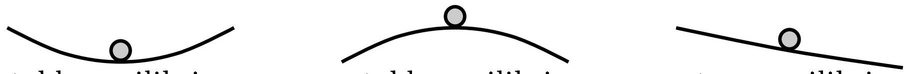
7 Many players, entry cost, and non-robustness

7a Small but important

The notion of equilibrium was used in physics several centuries earlier than in economics. There, an equilibrium can be stable or unstable.



stable equilibrium unstable equilibrium not an equilibrium

Unstable equilibria have no game-theoretic counterparts; in game theory, an equilibrium is always stable.[1] However, it does not guarantee that it is *robust.*

robust equilibrium robust equilibrium non-robust equilibrium

An equilibrium is robust, if a small modification of the given system (mechanical system, a game, etc.) causes only a small modification of the equilibrium.[2] Consider for example a first price auction with a small entry cost $c > 0$. I mean the symmetric auction introduced in 3g (with no reserve price) but generalized to n players, as described in 5d (page 61). Let signals be uniform on $(0, 1)$,

$$S_k \sim \mathrm{U}(0,1)\,,$$

then the equation $(s_0 - r)F_X(s_0) = c$ (where $X = \max(S_2, \dots, S_n)$) for the participation threshold s_0 becomes $s_0 \cdot s_0^{n-1} = c$, thus

$$s_0 = \sqrt[n]{c}\,.$$

The equilibrium strategy is $A = \varphi(S)$ where

$$\begin{aligned}\varphi(s) = \mathbb{E}\big(h(X) \,\big|\, X \le s \big) &= \mathbb{E}\big(X \,\big|\, s_0 \le X \le s \big)\, \mathbb{P}\big(s_0 \le X \le s \,\big|\, X \le s \big) = \\ &= \frac{1}{s^{n-1}} \int_{s_0}^{s} x \, dx^{n-1} = \frac{n-1}{n} \frac{s^n - s_0^n}{s^{n-1}} = \Big(1 - \frac{1}{n}\Big) s \bigg(1 - \Big(\frac{s_0}{s}\Big)^n\bigg)\end{aligned}$$

for $s > \sqrt[n]{c}$, otherwise $\varphi(s) = 0$. That is,

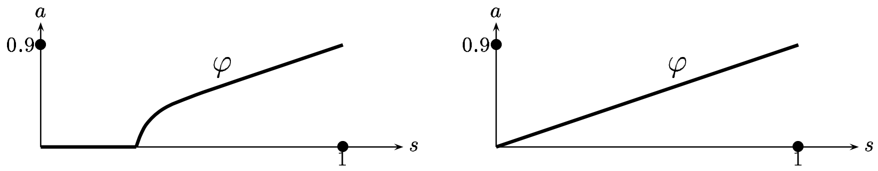
$$\varphi(s) = \max\bigg(0, \frac{n-1}{n} s \Big(1 - \frac{c}{s^n}\Big)\bigg)\,.$$

The limit for $c \to 0$ is $A = \frac{n-1}{n} S$, just the equilibrium for the auction with no entry cost. Thus, if the entry cost is small, we may neglect it; do you agree?

[1] Well, sometimes it is indifferent, but never unstable.

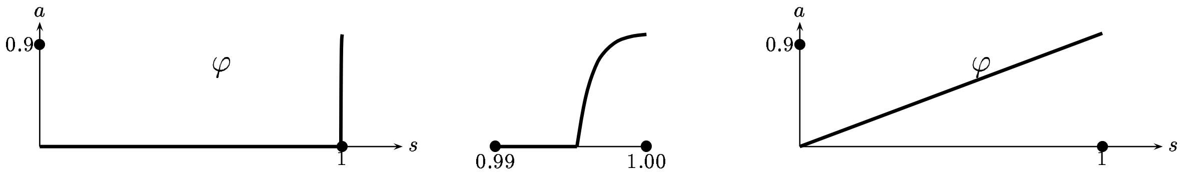
[2] That is an intuitive idea, not a definition.

Say, for $c = 10^{-5} = 0.000\,01$ and $n = 10$ we have $s_0 = (10^{-5})^{1/10} = 1/\sqrt{10} \approx 0.31$ and $\varphi(s) \approx 0.9s\bigl(1 - \bigl(\frac{0.31}{s}\bigr)^{10}\bigr)$ for $s > s_0$.



c=0.000 01, n=10 c=0, n=10

Oops! Such a noticeable effect of such a small cause! Now, what happens for $n = 1\,000$ and $c = 0.01$?



c=0.01, n=1 000 c=0, n=1 000

A dramatic effect of only 1% entry cost!

7b Many players but few participants

The equilibrium strategy $A = \varphi(S)$ depends on n and c; $A = \varphi_{n,c}(S)$. What happens when $n \to \infty$?

$$\begin{aligned} c = 0 : &\qquad \varphi_{n,0}(s) = \frac{n-1}{n} s \xrightarrow[n\to\infty]{} s = \varphi_{\infty,0}(s) ; \\ c > 0 : &\qquad \varphi_{n,c}(s) \xrightarrow[n\to\infty]{} 0 = \varphi_{\infty,c}(s) , \end{aligned}$$

since $\sqrt[n]{c} \xrightarrow[n\to\infty]{} 1$; you see, $\varphi_{n,c}(s) = 0$ when $s^n < c$, which holds for n large enough, namely, for

$$n > \frac{\ln c}{\ln s}$$

(or we may write $\ln(1/c)/\ln(1/s)$.) So,

$$\varphi_{\infty,c}(s) = \begin{cases} s & \text{if } c = 0, \\ 0 & \text{if } c > 0; \end{cases}$$

the discontinuity at $c = 0$ is a manifestation of the non-robustness.

A meaningful description of $\varphi_{n,c}$ for large n is given by an asymptotic formula with rescaled argument:

$$\varphi_{n,c}\Bigl(1 - \frac{1}{n}x\Bigr) = \max(0, 1 - ce^x) ;$$

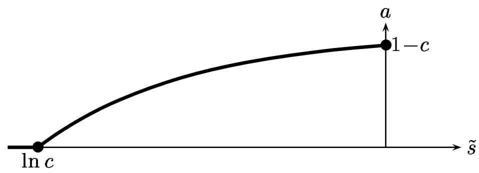
indeed,[3]

$$\varphi_{n,c}\Big(1-\frac{1}{n}x\Big) = \max\left(0, \Big(1-\frac1n\Big)\Big(1-\frac{x}{n}\Big)\Big(1-\frac{c}{(1-\frac{x}{n})^n}\Big)\right) \xrightarrow[n\to\infty]{} \\ \xrightarrow[n\to\infty]{} \max\Big(0, 1-\frac{c}{e^{-x}}\Big) = \max\big(0, 1-ce^x\big)\,.$$

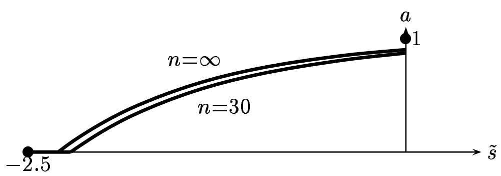
We may write it in the form

$$\varphi_{n,c}\Big(1+\frac1n \tilde s\Big) \xrightarrow[n\to\infty]{} \max\big(0,1-ce^{-\tilde s}\big) \qquad \text{for } \tilde s \in (-\infty, 0)\,,$$

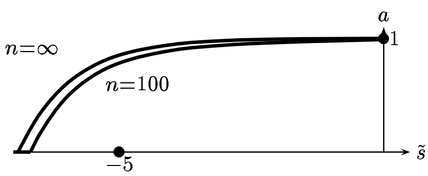
treating $\tilde S = n(S-1)$ as a rescaled signal.[4] I prefer the negative $n(S-1)$ to the positive $n(1-S)$ for preserving the orientation: the higher $\tilde S$, the better (for the player).



Some numeric examples give an idea of the error of the approximation:



c=0.1



c=0.001

A player participates with the probability

$$\mathbb{P}\big(\,A>0\,\big) = \mathbb{P}\big(\,S > \sqrt[n]{c}\,\big) = 1-\sqrt[n]{c}\,,$$

thus, the number of participants is distributed binomially, and the limiting distribution is Poissonian,

$$\#\{k : A_k > 0\} \sim \operatorname{Binom}(n, 1-\sqrt[n]{c}) \xrightarrow[n\to\infty]{} \operatorname{Poisson}(\ln\frac1c)\,,$$

since $n(1-\sqrt[n]{c}) \to -\ln c$.[5] In other words,

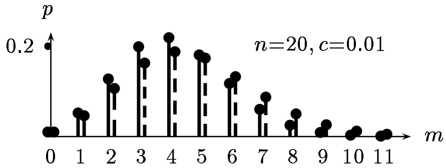
$$\mathbb{P}\big(\,\#\{k : A_k > 0\} = m\,\big) \xrightarrow[n\to\infty]{} c\frac{(\ln 1/c)^m}{m!} \text{ for } m = 0,1,\dots$$

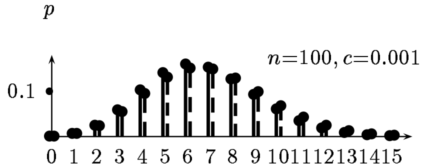
[3] Recall that $\big(1+\frac{x}{n}\big)^n \to e^x$ when $n\to\infty$, for every $x\in\mathbb{R}$.

[4] Note that the expected number of stronger competitors is $(n-1)(1-S) \approx -\tilde S$, which sheds a light on the meaning of $\tilde S$.

[5] Indeed, $\sqrt[n]{c} = c^{1/n} = \exp(\frac1n \ln c) = 1 + \frac1n \ln c + o(\frac1n)$, thus $n(1-\sqrt[n]{c}) = -\ln c + o(1)$.

Some numeric examples:





The maximal signal $\max(S_1, \dots, S_n)$ is close to 1 for large n; however, the maximal rescaled signal $n\big(\max(S_1, \dots, S_n) - 1\big) = \max(\tilde S_1, \dots, \tilde S_n)$ has a non-degenerate limiting distribution:

$$\mathbb{P}\left(\max(\tilde S_1, \dots, \tilde S_n) \le \tilde s \right) = \Big(\mathbb{P}\left(n(S-1) \le \tilde s \right) \Big)^n = \\ = \Big(\mathbb{P}\left(S \le 1 + \frac{\tilde s}{n} \right) \Big)^n = \Big(1 + \frac{\tilde s}{n} \Big)^n \xrightarrow[n\to\infty]{} \exp \tilde s$$

for $\tilde s \in (-\infty, 0]$; that is, the distribution of $-\max(\tilde S_1, \dots, \tilde S_n)$ is approximately exponential, $\mathrm{Exp}(1)$, for large n.

The winning bid (that is, maximal action) is a function of the maximal signal;

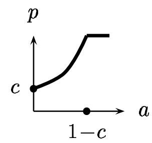
$$\max(A_1, \dots, A_n) = \varphi_{n,c}\big(1 + \frac{1}{n} \max(\tilde S_1, \dots, \tilde S_n)\big) \approx \max\big(0, 1 - c\exp(-\max(\tilde S_1, \dots, \tilde S_n))\big) ,$$

thus[6]

$$\mathbb{P}\left(\max(A_1, \dots, A_n) \le a \right) \xrightarrow[n\to\infty]{} \mathbb{P}\left(1 - ce^Z \le a \right) ,$$

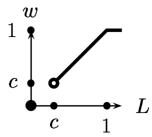
where $Z \sim \mathrm{Exp}(1)$; and finally,

$$\mathbb{P}\left(\max(A_1, \dots, A_n) \le a \right) \xrightarrow[n\to\infty]{} \min\Big(1, \frac{c}{1-a}\Big) \quad \text{for } a \in [0,1] .$$

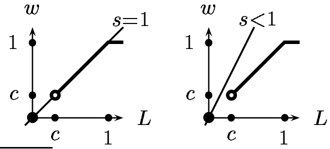


7b1. Exercise. Show that the winning probability function $W(a)$ (recall (5c15)) also converges to $\min\big(1, \frac{c}{1-a}\big)$ for $n \to \infty$.

What happens on the plane (expected loss, winning probability) introduced in 2b? For $0 < a \le 1-c$ we have the winning probability $W(a) \approx \frac{c}{1-a}$ and the expected loss $c + aW(a) \approx c + a\frac{c}{1-a} = \frac{c}{1-a}$; you see, these are approximately equal.



For every finite n the curve is strictly concave, however, its limit for $n \to \infty$ is linear. The winner, having a signal close to 1, is nearly indifferent; every action gives him nearly 0. A typical player, having a signal not close to 1, does not want to participate.



[6] Uniform (in $\tilde s \in [0,1]$) convergence of $\varphi_{n,c}(1 + \frac{1}{n}\tilde s)$ to $\max(0, 1 - ce^{-\tilde s})$ is used.

7b2. Exercise. Generalize the theory for an arbitrary distribution of signals, nonatomic, with a compact support; show that (for large n)

$$\begin{gathered} F_S(s_0) \approx 1 - \frac{1}{n} \ln \frac{s^{\max}}{c} \, ; \\ \varphi(s) \approx \max \Big(0, s^{\max} - c \exp\big(n(1 - F_S(s)) \big) \Big) \, ; \\ W(a) \approx \min \Big(1, \frac{c}{s^{\max} - a} \Big) \, ; \\ c + aW(a) \approx s^{\max} W(a) \quad \text{for } a < s^{\max} - c \end{gathered}$$

where $s^{\max} = \sup\{ s : F_S(s) < 1 \}$. Also, the number of participants is distributed approximately

$$\operatorname{Poisson} \Big(\ln \frac{s^{\max}}{c} \Big) \, .$$

Hint: $\varphi(s) = \underbrace{\mathbb{E} \left(X \,\middle|\, s_0 \le X \le s \right)}_{\approx s^{\max}} \mathbb{P} \left(s_0 \le X \le s \,\middle|\, X \le s \right).$

So, for a positive entry cost c we have a satisfactory theory of an 'unlimited' auction with infinitely many players (but finitely many participants), simpler than the exact theory for a finite n, and giving a good approximation for large n. The 'unlimited' auction diverges for $c \to 0$, but very slowly. In reality, I believe, c is never less than, say, $10^{-5} s^{\max}$, therefore the mean number of participants[7] never exceeds $\ln 10^5 \approx 11.5$ or so.

7c A more general approach

The non-robustness pointed out in 7a is of quite general nature, as we'll see soon. Consider a symmetric game of n players, described by (recall 5a)

$$(\mathcal{S}, \mathcal{A}, P, \mathbf{\Pi}, n) \, ;$$

here the signal space $\mathcal{S}$ is arbitrary (possibly multidimensional); the action space $\mathcal{A}$ is one-dimensional,

$$\mathcal{A} \subset [0, \infty) \, , \qquad 0 \in \mathcal{A} \, ; \tag{7c1}$$

the signal distribution P is arbitrary; the payoff function $\mathbf{\Pi}$ is arbitrary but satisfying

$$\mathbf{\Pi}(0, s_1; a_2, s_2; \dots; a_n, s_n) = 0 \quad \text{for all } s_1, \dots, s_n, a_2, \dots, a_n \, ; \tag{7c2}$$

it means that a player may quit, if he wants. (As before, signals are independent.)

Assume that

$$\mathbf{\Pi}(a_1, s_1; \dots; a_n, s_n) = \mathbf{G}(a_1, s_1; \dots; a_n, s_n) - \mathbf{L}(a_1, s_1; \dots; a_n, s_n) \tag{7c3}$$

[7] In a *single unit* auction, of course.

(this time, both the gain and the loss may depend on all signals),

(7c4) $$\mathbf{G}(a_1, s_1; \dots; a_n, s_n) \le G^{\max} \quad \text{for all } a_1, s_1, \dots, a_n, s_n$$

for some $G^{\max} < \infty$ ('maximal gain'), and

(7c5) $$\mathbf{L}(a_1, s_1; \dots; a_n, s_n) \ge L_{\min}(a_1, s_1) \quad \text{for all } a_1, s_1, \dots, a_n, s_n$$

for some function $L_{\min} : \mathcal{A} \times \mathcal{S} \to \mathbb{R}$ ('inescapable loss').[8]

The last assumption generalizes the idea of a *single unit* auction (with the standard allocation rule):

(7c6) $$\mathbf{G}(a_1, s_1; \dots; a_n, s_n) = 0 \quad \text{whenever } a_1 < \max(a_1, \dots, a_n)\,.$$

7c7. Theorem. Assuming (7c1)–(7c6) and a symmetric equilibrium, we have

$$L_{\min}(A_1, S_1) \le G^{\max} F_A^{n-1}(A_1)$$

almost surely.

Proof.

$$\mathbf{\Pi}_1(A_1, S_1) = \sup_a \mathbf{\Pi}_1(a, S_1) \ge \mathbf{\Pi}_1(0, S_1) = 0$$

almost surely (recall 2a18(c)); here, as before, $\mathbf{\Pi}_1(a_1, s_1) = \mathbf{\Pi}_1(a_1, s_1; \mu; \dots; \mu)$, etc. Hence

$$\mathbf{G}(A_1, S_1) \ge \mathbf{L}(A_1, S_1) \ge L_{\min}(A_1, S_1)\,.$$

On the other hand, $\mathbf{G}(a_1, s_1; \dots; a_n, s_n) \le G^{\max} \mathbf{1}_{a_1 = \max(a_1, \dots, a_n)} = \begin{cases} G^{\max} & \text{if } a_1 = \max(a_1, \dots, a_n), \\ 0 & \text{otherwise} \end{cases}$

always, therefore

$$\mathbf{G}(a_1, s_1) = \mathbb{E}\,\mathbf{G}(a_1, s_1; A_2, S_2; \dots; A_n, S_n) \le \\ \le G^{\max} \mathbb{P}\,\big(a_1 = \max(a_1, A_2, \dots, A_n) \big) = G^{\max} F_A^{n-1}(a_1)\,.$$

□

7c8. Corollary. Assuming (7c1)–(7c6) and a symmetric equilibrium, we have

$$\mathbb{P}\,\big(L_{\min}(A_1, S_1) \le G^{\max} F_A^{n-1}(a) \big) \ge F_A(a)$$

for every $a \in [0, \infty)$.

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[8] For an auction with entry cost c we may take $L_{\min}(a_1, s_1) = \begin{cases} c & \text{if } a_1 > 0, \\ 0 & \text{if } a_1 = 0, \end{cases}$ irrespective of s_1. For an all-pay auction we may take $L_{\min}(a_1, s_1) = a_1$.

Proof. First, note a general probabilistic fact: for every random variable X and every number x,

$$\mathbb{P}\,\big(\,F_X(X) \le F_X(x)\,\big) = F_X(x)\,,$$

since, on one hand, $X \le x \implies F_X(X) \le F_X(x)$ always, and on the other hand, $X > x \implies F_X(X) > F_X(x)$ almost surely.[9]

In particular,

$$\mathbb{P}\,\big(\,F_A(A_1) \le F_A(a)\,\big) = F_A(a)\,.$$

By Theorem 7c7,

$$F_A(A_1) \le F_A(a) \quad\implies\quad L_{\min}(A_1, S_1) \le G^{\max} F_A^{n-1}(a)$$

a.s.; so,

$$F_A(a) = \mathbb{P}\,\big(\,F_A(A_1) \le F_A(a)\,\big) \le \mathbb{P}\,\big(\,L_{\min}(A_1, S_1) \le G^{\max} F_A^{n-1}(a)\,\big)\,.$$

□

7c9. Exercise. If $c > 0$ is such that $L_{\min}(a_1, s_1) \ge c$ for all $a_1 > 0$,[10] and A_1 has no atoms on $(0, \infty)$, then A_1 has an atom at 0, moreover,

$$\mathbb{P}\,\big(\,A_1 = 0\,\big) \ge \sqrt[n-1]{\frac{c}{G^{\max}}}\,.$$

Prove it.

For large n show that the number of participants is distributed approximately Poisson(λ) with some $\lambda \le \ln(G^{\max}/c)$. Can we neglect c if it is only $10^{-5} G^{\max}$?

Hint. $F_A^{n-1}(A_1) \ge c/G^{\max}$ whenever $A_1 > 0$ (a.s.)

7c10. Exercise. If $L_{\min}(a_1, s_1) = a_1$ for all a_1, s_1,[11] then

$$F_A(a) \ge \sqrt[n-1]{\frac{a}{G^{\max}}}$$

for all $a \in [0, \infty)$.

Prove it.

For large n show that the number of players giving $A_k > a$ ('serious bidders') is distributed approximately Poisson(λ) with some $\lambda \le \ln(G^{\max}/a)$. Could you believe that 30 players (out of 1 000, say) gave bids $> 10^{-5} G^{\max}$?

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[9] Assume the contrary and get a contradiction . . .

[10] Which is the case for an auction with entry cost.

[11] Which is the case for an all-pay auction.

7c11. Exercise. Consider such a model with an individual entry cost. Each signal is two-dimensional, $S_k = (V_k, C_k)$, where C_k is the entry cost,

$$L_{\min}(A_1, S_1) = C_1 \,,$$

and V_k is the valuation (or whatever).[12] Assume that

$$C_k \sim \mathrm{U}(0, c)$$

for a given $c > 0$. If A_1 has no atoms on $(0, \infty)$, then A_1 has an atom at 0, moreover,

$$\mathbb{P}\big(A_1 = 0 \big) \ge \sqrt[n-2]{\frac{c}{G^{\max}}} \,.$$

Prove it.
Compare it with 7c9.
Hint: $F_A(a) \le \mathbb{P}\big(C_1 \le G^{\max} F_A^{n-1}(a) \big) \le \frac{1}{c} G^{\max} F_A^{n-1}(a)$.

7c12. Exercise. Try to generalize the theory for multiple unit auctions. To this end, replace (7c6) with

$$\text{(7c13)} \qquad \mathbf{G}(a_1, s_1; \dots; a_n, s_n) = 0 \quad \text{whenever } a_1 \text{ is less than two (or more) of } a_1, \dots, a_n$$

or even

$$\text{(7c14)} \qquad \mathbf{G}(a_1, s_1; \dots; a_n, s_n) = 0 \quad \text{whenever } a_1 \text{ is less than } n/2 \text{ (or more) of } a_1, \dots, a_n \,.$$

In the latter case show that the number of participants is typically $\frac{1}{2} n + \text{const} \cdot \sqrt{n}$, where the constant does not depend on n. What does it depend on?

7d Unbounded gain

Estimations of 7c are based on the upper bound $G^{\max}$ of any possible gain. Usually, the gain depends on a random signal (or many random signals). In principle, we may believe that the random gain is bounded, that is, its distribution has a compact support. In practice, however, our knowledge about distributions usually comes from empirical observations, that is, sampling. A sample (not too small) gives us an idea of the main part of the distribution, but not of its tail.

Look at a typical sample. We may reasonably believe that the distribution is like this,

[12] V_1 and C_1 need not be independent.

and its expectation is approximately 2. However, what about boundedness? Any opinion about an upper bound is much more risky than that about the expectation.[13]

You could say: anyway, values higher than (say) 7 are very improbable; let us neglect them, thus making the support compact. However, the question is, whether the tail may be neglected, or not. In other words: whether the investigated phenomenon is especially sensitive to the tail, or not. There are two possibilities.

- The phenomenon is not sensitive. Its theory can be generalized to unbounded distributions.
- The phenomenon is sensitive. Its theory holds for bounded distributions only. Such a theory cannot be used in practice unless we have a reliable information about the support.

The distinction is essential. This is why I do not like saying something like

> "We add the technical assumption of compact supports (since in reality everything is bounded, anyway)."

We still assume (7c1)–(7c3) and (7c5)–(7c6), but not (7c4); the gain need not be bounded. We consider a symmetric equilibrium such that

$$A_1 \text{ has no atoms on } (0,\infty)\,, \tag{7d1}$$

denote

$$\mathbb{P}\big(A_1 = 0\big) = p_0 \in [0,1]$$

and strive to show that (similarly to 7c9, 7c11) p_0 must be close to 1 if n is large.

The random variable $G = \mathbf{G}(A_1, S_1; \dots; A_n, S_n)$ has its quantile function $G^* : (0,1) \to [0,\infty)$. Also the random variable $L_{\min} = L_{\min}(A_1, S_1)$ has its quantile function $L^*_{\min} : (0,1) \to [0,\infty)$.

7d2. Lemma. Assuming (7c1)–(7c3), (7c5)–(7c6), a symmetric equilibrium and (7d1), we have

$$\int_{p_0}^{p_1} L^*_{\min}(p)\,dp \le \int_{1-\frac1n(p_1^n - p_0^n)}^{1} G^*(p)\,dp$$

for all $p_1 \in [p_0, 1]$.

Proof. We take a_1 such that $F_A(a_1) = p_1$ and consider events

$$E_0 = \{A_1 = 0\}\,, \qquad E_1 = \{0 < A_1 < a_1\}\,.$$

We have $\mathbb{P}\big(E_0\big) = p_0$, $\mathbb{P}\big(E_1\big) = p_1 - p_0$, and

$$\mathbb{E}\big(L_{\min}\mathbf{1}_{E_1}\big) \ge \int_{p_0}^{p_1} L^*_{\min}(p)\,dp\,, \tag{7d3}$$

[13] And about the expectation, it is still risky. It may happen that our distribution consists of 99.99% of a 'good' distribution with a finite expectation and 0.01% of a 'bad' distribution, with infinite expectation. A sample has little chance to reveal the 'bad' component.

since[14]

$$\mathbb{E}\left(L_{\min}\mathbf{1}_{E_1}\right) = \mathbb{E}\left(L_{\min}\mathbf{1}_{E_0\cup E_1}\right) \ge \int_0^{\mathbb{P}(E_0\cup E_1)} L^*_{\min}(p)\,dp = \\ = \int_0^{p_1} L^*_{\min}(p)\,dp = \int_{p_0}^{p_1} L^*_{\min}(p)\,dp\,.$$

Introduce one more event

$$E_2 = \{A_1 = \max(A_1,\dots,A_n) > 0\}\,;$$

$\mathbb{P}\left(E_2\right) = \int_{(0,\infty)} F_A^{n-1}(a)\,dF_A(a) = \frac{1}{n}(1-p_0^n)$; similarly, $\mathbb{P}\left(E_1\cap E_2\right) = \frac{1}{n}(p_1^n - p_0^n)$. We have

$$\mathbb{E}\left(G\mathbf{1}_{E_1}\right) \le \int_{1-\frac{1}{n}(p_1^n-p_0^n)}^1 G^*(p)\,dp\,, \tag{7d4}$$

since[15]

$$\mathbb{E}\left(G\mathbf{1}_{E_1}\right) = \mathbb{E}\left(G\mathbf{1}_{E_1\cap E_2}\right) \le \int_{1-\mathbb{P}(E_1\cap E_2)}^1 G^*(p)\,dp = \int_{1-\frac{1}{n}(p_1^n-p_0^n)}^1 G^*(p)\,dp\,.$$

On the other hand, $G(A_1,S_1) \ge L_{\min}(A_1,S_1)$, that is,

$$\mathbb{E}\left(G \,\middle|\, A_1,S_1\right) \ge L_{\min}\,,$$

which was shown in the beginning of the proof of Theorem 7c7 (without using (7c4)). We have

$$\mathbb{E}\left(\mathbf{1}_{E_1}L_{\min}\right) \le \mathbb{E}\left(\mathbf{1}_{E_1}\mathbb{E}\left(G \,\middle|\, A_1,S_1\right)\right) = \mathbb{E}\left(\mathbf{1}_{E_1}G\right);$$

it remains to combine it with (7d3) and (7d4). □

For a single unit auction, the gain G of the first player cannot exceed his valuation V_1. The latter is a random variable; it need not be bounded, but its expectation $\mathbb{E}\,V_1$ should be finite, and not growing for $n\to\infty$. Markov inequality gives $\mathbb{P}\left(V_1 > x\right) \le \frac{\mathbb{E}\,V_1}{x}$ for any $x\in(0,\infty)$, therefore $G^*(p) \le \frac{\mathbb{E}\,V_1}{1-p}$. If V_1 has also a finite second moment, we get more:

$$\mathbb{P}\left(V_1 > x\right) \le \frac{\mathbb{E}\,V_1^2}{x^2}\,, \quad \text{therefore} \quad G^*(p) \le \frac{\mathbb{E}\,V_1^2}{\sqrt{1-p}}\,,$$

etc. Taking all that into account and returning to our general framework, we assume that

$$\int_{1-\varepsilon}^1 G^*(p)\,dp \le M\varepsilon^\alpha \quad \text{for all } \varepsilon\in(0,1)\,; \tag{7d5}$$

here $\alpha\in(0,1)$ and $M\in(0,\infty)$ are constants (not depending on n). Now, Lemma 7d2 gives

$$\int_{p_0}^{p_1} L^*_{\min}(p)\,dp \le M\left(\frac{p_1^n-p_0^n}{n}\right)^\alpha \quad \text{for all } p_1\in[p_0,1]\,. \tag{7d6}$$

[14] $L_{\min}=0$ on E_0; accordingly, $L^*_{\min}=0$ on $(0,p_0)$.
[15] $G=0$ outside of E_2.

7d7. Exercise. If $c > 0$ is such that $L_{\min}(a_1, s_1) \ge c$ for all $a_1 > 0$,[16] then

$$p_0^{\alpha n-1} \ge \frac{cn^\alpha}{M \inf_{x\in(1,1/p_0)} \frac{(x^n-1)^\alpha}{x-1}} .$$

Prove it.

Hint: $c(p_1 - p_0) \le M\big(\frac{p_1^n - p_0^n}{n}\big)^\alpha$; denote $p/p_0 = x$.

For large n we have

$$\inf_{x\in(1,1/p_0)} \frac{(x^n-1)^\alpha}{x-1} = \inf_{y\in(1,1/p_0^n)} \frac{(y-1)^\alpha}{\sqrt[n]{y}-1} \approx n \inf_{y\in(1,\infty)} \frac{(y-1)^\alpha}{\ln y},$$

since $\sqrt[n]{y} - 1 \approx \frac{1}{n}\ln y$, and $\frac{(y-1)^\alpha}{\ln \alpha}$ reaches its infimum at a (finite) point.[17] Thus,

$$p_0^{\alpha n-1} \ge \text{const} \cdot \frac{c}{n^{1-\alpha} M} \quad \text{for large } n\,,$$

where "const" depends on α only. It follows that

$$p_0 \ge 1 - \frac{1}{\alpha n} \ln\Big(\text{const} \cdot \frac{n^{1-\alpha} M}{c}\Big) \quad \text{for large } n\,,$$

which generalizes 7c9 for unbounded gain.

The number of participants is distributed approximately Poisson(λ) with $\lambda \le \frac{1}{\alpha}\ln\big(\text{const}\cdot \frac{n^{1-\alpha}M}{c}\big)$. The entry cost c appears under the logarithm; therefore, even a small c is important (as before). This time, however, λ can grow when $n \to \infty$, but very slowly (logarithmically).

7d8. Exercise. Think, which statements of 7a–7b are sensitive (to the tail of the distribution, recall the beginning of 7d) and which are not.

7d9. Exercise. Try to generalize 7c10.

7d10. Exercise. Try to generalize 7c11.

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[16] Which is the case for an auction with entry cost; recall 7c9.

[17] The argument works if $1/p_0^n \to \infty$ for $n \to \infty$; otherwise p_0 is even closer to 1 than it really must be.