-s} - 1\biggr) \quad \text{for } s \in (0, 0.1)\,,$$
$$f_s(s) = \frac{1}{9 \ln(10/9)} \quad \text{for } s \in (0.1, 1)\,.$$

For $s \in (0, 0.1)$ we get

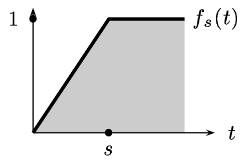
$$F_s(s) = \int_0^s f_s(t)\,dt = \frac{1}{\ln \frac{1}{1-s}} \int_0^s \biggl(\frac{1}{1-t} - 1\biggr)\,dt = 1 - \frac{s}{\ln \frac{1}{1-s}}\,.$$



We may also expand the function into powers:[10]

$$F_s(s) = \frac{1}{2}s + \frac{1}{12}s^2 + O(s^3) \qquad \text{for } s \to 0\,;$$

think, why $F_s(s) \approx \frac{1}{2}s$ for small s; hint:



For $s \in [0.1, 1)$ we get

$$F_s(s) = \int_0^s f_s(t)\,dt = \frac{1}{\ln(10/9)}\biggl(\int_0^{0.1} \Bigl(\frac{1}{1-t} - 1\Bigr)\,dt + \int_{0.1}^s \frac{1}{9}\,dt\biggr) =$$
$$= \frac{1}{\ln(10/9)}\biggl(\ln(10/9) - 0.1 + \frac{1}{9}(s - 0.1)\biggr) = 1 + \frac{s-1}{9\ln(10/9)}\,.$$

[10]Indeed, $-\ln(1-s) = s + \frac{1}{2}s^2 + \frac{1}{3}s^3 + O(s^4)$; $-\frac{1}{s}\ln(1-s) = 1 + \frac{1}{2}s + \frac{1}{3}s^2 + O(s^3)$; $\frac{s}{-\ln(1-s)} = \frac{1}{1+(\frac{1}{2}s+\frac{1}{3}s^2+O(s^3))} = 1 - (\frac{1}{2}s + \frac{1}{3}s^2 + O(s^3)) + (\frac{1}{2}s + O(s^2))^2 = 1 - \frac{1}{2}s - \frac{1}{3}s^2 + \frac{1}{4}s^2 + O(s^3)$.

We have to integrate the function

$$\frac{F_s'(s)}{F_s(s)} = \begin{cases} \frac{\frac{1}{1-s}-1}{\ln\frac{1}{1-s}-s} & \text{for } s \in (0,0.1), \\ \frac{1}{9\ln(10/9)+s-1} & \text{for } s \in (0.1,1). \end{cases}$$

Good luck: the integration can be performed explicitly. On the interval $(0.1, 1)$ we have

$$\int \frac{F_s'(s)}{F_s(s)}\,ds = \int \frac{ds}{s+9\ln(10/9)-1}\,ds = \ln(s+9\ln(10/9)-1) + \text{const}\,;$$

$\tilde F(s) = \text{const}\cdot(s+9\ln(10/9)-1)$; we choose the constant such that $\tilde F(1) = 1$:

$$\tilde F(s) = \frac{s+9\ln(10/9)-1}{9\ln(10/9)} = 1 - \frac{1-s}{9\ln(10/9)} \quad \text{for } s \in [0.1,1)\,.$$

On the interval $[0, 0.1]$ we have

$$\int \frac{F_s'(s)}{F_s(s)}\,ds = \int \frac{\frac{1}{1-s}-1}{\ln\frac{1}{1-s}-s}\,ds = \int \frac{\big(\ln\frac{1}{1-s}-s\big)'}{\ln\frac{1}{1-s}-s}\,ds = \ln\Big(\ln\frac{1}{1-s}-s\Big) + \text{const}\,;$$

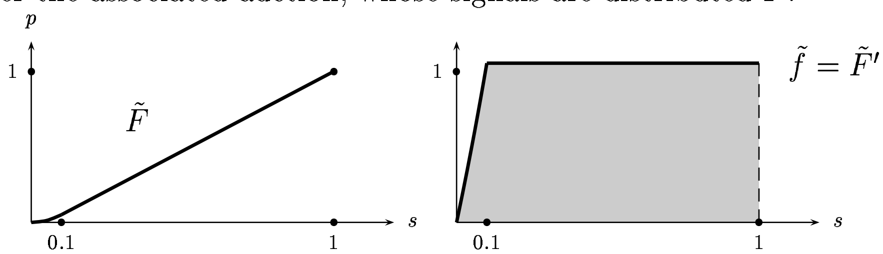
$\tilde F(s) = \text{const}\cdot\big(\ln\frac{1}{1-s}-s\big)$. We choose the constant by continuity at 0.1; namely, $\tilde F(0.1) = \frac{\ln(10/9)-0.1}{\ln(10/9)}$, thus

$$\tilde F(s) = \frac{1}{\ln(10/9)}\Big(\ln\frac{1}{1-s}-s\Big) \quad \text{for } s \in [0,0.1]\,.$$

Note also that $\tilde F(s) = \frac{1}{\ln(10/9)}\cdot\big(\frac12 s^2 + \frac13 s^3 + O(s^4)\big)$ for $s \to 0$. And here is the density:

$$\tilde F'(s) = \frac{1}{\ln(10/9)}\min\Big(\frac{s}{1-s},\frac19\Big)\,.$$

So, we consider the associated auction, whose signals are distributed $\tilde F$.



It appears to be close to the (unconditional, marginal) signal distribution of the original auction (too close for seeing the difference on a graph), but different. The equilibrium strategy of the associated auction is $A = \varphi(S)$ where (recall 3d3)

$$\varphi(s) = \frac{1}{\tilde F(s)}\int_0^s s\,d\tilde F(s)\,.$$

For $s \in (0, 0.1)$ we have

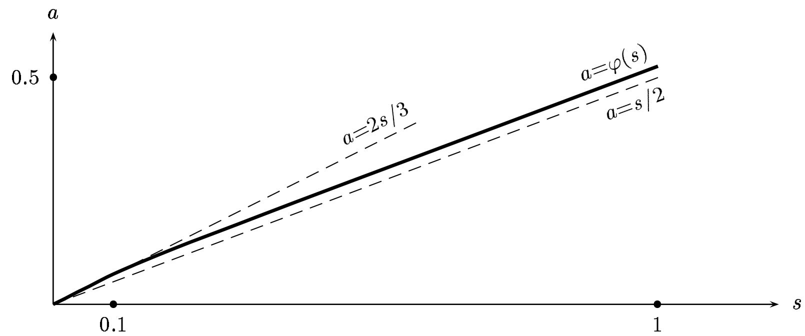
$$\Big(\ln \frac{1}{1-s} - s \Big) \varphi(s) = \int_0^s t \frac{t}{1-t} \, dt = \int_0^s \Big(\frac{1}{1-t} - 1 - t \Big) \, dt = \ln \frac{1}{1-s} - s - \frac12 s^2 \, ;$$

$$\varphi(s) = \frac{ \ln \frac{1}{1-s} - s - \frac12 s^2 }{ \ln \frac{1}{1-s} - s } \, .$$

Note that $\varphi(s) = \frac{ \frac13 s^3 + O(s^4) }{ \frac12 s^2 + O(s^3) } = \frac23 s + O(s^2)$ for $s \to 0$. For $s \in (0.1, 1)$ we have

$$\begin{aligned} \Big(\ln \frac{10}{9} - \frac{1-s}{9} \Big) \varphi(s) &= \Big(\ln \frac{10}{9} \Big) \int_0^{0.1} t \, d\tilde F(t) + \Big(\ln \frac{10}{9} \Big) \int_{0.1}^s t \, d\tilde F(t) = \\ &= \ln \frac{10}{9} - 0.1 - \frac{0.01}{2} + \int_{0.1}^s t \cdot \frac19 \, dt = \ln \frac{10}{9} - 0.105 + \frac1{18} (s^2 - 0.1^2) \, ; \end{aligned}$$

$$\varphi(s) = \frac{ \frac1{18} s^2 + \big(\ln \frac{10}{9} - 0.105 - \frac{0.01}{18} \big) }{ \frac19 s + \ln \frac{10}{9} - \frac19 } \approx \frac12 \frac{ s^2 - 0.0035 }{ s - 0.0518 } \, .$$



8d Monotone equilibria (formal)

We return to (8c1)–(8c6); an independent-signals auction is associated with the given (original) correlated-signals auction. Let φ^{assoc} be the equilibrium strategy function[11] of the associated auction, $\varphi^{\text{assoc}}(s) = \frac{1}{\tilde F(s)} \int_0^s t \, d\tilde F(t)$. We address questions:

- Is φ^{assoc} an equilibrium strategy function of the original auction?
- Have the original auction other monotone symmetric equilibria?

We assume that the two-dimensional distribution of signals has a density f_{S_1,S_2}, concentrated on the square $(0, s^{\max}) \times (0, s^{\max})$, strictly positive and continuous on the square.[12]

[11] The equilibrium strategy is the joint distribution of random variables $S, \varphi^{\text{assoc}}(S)$.

[12] Not the most general case, of course.

Thus, we have

$$f_s(t) = F'_s(t) = \frac{f_{S_1,S_2}(s,t)}{f_{S_1}(s)} = \frac{f_{S_1,S_2}(s,t)}{\int_0^{s^{\max}} f_{S_1,S_2}(s,u)\,du}\,;$$

$$\frac{F'_s(t)}{F_s(t)} = \frac{f_{S_1,S_2}(s,t)}{\int_0^t f_{S_1,S_2}(s,u)\,du}\,; \quad \text{(a continuous function on the square)}$$

$$\tilde F(s) = \exp\Big(- \int_s^{s^{\max}} \frac{F'_u(u)}{F_u(u)}\,du \Big) \quad \text{for } s \in (0, s^{\max})\,.$$

Let $\varphi : (0, s^{\max}) \to \mathbb{R}$ be a strictly increasing continuous function. It describes a symmetric equilibrium if and only if (8c1) holds for almost all $s \in (0, s^{\max})$, therefore (by continuity), for all $s \in (0, s^{\max})$. That cannot happen unless

(8d1) $$\varphi(s) < s \quad \text{for all } s \in (0, s^{\max})\,.$$

Thus, φ maps $(0, s^{\max})$ onto $(0, a^{\max})$ for some $a^{\max} \in (0, s^{\max})$. If (8c1) holds for all $a \in (0, s^{\max})$ then it holds for all $a \in [0, \infty)$ (think, why). So, the combination of (8c2) and (8d1) is necessary and sufficient (for equilibrium).

Turning from (8c2) to (8c3) we must take care of the restriction $\varphi(t) < s$. We introduce

(8d2) $$\begin{gathered} X(s,t) = \ln F_s(s) - \ln F_s(t)\,, \\ Y(s,t) = \begin{cases} \ln\big(s - \varphi(s)\big) - \ln\big(s - \varphi(t)\big) & \text{if } \varphi(t) < s, \\ +\infty & \text{otherwise,} \end{cases} \end{gathered}$$

then (8c3) becomes

(8d3) $$X(s,t) + Y(s,t) \ge 0 \quad \text{for all } s,t \in (0, s^{\max})\,.$$

In combination with (8d1) it is necessary and sufficient (for equilibrium).

Condition (8c4) is an infinitesimal form of (8c3), for s, t infinitesimally close. Does it imply (8c3)?

8d4. Lemma. $Y(r,s) + Y(s,t) \le Y(r,t)$ whenever $r \le s \le t$ or $r \ge s \ge t$.

Proof. Case $Y(r,s) + Y(s,t) = +\infty$. That cannot happen for $r \ge s \ge t$, thus we have $r \le s \le t$, and $r \le \varphi(s)$ or $s \le \varphi(t)$ (or both). In every case $r \le \varphi(t)$, therefore $Y(r,t) = +\infty$.

Case $Y(r,s) + Y(s,t) < \infty$. We rewrite the inequality:

$$\begin{gathered} \frac{r - \varphi(r)}{r - \varphi(s)} \cdot \frac{s - \varphi(s)}{s - \varphi(t)} \le \frac{r - \varphi(r)}{r - \varphi(t)}\,; \\ (r - \varphi(t))(s - \varphi(s)) \le (r - \varphi(s))(s - \varphi(t))\,; \\ -r\varphi(s) - s\varphi(t) \le -r\varphi(t) - s\varphi(s)\,; \\ 0 \le (s - r)(\varphi(t) - \varphi(s))\,; \end{gathered}$$

now it follows from monotonicity of φ. □

8d5. Lemma. The following two conditions are equivalent.

(8d6) $$X(r,s) + X(s,t) \le X(r,t) \quad \text{whenever } r \le s \le t \text{ or } r \ge s \ge t\,;$$

(8d7) $$\begin{vmatrix} F_r(s) & F_r(t) \\ F_s(s) & F_s(t) \end{vmatrix} \ge 0 \quad \text{and} \quad \begin{vmatrix} F_s(r) & F_s(s) \\ F_t(r) & F_t(s) \end{vmatrix} \ge 0 \quad \text{whenever } r \le s \le t\,.$$

They are weaker than the condition

(8d8) $$\begin{vmatrix} F_{s_1}(t_1) & F_{s_1}(t_2) \\ F_{s_2}(t_1) & F_{s_2}(t_2) \end{vmatrix} \ge 0 \quad \text{whenever } s_1 \le s_2 \text{ and } t_1 \le t_2\,.$$

Proof. Immediate. □

In fact, (8d8) is weaker than affiliation (recall 8b3); in the smooth case, (8d8) is equivalent to

(8d9) $$\frac{\partial^2}{\partial s \partial t} \ln F_s(t) \ge 0\,.$$

The relation between local and global is now simple. Assume that there exist ε and M such that

(8d10) $$X(s,t) + Y(s,t) \ge -M(s-t)^2 \quad \text{whenever } |s-t| \le \varepsilon\,.$$

Then (8d6) implies (8d3). Indeed, using 8d4, for m large enough we have

$$X(s,t) + Y(s,t) \ge \sum_{k=1}^{m} \Big(X\big(s + \tfrac{k-1}{m}(t-s), s + \tfrac{k}{m}(t-s)\big) + Y\big(s + \tfrac{k-1}{m}(t-s), s + \tfrac{k}{m}(t-s)\big) \ge$$
$$\ge -mM\Big(\frac{|s-t|}{m}\Big)^2 \xrightarrow[m\to\infty]{} 0\,.$$

8d11. Exercise. Assume that we have two auctions, described by $F_s^{(1)}(t)$ and $F_s^{(2)}(t)$ respectively, such that

(8d12) $$|X^{(1)}(s,t) - X^{(2)}(s,t)| \le M(s-t)^2 \quad \text{whenever } |s-t| \le \varepsilon$$

for some ε and M; here $X^{(1)}, X^{(2)}$ are defined as in (8d2). Assume also that (8d6) holds for both auctions. Then (8d3) is either satisfied for both auctions, or violated for both auctions.[13]

Prove it.

Hint: divide $[s,t]$ into m small intervals, as before.

We want to apply 8d11 to the given (original) auction and its associated auction. The latter satisfies (8d6) (think, why); the former is assumed to satisfy (8d6). And (8d12) follows from (8c5) provided that second derivatives $\frac{\partial^2}{\partial t^2} \ln F_s(t)$ and $\frac{\partial^2}{\partial t^2} \ln \tilde F(t)$ exist and are bounded.

[13] The same function φ is used for both auctions.

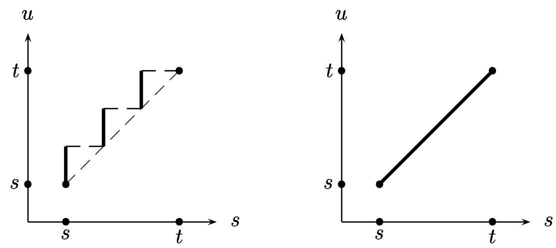
The latter smoothness condition is not really needed. Instead of (8d12), it is enough to satisfy

(8d13)
$$\sum_{k=1}^{m} \Big| X^{(1)}\big(s+\tfrac{k-1}{m}(t-s), s+\tfrac{k}{m}(t-s)\big) - X^{(2)}\big(s+\tfrac{k-1}{m}(t-s), s+\tfrac{k}{m}(t-s)\big) \Big| \xrightarrow[m\to\infty]{} 0\,,$$

where $X^{(1)}(s,t) = \ln F_s(s) - \ln F_s(t)$ and $X^{(2)}(s,t) = \ln \tilde F(s) - \ln \tilde F(t)$. We have

$$|X^{(1)}(s,t) - X^{(2)}(s,t)| = \bigg| \int_t^s \frac{F'_s(u)}{F_s(u)}\,du - \int_t^s \frac{\tilde F'(u)}{\tilde F(u)}\,du \bigg| \le \bigg| \int_t^s \Big(\frac{F'_s(u)}{F_s(u)} - \frac{F'_u(u)}{F_u(u)} \Big)\,du \bigg|\,.$$

Applying it to each one of the m small intervals in (8d13), we get the difference of two integrals.



(the case $m = 3, s < t$ is shown). When $m \to \infty$, the difference tends to 0 by the bounded convergence theorem, since the function $(s,u) \mapsto \frac{F'_s(u)}{F_s(u)}$ is continuous on $(0, s^{\max}) \times (0, s^{\max})$. So, (8d13) is satisfied, and we get the following result.

8d14. Theorem. Let the two-dimensional distribution of signals satisfy (8d8).[14] Then the first price auction[15] has one and only one symmetric equilibrium of the form $A = \varphi(S)$ where $\varphi : (0, s^{\max}) \to \mathbb{R}$ is a strictly increasing continuous function. Namely,

$$\varphi(s) = \frac{1}{\tilde F(s)} \int_0^s t \tilde F(t)\,dt\,,$$

where

$$\tilde F(s) = \exp\bigg(-\int_s^{s^{\max}} \frac{f_{S_1,S_2}(t,t)}{\int_0^t f_{S_1,S_2}(t,u)\,du} \bigg)\,.$$

8e Notes on more general auctions

Correlated signals are often used in combination with non-private values, which means

(8e1)
$$\mathbf{G}(a_1, s_1; a_2, s_2) = \begin{cases} 0 & \text{if } a_1 < a_2, \\ 0 & \text{if } a_1 = a_2 = 0, \\ \frac12 V(s_1,s_2) & \text{if } a_1 = a_2 > 0, \\ V(s_1,s_2) & \text{if } a_1 > a_2 \end{cases}$$

[14]Affiliation is sufficient for that. The distribution is assumed to be concentrated on $(0, s^{\max}) \times (0, s^{\max})$ and have a continuous, strictly positive density on the square.

[15]Symmetric, two players, private value.

instead of (3e1); here V is a given function, increasing in both arguments.[16] The approach of 3b–3d is applicable to a private value auction with unknown distributions; the parameter θ, unknown to a player, is treated as influencing signals of his competitors, but not his own valuation. Otherwise we must use a valuation $V(s_1, s_2)$ depending on both signals.

The theory of independent signals can be generalized for non-private values. Still, the best response μ_1 to a strategy μ_2 is a monotone (increasing) strategy. No matter whether μ_2 is monotone or not; anyway, μ_1 is in fact the best response to P_{A_2} rather than μ_2. Therefore, every equilibrium is monotone. A symmetric equilibrium is known to be unique, and given by $A = \varphi(S)$ where

$$\varphi(s) = \mathbb{E}\left(V(S,S) \,\middle|\, S \le s \right) . \tag{8e2}$$

(Note that the case $V(s,t) = s$ returns us to 3d3.)

The result of 8b can be generalized for non-private values. Still, the best response to an increasing strategy is an increasing strategy, provided that signals are affiliated.

What about generalizing the theory of correlated signals (8c, 8d) for non-private values?

Instead of $(s-a)F_{\varphi(S_2)|S_1=s}(a)$ (recall (8c1)) we get

$$\mathbb{E}\left((V(S_1,S_2) - a)\mathbf{1}_{[0,a]}(\varphi(S_2)) \,\middle|\, S_1 = s \right) ; \tag{8e3}$$

(8c2) becomes

$$\int_0^t \big(V(s,u) - \varphi(t)\big)\, dF_s(u) \le \int_0^s \big(V(s,u) - \varphi(s)\big)\, dF_s(u) ; \tag{8e4}$$

(8c3) turns into

$$\int_s^t V(s,u)\, dF_s(u) \le \varphi(t)F_s(t) - \varphi(s)F_s(s) , \tag{8e5}$$

the infinitesimal form being

$$V(s,s)F_s'(s) = \varphi(s)F_s'(s) - \varphi'(s)F_s(s) \tag{8e6}$$

instead of (8c4). Once again, the quotient $F_s'(s)/F_s(s)$ is the only relevant aspect of the two-dimensional distribution of signals. Therefore we may replace our auction with the associated auction, with independent signals distributed $\tilde{F}$, just the same $\tilde{F}$ as in Theorem 8d14. The associated auction has its equilibrium strategy function φ given by (8e2). For the original auction, φ satisfies the infinitesimal (first-order) condition of equilibrium (8e6), and is the only such function.

However, the relation between local and global via superadditivity (Lemmas 8d4, 8d5) does not want to work here (as far as I understand). In fact, the function φ given by (8e2) describes an equilibrium; however, the classical proof (Milgrom and Weber 1982) follows a different pattern: the explicit formula (8e2) is used in the proof (in combination with affiliation).

[16] For example: $V(s_1,s_2) = \frac{2}{3}s_1 + \frac{1}{3}s_2$. Another example, $V(s_1,s_2) = \frac{1}{2}s_1 + \frac{1}{2}s_2$, describes a common value auction. And $V(s_1,s_2) = s_1$ returns us to private values.

Another generalization, for n players, is also well-known. Basically, one only needs to replace the signal S_2 of the competitor with the signal $X = \max(S_2, \dots, S_n)$ of the strongest competitor. Affiliation of $S_1, \dots, S_n$ implies affiliation of S_1, X. The formula for $\tilde{F}$ becomes

$$\tilde{F}(s) = \exp\left(- \int_s^{s^{\max}} \frac{f_{S_1,X}(t,t)}{\int_0^t f_{S_1,X}(t,u)\,du} \right).$$