5 Not just two players

5a Basic models and notions, revisited

A game of n players is described by (recall (1b2))

(5a1) $$(\mathcal{S}_1,\dots,\mathcal{S}_n;\ \mathcal{A}_1,\dots,\mathcal{A}_n;\ P_1,\dots,P_n;\ \mathbf{\Pi}_1,\dots,\mathbf{\Pi}_n)\,;$$

the k-th player has its signal space $\mathcal{S}_k$, action space $\mathcal{A}_k$, signal distribution P_k and payoff function $\mathbf{\Pi}_k$. (In general, each payoff depends on all actions and signals.) The game is called symmetric, if $\mathcal{S}_1=\dots=\mathcal{S}_n$, $\mathcal{A}_1=\dots=\mathcal{A}_n$, $P_1=\dots=P_n$, and $\mathbf{\Pi}_k(a_1,s_1;a_2,s_2;\dots;a_n,s_n)=\mathbf{\Pi}_1(a_k,s_k;a_{i_1},s_{i_1};\dots;a_{i_{n-1}},s_{i_{n-1}})$ for every k and every $i_1,\dots,i_{n-1}$ such that $\{k,i_1,\dots,i_{n-1}\}=\{1,\dots,n\}$; in other words, $\{k,i_1,\dots,i_{n-1}\}$ is a permutation of $\{1,\dots,n\}$. A symmetric game of n players is described by $(\mathcal{S},\mathcal{A},P,\mathbf{\Pi},n)$.

The oligopoly game of Sect. 4 is a symmetric game of a specific form: $\mathcal{S}=\mathbb{R}$, $\mathcal{A}=[0,\infty)$, $\mathbb{P}\big([0,\infty)\big)=1$, and

(5a2) $$\mathbf{\Pi}(a_1,s_1;a_2,s_2;\dots;a_n,s_n)=\mathbf{G}(a_1,s_1;a_2+\dots+a_n)-\mathbf{L}(a_1;a_2+\dots+a_n)$$

(the additional restriction $n=2$ stipulated in Sect. 4 is now discarded). The function $\mathbf{\Pi}$ of $2n$ variables amounts to a function $\mathbf{G}$ of 3 variables and a function $\mathbf{L}$ of 2 variables.

We turn to auction games of 3d, 3f, 3g, 3h. Roughly speaking, they are of the form

$$\mathbf{\Pi}(a_1,s_1;a_2,s_2;\dots;a_n,s_n)=\mathbf{G}\big(a_1,s_1;\max(a_2,\dots,a_n)\big)-\mathbf{L}\big(a_1;\max(a_2,\dots,a_n)\big)\,,$$

however, ties breaking complicates the situation when $a_1=\max(a_2,\dots,a_n)$. Here the winning probability is $\frac{1}{M+1}$, where M is the multiplicity; $M=1$ if $a_1=a_k$ for a single $k\in\{2,\dots,n\}$; $M=2$ if there exist two such k; and so on; $M=n-1$ if $a_1=a_2=\dots=a_n$. If the first player wins ties breaking, he gets $\mathbf{G}\big(a_1+,s_1;\max(a_2,\dots,a_n)\big)-\mathbf{L}\big(a_1+;\max(a_2,\dots,a_n)\big)$; if he looses he gets $\mathbf{G}\big(a_1-,s_1;\max(a_2,\dots,a_n)\big)-\mathbf{L}\big(a_1-;\max(a_2,\dots,a_n)\big)$. It means that

(5a3) $$\mathbf{\Pi}(a_1,s_1;a_2,s_2;\dots;a_n,s_n)=$$
$$=\begin{cases}\mathbf{G}(a_1,s_1;b)-\mathbf{L}(a_1;b) & \text{if } a_1\ne b;\\ \big(\frac{M}{M+1}\mathbf{G}(a_1-,s_1;b)+\frac{1}{M+1}\mathbf{G}(a_1+,s_1;b)\big)-\big(\frac{M}{M+1}\mathbf{L}(a_1-;b)+\frac{1}{M+1}\mathbf{L}(a_1+;b)\big) & \text{if } a_1=b;\end{cases}$$

here $b=\max(a_2,\dots,a_n)$. Functions $\mathbf{G}(a_1,s_1;b)$ and $\mathbf{L}(a_1;b)$ are continuous in a_1 everywhere except for b; their values at $a_1=b$ will not be used.

Some assumptions must be added for ensuring integrability (finiteness of expectations). Having $\mathcal{S}_k=\mathbb{R}$ and $P_k([0,\infty))=1$ we assume in addition

$$\int_0^\infty s\,dP_k(s)<\infty\,,\qquad\text{that is,}\quad \mathbb{E}\,S_k<\infty\,;$$

boundedness of signals is evidently sufficient. For single unit auctions the assumption

$$\mathbf{\Pi}(a_1,s_1;\dots;a_n,s_n)\le s_1$$

holds and ensures that $+\infty$ does not appear in expectations; $-\infty$ may appear, which is harmless. For the oligopoly the situation is a bit more complicated: $\mathbf{\Pi}_1(a_1, s_1; \dots; a_n, s_n) \le a_1 s_1 - a_1^2 \le s_1^2/4$; one must assume $\mathbb{E}\, S^2 < \infty$; boundedness of signals is still sufficient.

We return to the general case. A strategy of player k is a probability distribution μ_k on $\mathcal{S}_k \times \mathcal{A}_k$ whose projection to $\mathcal{S}_k$ is P_k. Similarly to 1d,

$$\mathbb{E}\,(\Pi_1) = \mathbf{\Pi}_1(\mu_1; \mu_2; \dots; \mu_n) \tag{5a4}$$

if $\Pi_1 = \mathbf{\Pi}_1(A_1, S_1; \dots; A_n, S_n)$ where the pair (A_1, S_1) is distributed μ_1, and so on, (A_n, S_n) is distributed μ_n, and pairs $(A_1, S_1), \dots, (A_n, S_n)$ are independent.[1] Also,

$$\mathbb{E}\left(\Pi_1 \,\middle|\, A_1, S_1 \right) = \mathbf{\Pi}_1(A_1, S_1; \mu_2; \dots; \mu_n)\,, \tag{5a5}$$

etc.

A strategy μ_1 is called a best response (of the first player) to $\mu_2, \dots, \mu_n$, if

$$\mathbf{\Pi}_1(\mu_1; \mu_2; \dots; \mu_n) = \sup_{\mu_1'} \mathbf{\Pi}_1(\mu_1'; \mu_2; \dots; \mu_n)\,. \tag{5a6}$$

Best responses of other players are defined similarly. A sequence $(\mu_1, \dots, \mu_n)$ of n strategies is called an equilibrium, if for every k the strategy μ_k is a best response (of player k) to other strategies. For a symmetric game, an equilibrium $(\mu_1, \dots, \mu_n)$ is called symmetric, if $\mu_1 = \dots = \mu_n$.

5b A random number of players

Let N be a random variable taking on values $1, 2, \dots$ We may consider a game of N players, provided that it is a symmetric game. (It is meant that a player chooses his action without knowing N.) To this end we need a signal space $\mathcal{S}$, an action space $\mathcal{A}$, a signal distribution P, and many payoff functions $\mathbf{\Pi}_n$.[2] Namely, $\mathbf{\Pi}_n$ must be defined for every n such that $\mathbb{P}\left(N = n \right) > 0$; if N is bounded, we use finitely many payoff functions; but if N is unbounded, then infinitely many payoff functions are involved. Anyway, $\mathbf{\Pi}_n : (\mathcal{A} \times \mathcal{S})^n \to \mathbb{R}$. The case $n = 0$ is excluded, while the case $n = 1$ may be used or excluded, depending on the model.

5b1. Example. There are $n_{\max}$ potential players. Each one becomes a player at random, with a probability p_{play}, independently of others. The choice is made by nature (not by potential players). The number of players is a random variable distributed $\text{Binom}(n_{\max}, p_{\text{play}})$. However, that is not N. Rather, N is the number of players conditioned on the fact that a given potential player is chosen. Due to independence,[3] the number $N - 1$ of other players is distributed binomially,

$$N - 1 \sim \text{Binom}(n_{\max} - 1, p_{\text{play}})\,,$$

[1] No dependence *between* pairs, of course. Inside a pair dependence persists.

[2] Do not confuse $\mathbf{\Pi}_n$ here and $\mathbf{\Pi}_k$ of 5a. There, n is suppressed in the notation, and $\mathbf{\Pi}_k$ is the payoff function of player k. Here, n is the number of players, and $\mathbf{\Pi}_n$ is the payoff function of player 1, which is enough due to symmetry.

[3] Just for our example. Generally, a dependence is possible; think what happens if p_{play} is chosen (by nature) at random beforehand and is unknown to players (but its distribution is known).

which is the relevant specification of N. Say,[4] $\mathbb{P}\big(N = 1 \big) = (1 - p_{\text{play}})^{n_{\max}-1}$; $\mathbb{P}\big(N = 2 \big) = (n_{\max} - 1)p_{\text{play}}(1 - p_{\text{play}})^{n_{\max}-2}$; $\mathbb{P}\big(N = 0 \big) = 0$. Note that the expected number of players is $n_{\max}p_{\text{play}}$, while $\mathbb{E}N = 1 + (n_{\max} - 1)p_{\text{play}}$.

5b2. Example. The same as in 5b1, but $n_{\max}$ is large, p_{play} is small, while the expected number of players $n_{\max}p_{\text{play}}$ is the relevant parameter (neither small nor large). Then we may get rid of $n_{\max}$ and p_{play} by using Poisson approximation to the binomial distribution:

$$N - 1 \sim \text{Poisson}(\lambda)\,.$$

(The small difference between $n_{\max}p_{\text{play}}$ and $(n_{\max} - 1)p_{\text{play}}$ is neglected.) Say, $\mathbb{P}\big(N = 1 \big) = e^{-\lambda}$; $\mathbb{P}\big(N = 2 \big) = \lambda e^{-\lambda}$; $\mathbb{P}\big(N = 0 \big) = 0$.

A strategy μ supports a symmetric equilibrium of the game of N players, by definition, if

$$\text{(5b3)} \qquad \mathbb{E}\mathbf{\Pi}_N\big(\underbrace{\mu;\mu;\dots;\mu}_{N}\big) = \sup_{\mu'} \mathbb{E}\mathbf{\Pi}_N\big(\mu';\underbrace{\mu;\dots;\mu}_{N-1}\big)\,.$$

A fixed number of players may be treated as a special case of a random number of players, as far as only symmetric games and equilibria are considered.

5c Best response, revisited

Linearity of Π_1 in s_1 holds for all our games (single unit auctions and oligopoly), irrespective of the number of players. Similarly to (2a12) we define the function

$$\text{(5c1)} \qquad \mathbf{\Pi}_1^{\max}(s_1;\mu_2;\dots;\mu_n) = \sup_{a_1\in\mathcal{A}_1} \mathbf{\Pi}_1(a_1,s_1;\mu_2;\dots;\mu_n)$$

and observe its convexity in s_1. Still (recall 2a15),

$$\text{(5c2)} \qquad \mathbf{\Pi}_1(\mu_1;\mu_2;\dots;\mu_n) \le \mathbf{\Pi}_1^{\max}(P_{S_1};\mu_2;\dots;\mu_n)$$

for all μ_1, and (recall 2a18) the equality holds if and only if μ_1 is concentrated on the set of pairs (a_1, s_1) such that a_1 is an optimal action for s_1, which means

$$\text{(5c3)} \qquad \mathbf{\Pi}_1(a_1,s_1;\mu_2;\dots;\mu_n) = \mathbf{\Pi}_1^{\max}(s_1;\mu_2;\dots;\mu_n)\,.$$

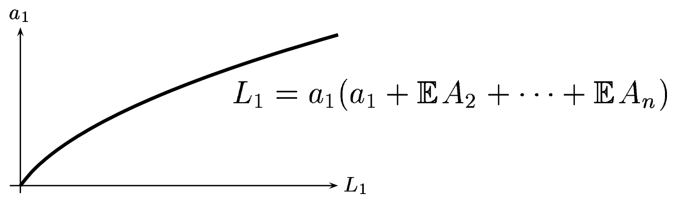
Still, every action a_1 determines a linear function $s_1 \mapsto \mathbf{\Pi}_1(a_1,s_1;\mu_2;\dots;\mu_n)$, write it $\mathbf{\Pi}_1(a_1,s_1)$ for short ($\mu_2,\dots,\mu_n$ being given). The linear function is determined by two numbers, $-\mathbf{\Pi}_1(a_1,0) = \mathbf{L}(a_1)$ and $\mathbf{\Pi}_1(a_1,1) - \mathbf{\Pi}_1(a_1,0) = \mathbf{G}(a_1,1)$, and may be represented by a point on the corresponding plane (recall 2b).

[4] Do not confuse it with the following (irrelevant) calculation: let $M \sim \text{Binom}(n_{\max}, p_{\text{play}})$, then

$$\mathbb{P}\big(M = 1 \,\big|\, M \ge 1 \big) = \frac{n_{\max}p_{\text{play}}(1 - p_{\text{play}})^{n_{\max}-1}}{1 - (1 - p_{\text{play}})^{n_{\max}}}\,.$$

For the oligopoly game, the situation is simple:

$$\begin{aligned}&\mathbf{G}(a_1,1)=a_1\,;\\&\mathbf{L}(a_1)=\mathbb{E}\big(a_1(a_1+A_2+\dots+A_n)\big)=a_1(a_1+\mathbb{E}A_2+\dots+\mathbb{E}A_n)\,.\end{aligned}\tag{5c4}$$



A single number $\mathbb{E}A_2 + \dots + \mathbb{E}A_n$ contains all the relevant information about $\mu_2, \dots, \mu_n$. Similarly to 4b, the best response is given by

$$A_1 = \frac{1}{2}\max\big(0, S_1 - (\mathbb{E}A_2 + \dots + \mathbb{E}A_n)\big)\,. \tag{5c5}$$

For auction games it is more complicated, especially if $A_2, \dots, A_n$ have atoms. Assume for a while that $A_2, \dots, A_n$ are nonatomic. Then the random variable $B = \max(A_2, \dots, A_n)$ is also nonatomic, and (5a3) gives

$$\mathbf{\Pi}_1(A_1, S_1; \mu_2; \dots; \mu_n) = \underbrace{\mathbb{E}\,\mathbf{G}(a_1, s_1; B)}_{\mathbf{G}(a_1,s_1)} - \underbrace{\mathbb{E}\,\mathbf{L}(a_1; B)}_{\mathbf{L}(a_1)}\,; \tag{5c6}$$

there is no ties breaking here, since $\mathbb{P}\big(B = a_1\big) = 0$. For the same reason $\mathbf{G}(a_1, s_1)$ and $\mathbf{L}(a_1)$ are continuous in a_1.[5] In fact, $\mathbf{G}(a_1, s_1) = s_1\mathbf{G}(a_1, 1)$, and $\mathbf{G}(a_1, 1)$ is the winning probability,

$$\begin{aligned}\mathbf{G}(a_1,1) = W_1(a_1) = \mathbb{P}\big(a_1 > \max(A_2,\dots,A_n)\big) &=\\ = \mathbb{P}\big(A_2 < a_1\big)\dots\mathbb{P}\big(A_n < a_1\big) &= F_{A_2}(a_1)\dots F_{A_n}(a_1)\,,\end{aligned}\tag{5c7}$$

a continuous function, indeed.

In general, $A_2, \dots, A_n$ may have atoms; taking (5a3) into account, we define

$$\mathbf{L}(a_1) = \mathbb{E}\Big(\frac{M}{M+1}\mathbf{L}(a_1-; B) + \frac{1}{M+1}\mathbf{L}(a_1+; B)\Big)\,; \tag{5c8}$$

here $B = \max(A_2, \dots, A_n)$, and M is the multiplicity, introduced before (5a3) for the case $a_1 = B$; for the other case ($a_1 \ne B$) the value of M does not matter, since here $\mathbf{L}(a_1-, B) = \mathbf{L}(a_1+, B)$. Equivalently,

$$\begin{aligned}&\mathbf{L}(a_1) = \mathbb{E}\big(\mathbf{L}(a_1; B)\,\big|\,B \ne a_1\big)\,\mathbb{P}\big(B \ne a_1\big) +\\ &+\mathbf{L}(a_1-; a_1)\mathbb{E}\Big(\frac{M}{M+1}\,\Big|\,B = a_1\Big)\,\mathbb{P}\big(B = a_1\big) + \mathbf{L}(a_1+; a_1)\mathbb{E}\Big(\frac{1}{M+1}\,\Big|\,B = a_1\Big)\,\mathbb{P}\big(B = a_1\big)\,.\end{aligned}\tag{5c9}$$

[5] If $a_n \to a$ then $\mathbf{L}(a_n, b) \to \mathbf{L}(a, b)$ for all b except for $b = a$; therefore $\mathbf{L}(a_n, B) \to \mathbf{L}(a, B)$ almost surely; under appropriate integrability assumptions, dominated convergence theorem is applicable, giving $\mathbb{E}\mathbf{L}(a_n, B) \to \mathbb{E}\mathbf{L}(a, B)$.

Similarly, we define

$$\mathbf{G}(a_1, s_1) = \mathbb{E}\Big(\frac{M}{M+1} \mathbf{G}(a_1-, s_1; B) + \frac{1}{M+1} \mathbf{G}(a_1+, s_1; B) \Big) \, ; \tag{5c10}$$

or equivalently,

(5c11) $\mathbf{G}(a_1, s_1) = \mathbb{E}\left(\mathbf{G}(a_1, s_1; B) \,\middle|\, B \ne a_1 \right) \mathbb{P}\left(B \ne a_1 \right) +$
$+\mathbf{G}(a_1-, s_1; a_1) \mathbb{E}\left(\frac{M}{M+1} \,\middle|\, B = a_1 \right) \mathbb{P}\left(B = a_1 \right) + \mathbf{G}(a_1+; a_1) \mathbb{E}\left(\frac{1}{M+1} \,\middle|\, B = a_1 \right) \mathbb{P}\left(B = a_1 \right) .$

The framework of 3a leaves the payment rule unspecified, but specifies the (standard) allocation rule. Thus $\mathbf{L}$ is just some function satisfying our assumptions, while

$$\begin{aligned} \mathbf{G}(a_1, s_1; b) &= 0 \quad \text{for } a_1 < b \, , \\ \mathbf{G}(a_1, s_1; b) &= s_1 \quad \text{for } a_1 > b \, . \end{aligned} \tag{5c12}$$

Therefore

$$\mathbf{G}(a_1, s_1) = s_1 W(a_1) \, , \tag{5c13}$$

where

$$W(a_1) = \mathbb{P}\left(B < a_1 \right) + \mathbb{E}\left(\frac{1}{M+1} \,\middle|\, B = a_1 \right) \mathbb{P}\left(B = a_1 \right) \tag{5c14}$$

is the winning probability. Still,

$$W(a_1) = F_{A_2}(a_1) \dots F_{A_n}(a_1) \quad \text{if } a_1 \text{ is a point of continuity.} \tag{5c15}$$

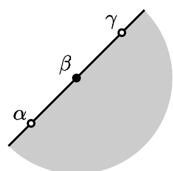
In the simple case $n = 2$ we have $M = 1$ always, and $W(a_1) = \mathbb{P}\left(A_2 < a_1 \right) + \frac{1}{2} \mathbb{P}\left(A_2 = a_1 \right) = \frac{1}{2} W(a_1-) + \frac{1}{2} W(a_1+)$. In general, $W(a_1)$ is not the center of the interval $[W(a_1-), W(a_1+)]$. Rather, it divides the interval according to

$$\frac{W(a_1) - W(a_1-)}{W(a_1+) - W(a_1-)} = \mathbb{E}\left(\frac{1}{M+1} \,\middle|\, B = a_1 \right) \, . \tag{5c16}$$

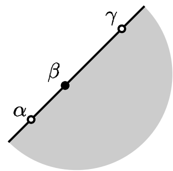
It is important that

$$\frac{\mathbf{G}(a_1, s_1) - \mathbf{G}(a_1-, s_1)}{\mathbf{G}(a_1+, s_1) - \mathbf{G}(a_1-, s_1)} = \frac{\mathbf{L}(a_1) - \mathbf{L}(a_1-)}{\mathbf{L}(a_1+) - \mathbf{L}(a_1-)} \tag{5c17}$$

since both quotients are equal to $\mathbb{E}\left(\frac{1}{M+1} \,\middle|\, B = a_1 \right)$, the winning probability in the case of tie breaking. We conclude that the nonatomicity argument of 3e2,



is still applicable; though, the point β is now not in the middle, but still on the interval,



Thus β can be optimal for a single s only, which proves the following generalization of 3e2.

5c18. Lemma. If a_1 is an optimal action (against given $\mu_2, \dots, \mu_n$) for more than a single s_1, and $a_1 \ne 0$, then a_1 is not an atom of $\max(A_2, \dots, A_n)$.

Lemma 2b10 (about gaps) is easily generalized to n players; a gap of $\max(A_2, \dots, A_n)$ is considered instead of a gap of A_2.

It is much easier, to generalize results (and proofs) of subsections 2c (weak monotonicity) and 2d (integral of winning probability), since these are based on the linearity (of Π_1 in s_1) only. Possible actions are represented by straight lines on one plane, or points on another plane. Any set of lines (points) is acceptable, no matter how it is generated by strategies of competitors, and how many competitors exist. Anyway, the joint distribution of S_1 and $\mathbf{G}(A_1) = \mathbf{\Pi}_1(A_1, 1) - \mathbf{\Pi}_1(A_1, 0)$ is weakly increasing, and an increment of $\mathbf{\Pi}_1^{\max}$ is the integral of winning probability.

We turn to symmetric auctions. Having in mind symmetric equilibria, we consider now best response to $\mu_2 = \dots = \mu_n$, that is, $F_{A_2} = \dots = F_{A_n}$. The winning probability is

(5c19) $$W(a) = F_{A_2}^{n-1}(a) \qquad \text{if } a \text{ is a point of continuity.}$$

If a is an atom of A_2, that is, $F(a-) < F(a+)$ (here $F = F_{A_2}$), then[6]

(5c20) $$W(a) = F^{n-1}(a+) \frac{1}{n} \frac{1}{1 - \frac{F(a-)}{F(a+)}} \left(1 - \frac{F^n(a-)}{F^n(a+)} \right)$$

which, however, will not be used, except for example 5c22 below. You can derive from (5c20) the ratio

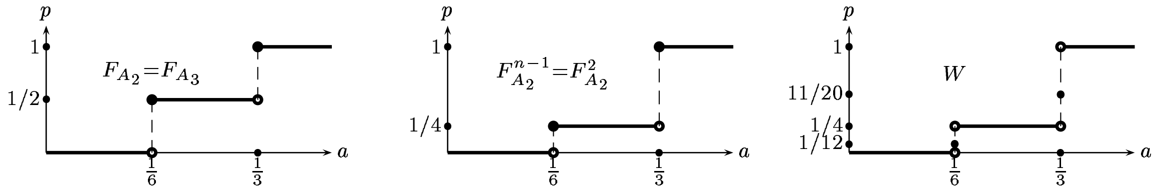
(5c21) $$\begin{aligned} \frac{W(a) - W(a-)}{W(a+) - W(a-)} &= \frac{1}{2} \quad \text{for } n = 2 \,, \\ \frac{W(a) - W(a-)}{W(a+) - W(a-)} &= \frac{1}{3} + \frac{1}{3} \frac{F(a-)}{F(a+)} \frac{1}{1 + \frac{F(a-)}{F(a+)}} \quad \text{for } n = 3 \end{aligned}$$

and think, why it exceeds $1/n$ for $n = 3$ but not for $n = 2$.

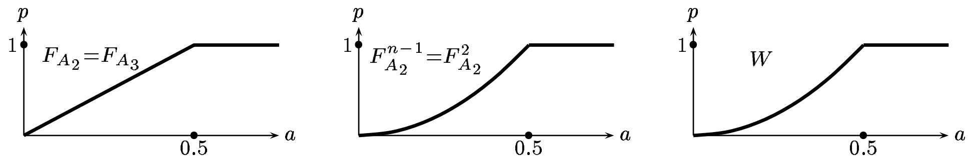
[6]Here is the idea of a proof (if you want). Let $U_1, \dots, U_n$ be independent random variables distributed uniformly on $(0, 1)$, and $p \in (0, 1)$. Then

$$\mathbb{P}\left(U_1 > \max(U_2, \dots, U_n) \,\middle|\, U_1 > 1 - p \right) = \frac{1}{p} \int_{1-p}^{1} u^{n-1} \, du = \frac{1}{np} \left(1 - (1-p)^n \right).$$

5c22. Example. Let A_2 be as in 2b1 (discrete), and $n = 3$, then we have



5c23. Example. Let $A_2 \sim \mathrm{U}(0, 1/2)$ as in 2b2, and $n = 3$, then we have



We turn to a random number of players, N. Formula (5c16) still holds, with $B = \max(A_2, \dots, A_N)$.[7] Thus, Lemma 5c18 holds. Results of Sections 2c (weak monotonicity) and 2d (integral of winning probability) hold for the same reason as before. The winning probability is (recall (5c19))

(5c24) $$W(a) = \mathbb{E} F^{N-1}(a) \qquad \text{if } a \text{ is a point of continuity.}$$

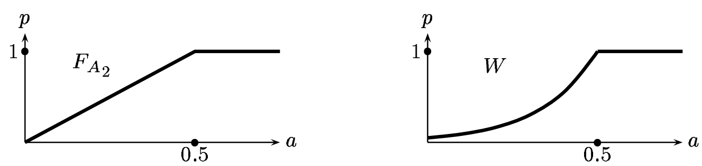
5c25. Example. Let $N - 1$ have Poisson distribution $P(\lambda)$, that is (recall 5b2),

$$\mathbb{P}\big(N = n + 1 \big) = e^{-\lambda} \frac{\lambda^n}{n!} \qquad \text{for } n = 0, 1, \dots$$

The player knows that he is a player, but does not know, whether there are more players, or not. Of course, when $N = 1$, the winning probability is equal to 1. If a is not an atom of A_2 then

$$W(a) = \mathbb{P}\big(\max(A_2, \dots, A_N) < a \,\big|\, N \ge 1 \big) = \mathbb{E} F^{N-1}(a) =$$
$$= \sum_{n=0}^{\infty} F^n(a) \mathbb{P}\big(N = n+1 \big) = \sum_{n=1}^{\infty} F^n(a) e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \exp(\lambda F(a)) = e^{-\lambda(1 - F(a))} \, .$$

Say, if $A_2 \sim \mathrm{U}(0, 1/2)$ and $\lambda = 3$ (which means $\mathbb{E} N = 3$), we get



Note that $W(0+) = e^{-\lambda} = \mathbb{P}\big(N = 1 \big)$.

[7] Do not confuse the conditional distribution of N given $B = a_1$ with the unconditional distribution of N.

For the symmetric oligopoly game with a random number of players,[8] we have (recall (5c4))

$$\mathbf{L}(a_1) = \mathbb{E}\big(a_1(a_1 + A_2 + \dots + A_N)\big) = a_1\big(a_1 + \mathbb{E}\,(A_2 + \dots + A_N)\big)\,.$$

However,

$$\mathbb{E}\,(A_2 + \dots + A_N) = \mathbb{E}\big(\mathbb{E}\,(\,A_2 + \dots + A_N \,|\, N\,)\,\big) = \mathbb{E}\big((N-1)\mathbb{E}\,A_2\big) = (\mathbb{E}\,N - 1)(\mathbb{E}\,A_2)\,.$$

Of course, we need $\mathbb{E}\,N < \infty$. So,

$$\mathbf{L}(a) = a\big(a + (\mathbb{E}\,N - 1)(\mathbb{E}\,A_2)\big)\,,$$

and (5c5) turns into

$$A_1 = \frac{1}{2}\max\big(0, S_1 - (\mathbb{E}\,N - 1)(\mathbb{E}\,A_2)\big)\,.$$

For a fixed number n of players, of course,

$$A_1 = \frac{1}{2}\max\big(0, S_1 - (n-1)\mathbb{E}\,A_2\big)\,.$$

In presence of an entry cost, results of 4d remain valid with $\mathbb{E}\,A_2$ replaced by $(\mathbb{E}\,N - 1)(\mathbb{E}\,A_2)$. The best response is

$$A_1 = \begin{cases} 0 & \text{if } S_1 < 2\sqrt{c} + (\mathbb{E}\,N - 1)(\mathbb{E}\,A_2), \\ \frac{1}{2}\big(S_1 - (\mathbb{E}\,N - 1)(\mathbb{E}\,A_2)\big) & \text{if } S_1 > 2\sqrt{c} + (\mathbb{E}\,N - 1)(\mathbb{E}\,A_2); \end{cases}$$

both actions may be used when $S_1 = 2\sqrt{c} + (\mathbb{E}\,N - 1)(\mathbb{E}\,A_2)$.

5d Symmetric equilibria and revenue equivalence, revisited

Lemma 3b1 was established for the function $W(x) = \frac{1}{2}F_X(x-) + \frac{1}{2}F_X(x+)$. Now we modify W according to (5c19): $W(x) = F^{n-1}(x)$ for continuity points x; still a strictly increasing one-to-one correspondence between $W(x)$ and $F(x)$. For discontinuity points we have $W(x-) < W(x) < W(x+)$, which is enough for the proof of the lemma. Therefore, Theorem 3b2 and Corollary 3b3 hold for any n:

- Every symmetric equilibrium is supported by a weakly increasing strategy.
- If the distribution of signals is nonatomic then every symmetric equilibrium is supported by an increasing pure strategy.

All that remains valid for a random number of players. Indeed, (5c24) shows that $W(a) = \alpha(F(x))$ for continuity points a; here $\alpha(p) = \mathbb{E}p^{N-1}$ is a strictly increasing continuous function. For discontinuity points $W(x-) < W(x) < W(x+)$, once again.

Similarly to 3c (page 28), assume for a moment that the optimal action is a *strictly increasing function* φ of a signal; that is, for every s there exists one and only one optimal

[8] The case $N = 1$ is acceptable here; we need $1 < \mathbb{E}\,N < \infty$.

action $a = \varphi(s)$, and in addition, s is not an atom of P_S. Then a is not an atom of P_A, and (5c19) gives

$$\mathbf{p}^{\text{win}}(s) = W(\varphi(s)) = F_A^{n-1}(\varphi(s)) = \Big(\mathbb{P}\big(\,\varphi(S) < \varphi(s)\,\big)\Big)^{n-1} = \Big(\mathbb{P}\big(\,S < s\,\big)\Big)^{n-1} = F_S^{n-1}(s)\,.$$

Adapting the proof of Lemma 3c3 accordingly, we get

$$\mathbf{p}^{\text{win}}(s_0) = F_S^{n-1}(s_0)$$

whenever the conditions of the lemma are satisfied. (A bunch is defined exactly as in Sect. 3.) For a random number of players,

$$\mathbf{p}^{\text{win}}(s_0) = \mathbb{E} F_S^{N-1}(s_0)$$

is obtained in the same way, using (5c24) instead of (5c19). Thus, Theorem 3c4 is generalized as follows.

5d1. Theorem. Let μ be a strategy supporting a symmetric equilibrium, with no bunch of positive probability (w.r.t. P_S). Then

$$\mathbf{\Pi}^{\max}(s) - \mathbf{\Pi}^{\max}(0) = \int_0^s \alpha(F_S(s'))\,ds'$$

for all $s \in [0,\infty)$; here $\alpha(p) = p^{n-1}$ for a fixed number n of players, and $\alpha(p) = \mathbb{E} p^{N-1}$ for a random number N of players.

5d2. Example. Let $N-1$ have Poisson distribution $P(\lambda)$ (recall 5c25). Then $\alpha(p) = \mathbb{E} p^{N-1} = e^{-\lambda(1-p)}$, and so,

$$\mathbf{\Pi}^{\max}(s) - \mathbf{\Pi}^{\max}(0) = \int_0^s e^{-\lambda(1-F_S(s'))}\,ds'\,.$$

Corollary 3c6 is generalized evidently. Generalization of Theorem 3c10 is straightforward; the convex function $s \mapsto \int_0^s F_S(s')\,ds'$ is replaced with another convex function, $s \mapsto \int_0^s \alpha(F_S(s'))\,ds'$. Condition 3c10(b) becomes

$$\mathbf{L}(\varphi(s)) = s\alpha(F_S(s)) - \int_0^s \alpha(F_S(s'))\,ds'\,.$$

Taking into account that $\alpha(F_S(\cdot))$ is the distribution function of the random variable $X = \max(S_2,\dots,S_N)$, we get (recall (3c8) and (3c9))

$$\mathbf{L}(\varphi(s)) = \mathbb{E}\big(\,X\,\big|\,X \le s\,\big)\cdot\mathbb{P}\big(\,X \le s\,\big)\,.$$

In order to avoid some complications we assume that $N \ge 2$ with probability 1.

We apply all that to the first price auction considered (for $n = 2$, of course) in 3d. General arguments work as before, and we get

$$\underbrace{\varphi(s)\overbrace{\mathbb{P}\big(\,X \le s\,\big)}^{\mathbf{p}^{\text{win}}(s)}}_{\mathbf{L}(\varphi(s))} = \mathbb{E}\big(\,X\,\big|\,X \le s\,\big)\,\mathbb{P}\big(\,X \le s\,\big)\,.$$

Theorem 3d3 is thus generalized, with

$$\varphi(s) = \mathbb{E}\left(X \,\middle|\, X \le s \right) , \qquad X = \max(S_2, \dots, S_N) .$$

For the second price auction (as in 3f) we get

$$\underbrace{\mathbb{E}\left(\varphi(X) \,\middle|\, X \le s \right) \mathbb{P}\left(X \le s \right)}_{\mathbf{L}(\varphi(s))} = \mathbb{E}\left(X \,\middle|\, X \le s \right) \mathbb{P}\left(X \le s \right) .$$

It means that $\varphi(X) = X$ for almost all X, which implies (in fact, is equivalent to) $\varphi(S) = S$ for almost all S.[9]

For the first price auction with reserve price and entry cost (as in 3g), Lemma 3g3 becomes

$$\mathbf{p}^{\mathrm{win}}(s) = \begin{cases} 0 & \text{for } s < s_0, \\ \alpha(F_S(s)) & \text{for } s > s_0, \end{cases}$$

and (3g6) becomes

$$\mathbf{\Pi}^{\max}(s) = \begin{cases} 0 & \text{for } s \le s_0, \\ \int_{s_0}^s \alpha(F_S(s'))\, ds' & \text{for } s \ge s_0. \end{cases}$$

Still $\mathbf{L}(0+) = c + rp_0$, but now $p_0 = \alpha(F_S(s_0)) = \mathbb{P}\left(X \le s_0 \right) = \mathbb{P}\left(A_2 = \dots = A_N = 0 \right)$, thus (3g10) becomes

$$(s_0 - r)\alpha(F_S(s_0)) = c .$$

No problem with the equation, since $\alpha(\cdot)$ increases on $[0,1]$, $\alpha(0) = 0$, $\alpha(1) = 1$. Condition 3g11(d) becomes $\mathbf{L}(\varphi(s)) = s\alpha(F_S(s)) - \int_{s_0}^s \alpha(F_S(s'))\, ds' = sF_X(s) - \int_{s_0}^s F_X(s')\, ds'$ for P_S-almost all $s > s_0$. However, $\mathbf{L}(a_1) = c + (r + a_1)\alpha(F_A(a_1))$ for $a_1 > 0$, and (3g12) becomes

$$\varphi(s) = -r + \frac{1}{F_X(s)} \biggl(-c + sF_X(s) - \int_{s_0}^s F_X(s')\, ds' \biggr)$$

for $s > s_0$. The bracketed expression is equal to $\mathbb{E}\left(h(X) \mathbf{1}_{[0,s]}(X) \right)$ by the same argument as on page 40, but for X instead of S. Theorem 3g13 is thus generalized, with

$$\varphi(s) = \begin{cases} 0 & \text{for } s < s_0, \\ -r + \mathbb{E}\left(h(X) \,\middle|\, X \le s \right) & \text{for } s > s_0; \end{cases} \qquad h(s) = \begin{cases} r & \text{for } s < s_0, \\ s & \text{for } s > s_0; \end{cases}$$

$$(s_0 - r)F_X(s_0) = c ; \qquad X = \max(S_2, \dots, S_N) .$$

[9] Distributions of X and S are equivalent (mutually absolutely continuous), and the corresponding density can be calculated explicitly:

$$\frac{dP_X}{dP_S}(s) = \sum_{n=2}^{\infty} (n-1) F^{n-2}(s) \mathbb{P}\left(N = n \right) .$$

5d3. Exercise. Generalize 3h2 (all-pay auctions).

Finally, we turn to the symmetric oligopoly game. Theorem 4c3 (and its proof) remains valid with the following equation for a_0:

$$2a_0 = \mathbb{E}\max\bigl(0, S - (\mathbb{E}N - 1)a_0\bigr)\,;$$

for a fixed number of players it becomes

$$2a_0 = \mathbb{E}\max\bigl(0, S - (n - 1)a_0\bigr)\,.$$

Example 4c4 ($S = s_0$ always) becomes $A = a_0 = \frac{s_0}{\mathbb{E}N+1}$ (or $\frac{s_0}{n+1}$); the mean aggregate supply is $\mathbb{E}(A_1 + \dots + A_N) = a_0 \mathbb{E}N = \frac{\mathbb{E}N}{\mathbb{E}N+1} s_0$ (or $\frac{n}{n+1} s_0$).

In presence of an entry cost, a symmetric equilibrium is a strategy $\mu = \mu_{a_0,p}$ described by

$$A = \begin{cases} 0 & \text{if } S < 2\sqrt{c} + (\mathbb{E}N - 1)a_0, \\ \frac{1}{2}\bigl(S - (\mathbb{E}N - 1)a_0\bigr) & \text{if } S > 2\sqrt{c} + (\mathbb{E}N - 1)a_0, \end{cases}$$

$$\mathbb{P}\bigl(\, A = 0 \,\big|\, S = 2\sqrt{c} + (\mathbb{E}N - 1)a_0 \,\bigr) = 1 - p\,,$$

$$\mathbb{P}\bigl(\, A = \sqrt{c} \,\big|\, S = 2\sqrt{c} + (\mathbb{E}N - 1)a_0 \,\bigr) = p\,;$$

the parameters a_0, p must be chosen such that $\mathbb{E}A = a_0$.

5d4. Exercise. Generalize the last part of 4d (the game of complete information, $S = s_0$ always). Show that the mean number of participants (players that do not quit) is equal to

$$\frac{\mathbb{E}N}{\mathbb{E}N - 1}\left(\frac{s_0}{\sqrt{c}} - 2\right)$$

whenever the latter belongs to the interval $(0, \mathbb{E}N)$. In that case, the mean aggregate supply is $\mathbb{E}(A_1 + \dots + A_N) = \frac{\mathbb{E}N}{\mathbb{E}N-1}(s_0 - 2\sqrt{c})$.