4 Oligopoly: Cournot competition as a game of incomplete information

4a Our framework

Two firms produce an identical product. The product has an initial price. The cost to produce a single unit is less than the initial price according to a signal:

$$\begin{aligned}(\text{cost of firm } 1) &= (\text{initial price}) - S_1\,,\\ (\text{cost of firm } 2) &= (\text{initial price}) - S_2\,.\end{aligned}$$

Firms simultaneously choose the quantities they will produce; these are actions:

$$\begin{aligned}A_1 &= (\text{level of supply of firm } 1)\,,\\ A_2 &= (\text{level of supply of firm } 2)\,.\end{aligned}$$

The aggregate supply $A_1 + A_2$ lowers the market price; we use the linear approximation,[1]

$$(\text{market price}) = (\text{initial price}) - (A_1 + A_2)\,;$$

a coefficient before $(A_1 + A_2)$ could be stipulated, however, we eliminate it by an appropriate choice of units (of prices and supply). Therefore

$$(\text{market price}) - (\text{cost of firm } 1) = S_1 - (A_1 + A_2)\,,$$

and the profit of the first firm is

$$\Pi_1 = A_1(S_1 - A_1 - A_2)\,.$$

So, the game is described by the profit function

$$\mathbf{\Pi}(a_1, s_1; a_2) = \mathbf{G}(a_1, s_1; a_2) - \mathbf{L}(a_1; a_2)$$

with the following gain function and loss function:[2]

$$\begin{aligned}\mathbf{G}(a_1, s_1; a_2) &= a_1 s_1\,;\\ \mathbf{L}(a_1; a_2) &= a_1(a_1 + a_2)\,.\end{aligned}$$

Symmetry of the game means that $\mathbf{\Pi}_1 = \mathbf{\Pi}_2 = \mathbf{\Pi}$. Action spaces and signal spaces still are

$$\mathcal{S}_1 = \mathcal{S}_2 = \mathbb{R}\,; \qquad \mathcal{A}_1 = \mathcal{A}_2 = [0, \infty)\,.$$

[1]The linear approximation predicts a negative market price for a large supply, which is incorrect, of course. However, firms do not want to lower the market price under their cost prices, which restricts their supply. It is quite improbable that the market price will approach zero.

[2]You may note that the true gain is rather $\big((\text{initial price}) - (a_1 + a_2)\big)a_1 = (\text{initial price})a_1 - \mathbf{L}(a_1; a_2)$ and the true loss is $\big((\text{initial price}) - s_1\big)a_1 = (\text{initial price})a_1 - \mathbf{G}(a_1, s_1; a_2)$. I intentionally reformulate the game in order to exclude the initial price (large and irrelevant), making the formalism closer to that of Sect. 3.

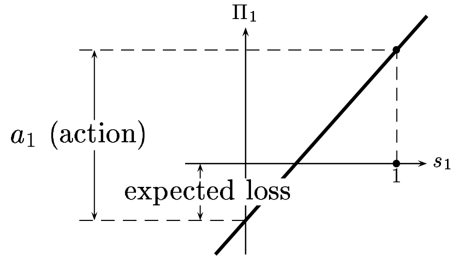
Signals are independent, and their distribution P_S $(= P_{S_1} = P_{S_2})$ is arbitrary, except for assuming $P_S\big([0,\infty)\big) = 1$, that is,

$$\mathbb{P}\big(\, S \ge 0 \,\big) = 1\,.$$

As before (in Sect. 3), the loss does not depend on signals, while the gain is proportional to s_1; therefore the profit is linear in s_1. Still, $\mathbf{\Pi}(a_1, 1; a_2) - \mathbf{\Pi}(a_1, 0; a_2) = \mathbf{G}(a_1, 1; a_2)$, but now it is equal to a_1 (not at all a winning probability). Still, $-\mathbf{\Pi}(a_1, 0; a_2) = \mathbf{L}(a_1; a_2)$ is the loss. And, naturally, the two firms will be called players.

4b Best response

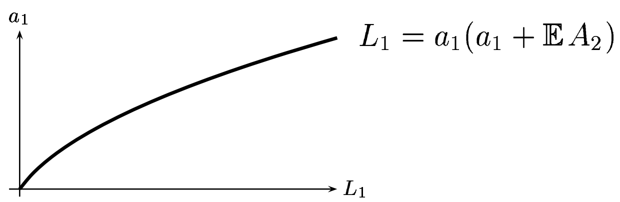
Given a distribution P_{A_2} of the action A_2 of the second player, each (possible) action a_1 of the first player determines a linear function of his signal s_1:



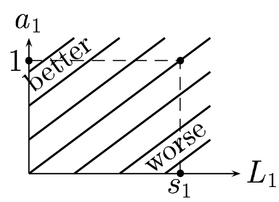
Compare it with a similar picture on page 12; the action a_1 plays now the role of the winning probability. Thus, the plane (expected loss, winning probability) introduced in 2b becomes now the plane (expected loss, action). The expected loss is

$$\mathbf{L}(a_1; P_{A_2}) = \mathbb{E}\,\mathbf{L}(a_1; A_2) = \mathbb{E}\big(a_1(a_1 + A_2)\big) = a_1(a_1 + \mathbb{E}\,A_2)\,.$$

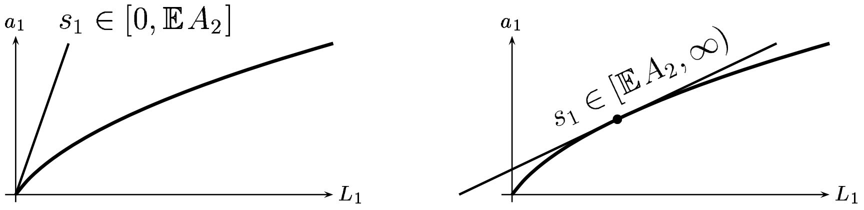
The situation is much simpler than before (for auctions); not the whole distribution P_{A_2} matters for the first player, but only the single number $\mathbb{E}\,A_2$. The number determines a parabola on the plane (expected loss, action):



An optimal action maximizes the goal function $\mathbf{\Pi}(a_1, s_1; P_{A_2}) = s_1 a_1 - L_1$ linear on the plane (L_1, a_1);



Case $s_1 \in [0, \mathbb{E} A_2]$. The action $a_1 = 0$ is optimal. No other action is optimal.



Case $s_1 \in [\mathbb{E} A_2, \infty)$. The action $a_1 = \frac{1}{2}(s_1 - \mathbb{E} A_2)$ is optimal. No other action is optimal. The optimum can be found as follows:

$$\frac{d}{da_1}\big(a_1(a_1 + \mathbb{E} A_2)\big) = s_1 ; \qquad 2a_1 + \mathbb{E} A_2 = s_1 .$$

Applying Lemma 2a18 we see that, irrespective of P_{S_1}, there exists one and only one best response to P_{A_2} (or rather, to $\mathbb{E} A_2$). Basically, it is given by

$$A_1 = \frac{1}{2} \max(0, S_1 - \mathbb{E} A_2) .$$

More formally, it is the joint distribution P_{S_1,A_1} of random variables S_1 (distributed P_{S_1}, as required) and A_1 (defined as $\frac{1}{2} \max(0, S_1 - \mathbb{E} A_2)$).

4c Symmetric equilibria

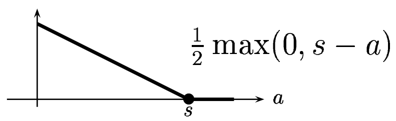
A strategy μ supports a symmetric equilibrium, if it is a best response to itself; in our case it means

$$A = \frac{1}{2} \max(0, S - \mathbb{E} A) . \tag{4c1}$$

In order to find such a strategy, we only need to find a number a such that

$$a = \frac{1}{2} \mathbb{E} \max(0, S - a) . \tag{4c2}$$

Every possible value s of S contributes to the right-hand side such a function:



The right-hand side $\frac{1}{2}\mathbb{E} \max(0, S - a)$ is a continuous decreasing function of a, strictly positive at $a = 0$ (unless $\mathbb{P}\big(S = 0\big) = 1$) and tending to 0 for $a \to +\infty$ (unless $\mathbb{E} S = +\infty$).[3] Therefore the equation (4c2) has one and only one solution $a \in (0, \infty)$, which proves the following result.

[3] $\max(0, s - a) \to 0$ for $a \to +\infty$ for every s, and $\max(0, s - a) \le s$ for all a and s, therefore $\mathbb{E} \max(0, S - a) \to 0$ for $a \to +\infty$ by the dominated convergence theorem.

4c3. Theorem. Let $0 < \mathbb{E}\, S < \infty$. Then the Cournot game has one and only one symmetric equilibrium, namely,

$$A = \varphi(S)\,, \quad \text{where } \varphi(s) = \frac{1}{2}\max(0, s - a_0)\,;$$

here $a_0 \in (0, \infty)$ is the unique solution of the equation

$$2a_0 = \mathbb{E}\,\max(0, S - a_0)\,.$$

4c4. Example. Let P_S be concentrated at a single atom s_0; then $A = a_0 = \frac{1}{3}s_0$.

4c5. Example. Let $S \sim \mathrm{U}(0,1)$, then the equation for a_0 becomes $2a_0 = \frac{1}{2}(1 - a_0)^2$ (for $0 < a_0 < 1$), hence $a_0 = 3 - \sqrt{8} \approx 0.17$. Thus $\mathbb{P}\,\big(\,A = 0\,\big) = \mathbb{P}\,\big(\,S < a_0\,\big) \approx 0.17$.

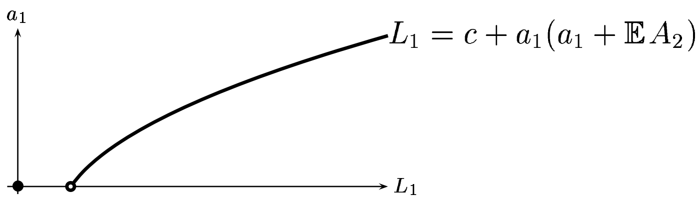
4d Entry cost

In order to produce any amount of the product, a firm must pay an entry cost c. Or else it may quit, paying nothing and producing nothing. It means the following modification of our loss function:

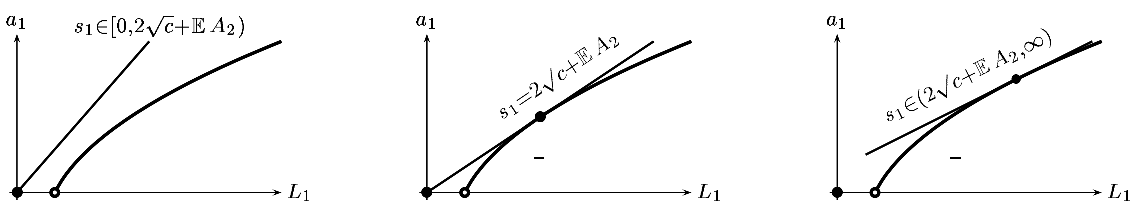
$$\mathbf{L}(a_1; a_2) = \begin{cases} 0 & \text{if } a_1 = 0, \\ c + a_1(a_1 + a_2) & \text{if } a_1 \ne 0. \end{cases}$$

The expected loss is now

$$\mathbf{L}(a_1; P_{A_2}) = \begin{cases} 0 & \text{if } a_1 = 0, \\ c + a_1(a_1 + \mathbb{E}\,A_2) & \text{if } a_1 \ne 0. \end{cases}$$



Case $s_1 \in [0, 2\sqrt{c} + \mathbb{E}\,A_2)$**.** The action $a_1 = 0$ is optimal. No other action is optimal.



The critical value $2\sqrt{c}+\mathbb{E}A_2$ may be found as follows:[4]

$$\begin{cases} c + a_1(a_1+\mathbb{E}A_2) = s_1 a_1\,, \\ \dfrac{d}{da_1}\big(c+a_1(a_1+\mathbb{E}A_2)\big) = \dfrac{d}{da_1}(s_1 a_1)\,; \end{cases} \qquad \begin{cases} \dfrac{c}{a_1} + a_1 + \mathbb{E}A_2 = s_1\,, \\ 2a_1 + \mathbb{E}A_2 = s_1\,; \end{cases}$$

$$\frac{c}{a_1} - a_1 = 0\,; \qquad a_1 = \sqrt{c}\,; \qquad s_1 = 2\sqrt{c}+\mathbb{E}A_2\,.$$

Case $s_1 = 2\sqrt{c}+\mathbb{E}A_2$**.** There are two optimal actions, $a_1 = 0$ and $a_1 = \sqrt{c}$. No other action is optimal.

Case $s_1 \in (2\sqrt{c}+\mathbb{E}A_2, \infty)$**.** The action $a_1 = \frac12(s_1 - \mathbb{E}A_2)$ is optimal. No other action is optimal. The optimum may be found as follows:

$$\frac{d}{da_1}\big(c+a_1(a_1+\mathbb{E}A_2)\big) = s_1\,; \qquad 2a_1 + \mathbb{E}A_2 = s_1\,.$$

Assuming that P_{S_1} is nonatomic and applying Lemma 2a18 we see that there exists one and only one best response to P_{A_2} (or rather, to $\mathbb{E}A_2$), given by

$$A_1 = \begin{cases} 0 & \text{if } S_1 < 2\sqrt{c}+\mathbb{E}A_2, \\ \frac12(S_1 - \mathbb{E}A_2) & \text{otherwise.} \end{cases}$$

The same holds as far as P_{S_1} has no atom at $2\sqrt{c}+\mathbb{E}A_2$. Assume however that P_{S_1} has an atom at $2\sqrt{c}+\mathbb{E}A_2$. Then we get a continuum of best responses, parametrized by $p \in [0,1]$ as follows:

$$A_1 = \begin{cases} 0 & \text{if } S_1 < 2\sqrt{c}+\mathbb{E}A_2, \\ \frac12(S_1 - \mathbb{E}A_2) & \text{if } S_1 > 2\sqrt{c}+\mathbb{E}A_2, \end{cases}$$

$$\mathbb{P}\big(A_1 = 0 \,\big|\, S_1 = 2\sqrt{c}+\mathbb{E}A_2 \big) = 1-p\,,$$

$$\mathbb{P}\big(A_1 = \sqrt{c} \,\big|\, S_1 = 2\sqrt{c}+\mathbb{E}A_2 \big) = p\,.$$

The best response is a mixed strategy when $p \in (0,1)$, but a pure strategy when $p = 0$ or $p = 1$.

A symmetric equilibrium is a strategy $\mu = \mu_{a_0,p}$ described by

$$A = \begin{cases} 0 & \text{if } S < 2\sqrt{c}+a_0, \\ \frac12(S - a_0) & \text{if } S > 2\sqrt{c}+a_0, \end{cases}$$

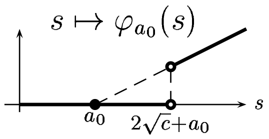
$$\mathbb{P}\big(A = 0 \,\big|\, S = 2\sqrt{c}+a_0 \big) = 1-p\,,$$

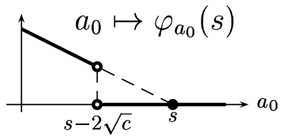
$$\mathbb{P}\big(A = \sqrt{c} \,\big|\, S = 2\sqrt{c}+a_0 \big) = p\,.$$

[4] You may wonder about $\sqrt{c}$, since its dimension seems to be "$\sqrt{\text{dollar}}$" rather than "dollar". The origin of the problem is the formula "(market price) = (initial price) − $(A_1 + A_2)$" (page 44); you see, the prices are dollars, while $A_1 + A_2$ is, say, kilograms. A coefficient of dimension (dollar)/(kilogram) could be inserted for making dimensions consistent.

Of course, the mixing parameter $p \in [0,1]$ matters only if S has an atom at $2\sqrt{c}+a_0$. Otherwise the strategy is pure and may be written in the form $A = \varphi_{a_0}(S)$. Anyway, the distribution P_A and its expectation $\mathbb{E}\,A$ depend on a_0 and p.

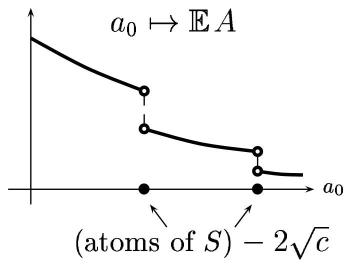
We start with the easier case: a_0 is such that $2\sqrt{c}+a_0$ is not an atom of S. Then $\mathbb{E}\,A = \mathbb{E}\,\varphi_{a_0}(S)$, and $\varphi_{a_0}(s)$ decreases in a_0 for every s.



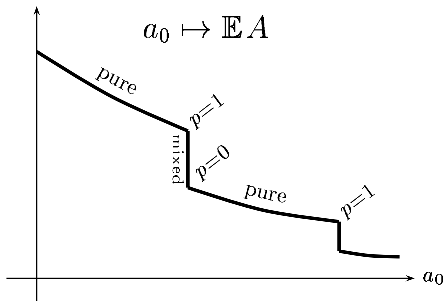


$$\varphi_{a_0}(s) = \begin{cases} 0 & \text{if } s < 2\sqrt{c}+a_0, \\ \frac{1}{2}(s-a_0) & \text{if } s > 2\sqrt{c}+a_0. \end{cases}$$

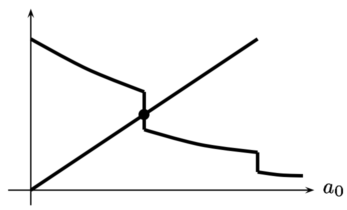
(In fact, $\varphi_{a_0}(s) = \varphi_0(s-a_0)$.) Thus, $\mathbb{E}\,A$ is a decreasing function of a_0. However, it is discontinuous, unless P_S is nonatomic.



Let s_1 be an atom of S. The limit of the pure strategy μ_{a_0} for $a_0 \to (s_1 - 2\sqrt{c})-$ is the pure strategy $\mu_{s_1-2\sqrt{c},1}$ (think, why); here 1 is the value of the mixing parameter p. The other limit, for $a_0 \to (s_1 - 2\sqrt{c})+$, is $\mu_{s_1-2\sqrt{c},0}$. Taking into account that $\mathbb{E}\,A$ is linear in the mixing parameter p we see that mixed strategies fill in the jump(s):



It is no more a function of a_0; rather, it is a weakly monotone relation between a_0 and $\mathbb{E}\,A$. Evidently, there exists one and only one point satisfying $\mathbb{E}\,A = a_0$.

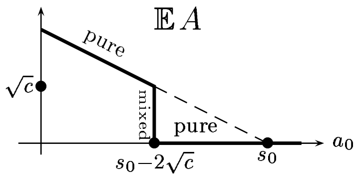


It may happen to be a pure strategy, but it may also be a mixed strategy.

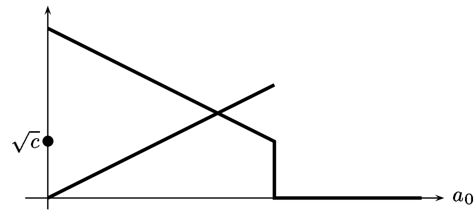
The rest of the subsection is devoted to the following example. Let P_S be concentrated at a single atom s_0 (which means a game of *complete* information). Then

$$\mathbb{E}A = \mathbb{E}\varphi_{a_0}(S) = \varphi_{a_0}(s_0) = \begin{cases} 0 & \text{if } s_0 < 2\sqrt{c} + a_0, \\ \frac{1}{2}(s_0 - a_0) & \text{if } s_0 > 2\sqrt{c} + a_0 \end{cases}$$

as far as $s_0 \neq 2\sqrt{c} + a_0$.



Case $s_0 > 3\sqrt{c}$. The equilibrium is supported by the pure strategy $A = \frac{1}{3}s_0$.

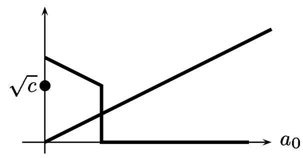


It may be seen as follows:

$$\begin{cases} a_0 = \dfrac{1}{2}(s_0 - a_0) \\ a_0 < s_0 - 2\sqrt{c} \end{cases} \qquad \begin{matrix} a_0 = \dfrac{1}{3}s_0 \\ \dfrac{1}{3}s_0 < s_0 - 2\sqrt{c} \end{matrix} \qquad 2\sqrt{c} < \frac{2}{3}s_0 \,.$$

Case $s_0 = 3\sqrt{c}$. The equilibrium is supported by the pure strategy $A = \sqrt{c}$.

Case $s_0 \in (2\sqrt{c}, 3\sqrt{c})$. The equilibrium is supported by the mixed strategy



$$\mathbb{P}\big(A = 0\big) = 3 - \frac{s_0}{\sqrt{c}}\,, \qquad \mathbb{P}\big(A = \sqrt{c}\big) = \frac{s_0}{\sqrt{c}} - 2\,.$$

It may be seen as follows:

$$\begin{cases} a_0 = s_0 - 2\sqrt{c} \\ a_0 = p \cdot \dfrac{1}{2}(s_0 - a_0) \\ 0 < p < 1 \end{cases} \qquad s_0 - 2\sqrt{c} = p\sqrt{c}\,; \qquad p = \frac{s_0}{\sqrt{c}} - 2\,.$$

Case $s_0 \in [0, 2\sqrt{c}]$. The equilibrium is supported by the trivial (pure) strategy $A = 0$ (always quitting).

Especially, let $s_0 = 2.5\sqrt{c}$, then $\mathbb{P}\big(A = \sqrt{c}\big) = 0.5 = \mathbb{P}\big(A = 0\big)$. With probability 0.25 both firms quit. With probability 0.25 they produce $\sqrt{c} + \sqrt{c}$ and get (each one) the profit $\sqrt{c} \cdot s_0 - c - \sqrt{c}(\sqrt{c} + \sqrt{c}) = 2.5c - c - 2c = -0.5c$. With probability 0.25 the second firm quits while the first firm produces $\sqrt{c}$ and gets the profit $\sqrt{c} \cdot s_0 - c - \sqrt{c}(\sqrt{c} + 0) = 2.5c - c - c = +0.5c$. The mean profit of each firm is 0, which must happen, since one of the two equally profitable actions is quitting. However, the 'always quitting' strategy does not support an equilibrium, since in that case producing $\sqrt{c}$ gives $0.5c > 0$.