# 3 Symmetric equilibria and revenue equivalence

## 3a Our framework

The game under consideration (throughout Section 3) is a symmetric single unit auction[1] with the standard allocation rule but an arbitrary payment rule. It means that the profit $\Pi$ is a gain $G$ minus a loss $L$ (for each player),

$$\mathbf{\Pi}(a_1, s_1; a_2) = \mathbf{G}(a_1, s_1; a_2) - \mathbf{L}(a_1; a_2)\,, \tag{3a1}$$

the gain being

$$\mathbf{G}(a_1, s_1; a_2) = \begin{cases} 0 & \text{if } a_1 < a_2, \\ \frac{1}{2}s_1 & \text{if } a_1 = a_2, \\ s_1 & \text{if } a_1 > a_2, \end{cases} \tag{3a2}$$

while the loss $\mathbf{L}(a_1; a_2)$ is an arbitrary function of the actions $a_1, a_2$. Note that the loss does not depend on signals, and the gain is proportional to $s_1$; therefore the profit is linear in $s_1$. Symmetry of the game means that $\mathbf{\Pi}_1 = \mathbf{\Pi}_2 = \mathbf{\Pi}$. Action spaces and signal spaces still are

$$\mathcal{S}_1 = \mathcal{S}_2 = \mathbb{R}\,; \qquad \mathcal{A}_1 = \mathcal{A}_2 = [0, \infty)\,.$$

Signals are still independent, but their distribution $P_S$ $(= P_{S_1} = P_{S_2})$ is now arbitrary, except for assuming $P_S\big([0,\infty)\big) = 1$, that is,

$$\mathbb{P}\big(\, S \ge 0 \,\big) = 1\,.$$

We have $\mathbf{\Pi}(a_1, 1; a_2) - \mathbf{\Pi}(a_1, 0; a_2) = \mathbf{G}(a_1, 1; a_2)$. The value $\mathbf{\Pi}(a_1, 1; P_{A_2}) - \mathbf{\Pi}(a_1, 0; P_{A_2}) = \mathbf{G}(a_1, 1; P_{A_2})$ is nothing but the winning probability.

We search for a symmetric equilibrium $(\mu, \mu)$, in other words, a strategy $\mu$ that is a best response to itself.

Of course, the 'very simple auction' game (1b4) is a special case, namely,

$$\mathbf{L}(a_1, a_2) = \begin{cases} 0 & \text{if } a_1 < a_2, \\ \frac{1}{2}a_1 & \text{if } a_1 = a_2, \\ a_1 & \text{if } a_1 > a_2, \end{cases} \qquad P_S = \mathrm{U}(0,1)\,.$$

## 3b Monotonicity

Every action $a$ has its winning probability

$$\mathbb{P}\big(\, A < a \,\big) + \frac{1}{2}\mathbb{P}\big(\, A = a \,\big) = \frac{1}{2}F_A(a-) + \frac{1}{2}F_A(a+)\,;$$

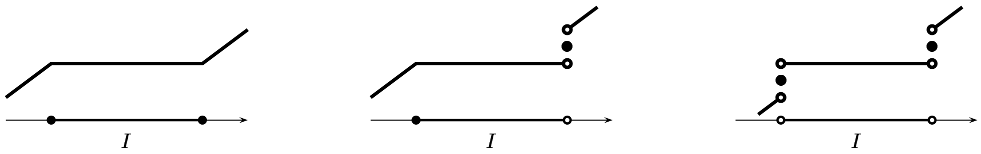
here $F_A$ is the cumulative distribution function of the marginal distribution $P_A$ (the projection of $\mu$ to the action space). The joint distribution $\mu$ of random variables $S$ and $A$

[1] Independent private values, sealed bid, symmetric single unit auction with two bidders.

determines the joint distribution of $S$ and the winning probability; the latter joint distribution is weakly increasing by Lemma 2c3. What about the former? The following general probabilistic lemma will help. It shows that an action is uniquely determined by its winning probability. Note that the argument works only for the symmetric case, $P_{A_1} = P_{A_2}$. Roughly speaking, a random variable never falls into a gap of its own distribution.

**3b1. Lemma.** For every random variable $X$ there exists an increasing function $\varphi$ such that $X = \varphi\big(W(X)\big)$ almost surely; here $W(x) = \frac{1}{2}F_X(x-) + \frac{1}{2}F_X(x+)$ for all $x$.

*Sketch of the proof.* The increasing function $W$ can be constant on some intervals. For every such interval $I$ we have $\mathbb{P}\,\big(\,X \in I\,\big) = 0$.

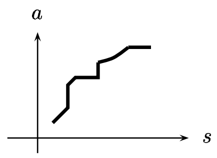



Denoting by $E$ the union of all such intervals, we have $\mathbb{P}\,\big(\,X \in E\,\big) = 0$.[2] The restriction $W|_{\mathbb{R}\setminus E}$ of $W$ to the complement of $E$ is strictly increasing. It remains to define $\varphi$ as the inverse function to $W|_{\mathbb{R}\setminus E}$. □

**3b2. Theorem.** Every symmetric equilibrium is supported by a weakly increasing strategy.

*Sketch of the proof.* We combine Lemma 2c3 and Lemma 3b1. □

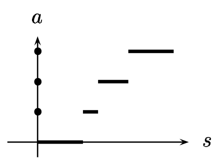
So, such a strategy is concentrated on a set with no incomparable points.




Some signals can correspond to many actions; however, these exceptional signals are at most a countable set, and may be neglected, unless at least one of them is an atom.

**3b3. Corollary.** If the distribution of signals is nonatomic, then every symmetric equilibrium is supported by an increasing pure strategy.

Note that the results hold for an arbitrary payment rule (in contrast to Corollary 2c5). In particular, the payment rule can stipulate a large fine (penalty) for submitting a non-integer as a bid,[3] which effectively turns the continuous action space $[0, \infty)$ into the discrete action space $\{0, 1, 2, \dots\}$.




Similarly, entry cost and reserve price are within the reach of Theorem 3b2 (and Corollary 3b3).

[2] Consider *maximal* constancy intervals. Their interiors are nonempty and pairwise disjoint. Therefore there exist at most countably many such intervals.

[3] You see, we did not assume monotonicity of the loss $\mathbf{L}_1(a_1, a_2)$ in $a_1$.

## 3c Revenue equivalence

Let a strategy $\mu$ support a symmetric equilibrium and $P_A$ be the corresponding action distribution. Being a best response to $P_A$, the strategy $\mu$ is concentrated on pairs $(s, a)$ such that $a$ is an optimal action for $s$, that is,

$$\mathbf{\Pi}(a, s; P_A) = \mathbf{\Pi}^{\max}(s; P_A) \, .$$

Existence of the best response ensures existence of optimal actions for $P_S$-almost all $s$, not for all $s$. Especially, outside the support of $P_S$, optimal actions need not exist. Nevertheless, the winning probability may be treated as an increasing function $\mathbf{p}^{\text{win}}$ defined for all $s$ except for at most countable set of jumps (recall 2d), and

$$\mathbf{\Pi}^{\max}(s''; P_A) - \mathbf{\Pi}^{\max}(s'; P_A) = \int_{s'}^{s''} \mathbf{p}^{\text{win}}(s) \, ds \tag{3c1}$$

for all $s', s''$.

Assume for a moment that the optimal action is a *strictly increasing function* $\varphi$ of a signal; that is, for every $s$ there exists one and only one optimal action $a = \varphi(s)$. Then

$$\mathbf{p}^{\text{win}}(s) = \mathbb{P}\big( A < \varphi(s) \big) + \frac{1}{2}\mathbb{P}\big( A = \varphi(s) \big) = \mathbb{P}\big( \varphi(S) < \varphi(s) \big) + \frac{1}{2}\mathbb{P}\big( \varphi(S) = \varphi(s) \big) =$$
$$= \mathbb{P}\big( S < s \big) + \frac{1}{2}\mathbb{P}\big( S = s \big) = \frac{1}{2}F_S(s-) + \frac{1}{2}F_S(s+) \, ;$$

the result is uniquely determined by the signal distribution only! It does not depend on

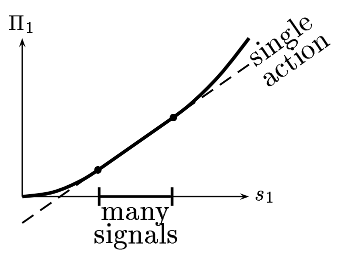
- the strategy $\mu$ (supporting a symmetric equilibrium),
- the payment rule $\mathbf{L}$.

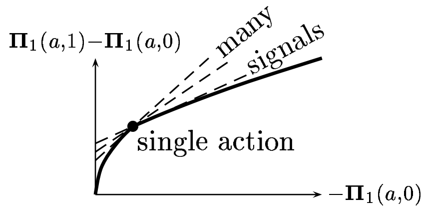
Knowing $\mathbf{p}^{\text{win}}(\cdot)$ we can find $\mathbf{\Pi}^{\max}(\cdot; P_A)$ by (3c1); then sometimes (but not always) we can find $\varphi(\cdot)$ such that

$$\mathbf{\Pi}(\varphi(s), s; P_A) = \mathbf{\Pi}^{\max}(s; P_A) \, .$$

and check that it is an equilibrium, thus proving its existence, but not uniqueness. Striving also to uniqueness, we abandon the assumption about a strictly increasing function $\varphi(\cdot)$ and return to the general case.

Recall that a single action can be optimal for many signals,




which is known as bunching. Here is the definition.

A *bunch* is a maximal interval of linearity of the function $s \mapsto \mathbf{\Pi}^{\max}(s; P_A)$.

Note that $P_S$ is not mentioned in the definition.

**3c2. Exercise.** Let $s', s''$ be two interior points[4] of a bunch. Then:

(a) All optimal actions (if any) for $s'$ are equivalent to each other.

(b) If $a$ is an optimal action for $s'$ then $a$ is also an optimal action for $s''$.

(c) If actions $a_1, a_2, \dots$ are such that $\mathbf{\Pi}(a_n, s') \xrightarrow[n\to\infty]{} \mathbf{\Pi}^{\max}(s')$ then also $\mathbf{\Pi}(a_n, s'') \xrightarrow[n\to\infty]{} \mathbf{\Pi}^{\max}(s'')$.[5]

(d) If the bunch is of positive probability (w.r.t. $P_S$) then an optimal action exists for $s'$.

Prove it.

Does it hold for endpoints of a bunch?

The set of all bunches is finite (maybe empty) or countable.[6]

It follows from (3c1) (and monotonicity of $\mathbf{p}^{\text{win}}$) that up to endpoints, bunches are maximal intervals of constancy of the function $\mathbf{p}^{\text{win}}$.

If $s_0$ does not belong to any bunch, then $\mathbf{p}^{\text{win}}(s') < \mathbf{p}^{\text{win}}(s_0) < \mathbf{p}^{\text{win}}(s'')$ whenever $s' < s_0 < s''$ (indeed, if $\mathbf{p}^{\text{win}}(s') = \mathbf{p}^{\text{win}}(s_0)$ then $[s', s_0]$ is contained in a bunch). Assume in addition that there exists an optimal action $a_0$ for $s_0$. Note that $\mathbf{p}^{\text{win}}(s_0) = W(a_0)$; here $W(a) = \frac{1}{2}F_A(a-) + \frac{1}{2}F_A(a+)$ is the winning probability function on the action space. If $a'$ is an optimal action for $s'$ and $s' < s_0$, then $W(a') = \mathbf{p}^{\text{win}}(s') < \mathbf{p}^{\text{win}}(s_0) = W(a_0)$, therefore $a' < a_0$ (indeed, monotonicity of $W$ means that $a' \ge a_0 \implies W(a') \ge W(a_0)$). Thus, $\mathbb{P}\big( S < s_0 \big) \le \mathbb{P}\big( A < a_0 \big)$, that is, $F_S(s_0-) \le F_A(a_0-)$. Similarly, $\mathbb{P}\big( S > s_0 \big) \le \mathbb{P}\big( A > a_0 \big)$, that is, $1 - F_S(s_0+) \le 1 - F_A(a_0+)$, and $F_S(s_0+) \ge F_A(a_0+)$. So, $[F_S(s_0-), F_S(s_0+)] \supset [F_A(a_0-), F_A(a_0+)]$. Assuming in addition that $s_0$ is not an atom of $P_S$ we get $F_A(a_0-) = F_A(a_0+) = F_S(s_0-) = F_S(s_0+)$. Taking into account that $\mathbf{p}^{\text{win}}(s_0) = W(a_0) = \frac{1}{2}F_A(a_0-) + \frac{1}{2}F_A(a_0+)$ we get

$$\mathbf{p}^{\text{win}}(s_0) = F_S(s_0)$$

whenever $s_0$ satisfies three conditions:

- $s_0$ does not belong to any bunch;
- there exists an optimal action for $s_0$;
- $s_0$ is not an atom of $P_S$.

The third condition is harmless; atoms of $P_S$, being at most a countable set, do not contribute to the integral of winning probability (3c1). The second condition is harmful, and will be eliminated in the next lemma. The first condition will be weakened. And another harmless condition will be added, namely, continuity of $\mathbf{p}^{\text{win}}(\cdot)$ at $s_0$ (it excludes at most a countable set). In fact, the latter condition can be eliminated, see 3c5.

**3c3. Lemma.** Assume that

(a) $s_0$ does not belong to a bunch of positive probability (w.r.t. $P_S$);[7]

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[4] Not endpoints.

[5] This item is especially useful when no optimal action exists.

[6] Since their interiors are disjoint nonempty open intervals.

[7] In other words: either $s_0$ does not belong to any bunch, or $s_0$ belongs to a bunch $[s', s'']$ such that $F_S(s'-) = F_S(s''+)$. Note that a bunch of positive probability is forbidden, even if all the positive probability is concentrated in an atom at an endpoint of the bunch, and even if $s_0$ is also an endpoint of the bunch.

(b) $s_0$ is not an atom of $P_S$;
(c) $\mathbf{p}^{\text{win}}(s_0-) = \mathbf{p}^{\text{win}}(s_0+)$.
Then

$$\mathbf{p}^{\text{win}}(s_0) = F_S(s_0) .$$

*Sketch of the proof.* Take a sequence $(a_n)$ such that (recall 2d and use (c))

$$\mathbf{\Pi}(a_n, s_0) \xrightarrow[n\to\infty]{} \mathbf{\Pi}^{\max}(s_0) ,$$
$$W(a_n) \xrightarrow[n\to\infty]{} \mathbf{p}^{\text{win}}(s_0) ;$$

here $W(a) = \frac{1}{2}F_A(a-) + \frac{1}{2}F_A(a+)$.

We may assume that the sequence $(a_n)$ is monotone (otherwise we take a monotone subsequence).[8] Assume increase,

$$a_1 \le a_2 \le \dots ;$$

the other (decreasing) case, being similar, is left to the reader.

On one hand, $P_S$-almost all $s$ such that $s < s_0$ satisfy $\mathbf{p}^{\text{win}}(s) < \mathbf{p}^{\text{win}}(s_0)$ due to (a). Thus, $\mu$-almost all pairs $(s, a)$ such that $s < s_0$ satisfy

$$W(a) = \mathbf{p}^{\text{win}}(s) < \mathbf{p}^{\text{win}}(s_0) = \lim_{n\to\infty} W(a_n) ,$$

which implies $W(a) < W(a_n)$ and $a < a_n$ for $n$ large enough.[9] Therefore $\mathbb{P}\big( S < s_0 \big) \le \lim_{n\to\infty} \mathbb{P}\big( A < a_n \big)$. However, $\mathbb{P}\big( A < a_n \big) \le W(a_n) \to \mathbf{p}^{\text{win}}(s_0)$. So,

$$F_S(s_0-) \le \mathbf{p}^{\text{win}}(s_0) .$$

On the other hand, $P_S$-almost all $s$ such that $s > s_0$ satisfy $\mathbf{p}^{\text{win}}(s) > \mathbf{p}^{\text{win}}(s_0)$ due to (a). Thus, $\mu$-almost all pairs $(s, a)$ such that $s > s_0$ satisfy

$$W(a) = \mathbf{p}^{\text{win}}(s) > \mathbf{p}^{\text{win}}(s_0) = \lim_{n\to\infty} W(a_n) ,$$

which implies $W(a) > W(a_n)$ and $a > a_n$ for all $n$. Therefore $\mathbb{P}\big( S > s_0 \big) \le \mathbb{P}\big( A > a_n \big)$ for all $n$. However, $\mathbb{P}\big( A > a_n \big) \le 1 - W(a_n) \to 1 - \mathbf{p}^{\text{win}}(s_0)$. So, $1 - F_S(s_0+) \le 1 - \mathbf{p}^{\text{win}}(s_0)$, that is,

$$F_S(s_0+) \ge \mathbf{p}^{\text{win}}(s_0) .$$

It remains to note that $F_S(s_0-) = F_S(s_0+)$ due to (b). □

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[8] Every sequence of real numbers has a monotone subsequence. Of course, an infinite sequence (and infinite subsequence) is meant. 'Monotone' does not mean 'strictly monotone'. The case $a_1 = a_2 = \dots$ allows treating an optimal action (when exists) in the same framework.

[9] Of course, the needed $n$ may depend on $(a, s)$.

**3c4. Theorem.** Let $\mu$ be a strategy supporting a symmetric equilibrium, with no bunch of positive probability (w.r.t. $P_S$). Then

$$\mathbf{\Pi}^{\max}(s;P_A) - \mathbf{\Pi}^{\max}(0;P_A) = \int_0^s F_S(s')\,ds'$$

for all $s \in [0,\infty)$.

*Proof.* Follows from (3c1) and Lemma 3c3. □

**3c5. Exercise.** Let $\mu$ be a strategy supporting a symmetric equilibrium, with no bunch of positive probability. Then functions $\mathbf{p}^{\text{win}}(\cdot)$ and $F_S(\cdot)$ have the same continuity points, and are equal at these points.

Prove it.

Hint: by 3c3, the two increasing functions coincide on a dense set, therefore $\mathbf{p}^{\text{win}}(s-) = F_S(s-)$ and $\mathbf{p}^{\text{win}}(s+) = F_S(s+)$ for all $s$.

**3c6. Corollary.** Let $\mu$ be a strategy supporting a symmetric equilibrium, with no bunch of positive probability. If $a$ is an optimal action for $s$, and $s$ is not an atom of $P_S$, then

$$\mathbf{L}(a;P_A) = sF_S(s) - \int_0^s F_S(s')\,ds' - \mathbf{\Pi}^{\max}(0;P_A)\,.$$

*Proof.* By Exercise 3c5, $\mathbf{p}^{\text{win}}(s-) = \mathbf{p}^{\text{win}}(s+)$. By Lemma 3c3, $\mathbf{p}^{\text{win}}(s) = F_S(s)$. Also, $\mathbf{p}^{\text{win}}(s) = W(a) = \frac12 F_A(a-) + \frac12 F_A(a+)$. We have $\mathbf{G}(a,s;P_A) = sW(a) = sF_S(s)$ and $\mathbf{L}(a;P_A) = \mathbf{G}(a,s;P_A) - \mathbf{\Pi}(a,s;P_A) = sF_S(s) - \mathbf{\Pi}^{\max}(s;P_A)$. It remains to use Theorem 3c4. □

**3c7. Exercise.** Assume that $\mathbf{L}(a_1;a_2) \ge 0$ for all $a_1,a_2 \in [0,\infty)$, and $\mathbf{L}(0;a_2) = 0$ for all $a_2 \in [0,\infty)$. Then $\mathbf{\Pi}^{\max}(0;P_A) = 0$ for every distribution $P_A$.

Prove it.

Hint: $\mathbf{\Pi}^{\max}(0;a_2) = 0$ for every $a_2 \in [0,\infty)$, since $\mathbf{G}(a_1,0;a_2) = 0$ and $\inf_{a_1} \mathbf{L}(a_1,a_2) = 0$.

Note that

$$\begin{aligned} sF_S(s) - \int_0^s F_S(s')\,ds' &= \int_0^s \big(F_S(s) - F_S(s')\big)\,ds' = \int_0^s \mathbb{P}\,\big(\, S \in (s',s] \,\big)\,ds' = \\ &= \int_0^s \mathbb{E}\,\mathbf{1}_{(s',s]}(S)\,ds' = \mathbb{E}\,\bigg( \int_0^s \mathbf{1}_{(s',s]}\,ds' \bigg)(S) = \mathbb{E}\,\big(S\mathbf{1}_{[0,s]}(S)\big) = \\ &= \mathbb{E}\,\big(\, S \,\big|\, 0 \le S \le s \,\big) \cdot \mathbb{P}\,\big(\, 0 \le S \le s \,\big)\,; \end{aligned}$$

though, the conditional expectation is well-defined only when $\mathbb{P}\,\big(\, 0 \le S \le s \,\big) > 0$; otherwise our expression vanishes anyway. Under the conditions of 3c6 and 3c7, and taking into account that $\mathbb{P}\,\big(\, S \ge 0 \,\big) = 1$, we get

$$\mathbf{L}(a;P_A) = \mathbb{E}\,\big(\, S \,\big|\, S \le s \,\big) \cdot \mathbb{P}\,\big(\, S \le s \,\big)\,. \tag{3c8}$$

Recall that $\mathbf{L}(\dots)$ is the mean loss, averaged over all situations (not only wins). If the player pays only when he wins, then the mean payment conditioned on winning is

$$\mathbb{E}\left( L_1 \,\middle|\, S_1 = s, \text{win} \right) = \frac{\mathbf{L}(a;P_A)}{\mathbb{P}\left( S \le s \right)} = \mathbb{E}\left( S \,\middle|\, S \le s \right) . \tag{3c9}$$

The mean payment is equal to the mean signal of the losing competitor! That holds for an arbitrary payment rule, as far as bunching does not appear.

All that is about necessary conditions for supporting a symmetric equilibrium. The next result gives a sufficient condition.

**3c10. Theorem.** Let $P_S$ be nonatomic, and $\mu$ be a pure strategy $A = \varphi(S)$ where $\varphi$ is an increasing function such that $P_A$ is nonatomic. Assume that $\mathbf{L}(a;P_A)$ is an increasing function of $a \in [0,\infty)$, and $\mathbf{L}(0;P_A) = 0$. Then the following two conditions are equivalent, and if they are satisfied then $\mu$ supports a symmetric equilibrium:

(a) $\mathbf{\Pi}(\varphi(s), s; P_A) = \int_0^s F_S(s')\,ds'$ for $P_S$-almost all $s$;

(b) $\mathbf{L}(\varphi(s); P_A) = sF_S(s) - \int_0^s F_S(s')\,ds'$ for $P_S$-almost all $s$.

*Proof.* For every $s$, $\mathbb{P}\left( A < \varphi(s) \right) \le \mathbb{P}\left( S < s \right) \le \mathbb{P}\left( S \le s \right) \le \mathbb{P}\left( A \le \varphi(s) \right) = \mathbb{P}\left( A < \varphi(s) \right)$, therefore $F_A\big(\varphi(s)\big) = F_S(s)$. We have $\mathbf{G}(\varphi(s), s; P_A) = sF_A\big(\varphi(s)\big) = sF_S(s)$, therefore $\mathbf{\Pi}(\varphi(s), s; P_A) = sF_S(s) - \mathbf{L}(\varphi(s); P_A)$, which shows that (a), (b) are equivalent. Assume that they are satisfied for all $s \in E$, $P_S(E) = 1$.

Let $s_0 \in E$ and $a_0 = \varphi(s_0)$. The linear function $s \mapsto \mathbf{\Pi}(a_0, s; P_A)$ and the convex function $s \mapsto \int_0^s F_S(s')\,ds'$ are equal at $s_0$. Their derivatives at $s_0$ are also equal, since the winning probability $F_A(a_0)$ is equal to $F_S(s_0)$. Therefore $\mathbf{\Pi}(a_0, s; P_A) \le \int_0^s F_S(s')\,ds'$ for all $s$. It means that

$$\mathbf{\Pi}(a, s; P_A) \le \mathbf{\Pi}(\varphi(s), s; P_A) \quad \text{for all } s \in E, a \in \varphi(E) .$$

In order to prove that $\mu$ supports a symmetric equilibrium, it suffices to prove that $\mathbf{\Pi}(a, s; P_A) \le \mathbf{\Pi}(\varphi(s), s; P_A)$ for all $s \in E$ and $a \in [0,\infty)$. Assume the contrary: $\mathbf{\Pi}(a, s; P_A) > \mathbf{\Pi}(\varphi(s), s; P_A)$ for some $s \in E$ and $a \in [0,\infty)$. Note that $\mathbf{\Pi}(\varphi(s), s; P_A) \ge 0$, therefore $\mathbf{\Pi}(a, s; P_A) > 0$, that is, $sF_A(a) - \mathbf{L}(a;P_A) > 0$; it follows that $F_A(a) > 0$ (since $\mathbf{L}(a;P_A) \ge \mathbf{L}(0;P_A) = 0$), in other words, $\mathbb{P}\left( \varphi(S) \le a \right) > 0$. We choose an increasing sequence $s_1 \le s_2 \le \dots$, $s_n \in E$, such that $\varphi(s_n) \le a$ and $F_S(s_n) \to F_A(a)$.[10] Denote $a_n = \varphi(s_n)$, then $a_n \le a$, $F_A(a_n) \to F_A(a)$, and $a_n \in \varphi(E)$. The latter implies $\mathbf{\Pi}(a_n, s; P_A) \le \mathbf{\Pi}(\varphi(s), s; P_A)$. However, $\mathbf{\Pi}(a_n, s; P_A) = sF_A(a_n) - \mathbf{L}(a_n; P_A) \ge sF_A(a_n) - \mathbf{L}(a;P_A)$ due to monotonicity of $\mathbf{L}(\cdot, P_A)$. Thus, $sF_A(a_n) - \mathbf{L}(a;P_A) \le \mathbf{\Pi}(\varphi(s), s; P_A)$ for all $n$; the limit $n \to \infty$ gives $sF_A(a) - \mathbf{L}(a;P_A) \le \mathbf{\Pi}(\varphi(s), s; P_A)$, that is, $\mathbf{\Pi}(a, s; P_A) \le \mathbf{\Pi}(\varphi(s), s; P_A)$. A contradiction. □

The monotonicity of $\mathbf{L}(\cdot; P_A)$ is essential. Consider for example a discount for integer $a$, or for large $a$ beyond the support of $A$.

[10] We do not claim that $\varphi(s_n) \to a$. You see, $a$ need not belong to the support of $A$.

## 3d First price auction

The simplest payment rule says, 'winner, pay your bid', which means the (expected) loss

$$\mathbf{L}(a_1;a_2) = \begin{cases} 0 & \text{if } a_1 < a_2, \\ \frac{1}{2}a_1 & \text{if } a_1 = a_2, \\ a_1 & \text{if } a_1 > a_2. \end{cases} \tag{3d1}$$

Assume that $P_S$ is nonatomic. Let $\mu$ be a strategy supporting a symmetric equilibrium, and denote by $P_A$ the corresponding action distribution. Corollary 3b3 (or 2c6) ensures that $\mu$ is an increasing pure strategy; it is concentrated on the graph of an increasing function, $A = \varphi(S)$. Exercise 3c7 shows that

$$\mathbf{\Pi}^{\max}(0;P_A) = 0\,.$$

Lemma 2b7 shows that $P_A$ cannot have atoms except, maybe, a single atom at the low end of the support. However, the only possible atom is excluded by 2b9. So, $P_A$ is nonatomic. Any bunch of nonzero probability would give an atom to $P_A$ due to 3c2. Therefore there is no such bunch, and Theorem 3c4 is applicable:

$$\mathbf{\Pi}^{\max}(s;P_A) = \int_0^s F_S(s')\,ds'$$

for all $s \in [0,\infty)$. It follows that never-winning actions are not in use (think, why; and formulate it more accurately). Theorem 3c10 is also applicable, and we get the following result.

**3d2. Lemma.** Let $P_S$ be nonatomic. Then a strategy $\mu$ supports a symmetric equilibrium if and only if it is a pure strategy $A = \varphi(S)$ where $\varphi$ is an increasing function such that $P_A$ is nonatomic and equivalent conditions 3c10(a,b) are satisfied.

Condition 3c10(b) may be rewritten using (3c8) as

$$\mathbf{L}(\varphi(s);P_A) = \mathbb{E}\left(\,S\,\middle|\,S \le s\,\right) \cdot \mathbb{P}\left(\,S \le s\,\right).$$

On the other hand, the 'first price' payment rule (3d1) gives

$$\mathbf{L}(\varphi(s);P_A) = \varphi(s)\mathbf{p}^{\text{win}}(s)$$

for $P_S$-almost all $s$. However, $\mathbf{p}^{\text{win}}(s) = \mathbb{P}\left(\,S \le s\,\right)$ for $P_S$-almost all $s$.[11] So, Condition 3c10(b) becomes

$$\varphi(s) = \mathbb{E}\left(\,S\,\middle|\,S \le s\,\right)$$

for $P_S$-almost all $s$. (Using monotonicity one can show that the equality holds for *every* $s$ such that $0 < F_S(s) < 1$, but we do not need it.) And so, Lemma 3d2 gives the next result.

[11] It follows easily from monotonicity of $\varphi$ and nonatomicity of $P_A$. Complicated arguments in the proof of 3c3 give much more: the same for all $s$ except for a countable set.

**3d3. Theorem.** If $P_S$ is nonatomic then the first price auction has one and only one symmetric equilibrium, namely,

$$A = \varphi(S)\,, \quad \text{where } \varphi(s) = \mathbb{E}\left( S \,\middle|\, S \le s \right)\,.$$

**3d4. Exercise.** Describe the support of $P_A$ for an arbitrary nonatomic $P_S$.
Hint. Prove and use continuity of $\varphi$. Or alternatively, use 2b10.

## 3e A note on nonatomicity and participation

Our allocation rule (3a2) is somewhat unnatural; namely, if $a_1 = a_2 = 0$, the auctioneer should keep the object; that is,

$$\text{(3e1)} \qquad \mathbf{G}(a_1, s_1; a_2) = \begin{cases} 0 & \text{if } a_1 < a_2, \\ 0 & \text{if } a_1 = a_2 = 0, \\ \frac12 s_1 & \text{if } a_1 = a_2 > 0, \\ s_1 & \text{if } a_1 > a_2. \end{cases}$$

From now on we use (3e1) instead of (3a2). It does not invalidate results of 3a–3d (only the function $W$ in 3b1 should be corrected at 0).

The action $a = 0$ is now interpreted as non-participation (quitting). An atom of $P_A$ at 0 can appear naturally (out of nonatomic $P_S$), if entry cost or/and reserve price is stipulated by the loss function $\mathbf{L}$. Any atom (including 0) is forbidden by 2b7, 2b9, but only for a single loss function (3d1). The argument still works after replacing (3a2) with (3e1) (think, why). However, we need a more general argument. Similarly to 2b7–2b9, it deals with optimal actions (not equilibria).

**3e2. Lemma.** Assume that the loss function $\mathbf{L}$ satisfies two conditions:

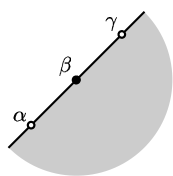
$$\text{(a)} \qquad \frac12 \mathbf{L}(a_1-; a_2) + \frac12 \mathbf{L}(a_1+; a_2) = \mathbf{L}(a_1; a_2) \qquad \text{for all } a_1 \in (0,\infty),\ a_2 \in [0,\infty)\,;$$

here $\mathbf{L}(a_1-; a_2) = \lim_{a\to a_1, a<a_1} \mathbf{L}(a; a_2)$ and $\mathbf{L}(a_1+; a_2) = \lim_{a\to a_1, a>a_1} \mathbf{L}(a; a_2)$ (existence of these limits is also required);

$$\text{(b)} \qquad \sup_{a_1 \in [0,M]} \sup_{a_2 \in [0,\infty)} \mathbf{L}(a_1; a_2) < \infty \quad \text{for all } M.$$

If $a_1$ is an optimal action (against $P_{A_2}$) for more than a single $s_1$, and $a_1 \ne 0$, then $a_1$ is not an atom of $P_{A_2}$.

*Sketch of the proof.* Assume the contrary: $a$ is an atom. Similarly to the proof of 2b7, on the plane (expected loss, winning probability) we have



where the points have the coordinates

$$\alpha = \big(\mathbf{L}(a_1-;P_{A_2}),\ F_{A_2}(a_1-)\big)\,,$$
$$\beta = \big(\mathbf{L}(a_1;P_{A_2}),\ \frac12 F_{A_2}(a_1-) + \frac12 F_{A_2}(a_1+)\big)\,,$$
$$\gamma = \big(\mathbf{L}(a_1+;P_{A_2}),\ F_{A_2}(a_1+)\big)\,.$$

However,

$$\mathbf{L}(a_1-;P_{A_2}) = \lim_{a\to a_1, a<a_1} \int \mathbf{L}(a;a_2)\,dP_{A_2}(a_2) =$$
$$= \int \Big( \lim_{a\to a_1, a<a_1} \mathbf{L}(a;a_2) \Big)\,dP_{A_2}(a_2) = \int \mathbf{L}(a_1-;a_2)\,dP_{A_2}(a_2)$$

by the bounded convergence theorem. The same for $a_1+$. It follows that $\frac12\mathbf{L}(a_1-;P_{A_2}) + \frac12\mathbf{L}(a_1+;P_{A_2}) = \mathbf{L}(a_1;P_{A_2})$. So, $\beta$ is the center of the interval $[\alpha,\gamma]$. The straight line containing the points determines a single signal; $a$ cannot be optimal for any other signal. $\square$

**3e3. Exercise.** Explain, why the proof does not work for $a=0$.

We return from optimal actions to a symmetric equilibrium supported by a strategy $\mu$.

**3e4. Corollary.** Assume that $P_S$ is nonatomic, and conditions 3e2(a,b) are satisfied by the loss function $\mathbf{L}$. Then $P_A$ has no atoms on $(0,\infty)$.

## 3f Second price auction

The second-price payment rule (for two players) says, 'winner, pay the bid of the loser', which means the (expected) loss

(3f1) $$\mathbf{L}(a_1;a_2) = \begin{cases} 0 & \text{if } a_1 < a_2, \\ \frac12 a_2 & \text{if } a_1 = a_2, \\ a_2 & \text{if } a_1 > a_2. \end{cases}$$

Assume that $P_S$ is nonatomic. Let $\mu$ be a strategy supporting a symmetric equilibrium, and denote by $P_A$ the corresponding action distribution. Corollary 3b3 (but not 2c6!) ensures that $\mu$ is an increasing pure strategy; it is concentrated on the graph of an increasing function, $A=\varphi(S)$. Exercise 3c7 shows that $\mathbf{\Pi}^{\max}(0;P_A) = 0$. The loss function (3f1) satisfies Conditions 3e2(a,b) (check it). Thus, Corollary 3e4 shows that $P_A$ cannot have atoms on $(0,\infty)$. In order to exclude an atom at 0 we need an additional argument.

**3f2. Exercise.** In addition to 3e2(a,b) assume that $\mathbf{L}$ satisfies the condition

$$\mathbf{L}(0+,a_2) = 0 \quad \text{for all } a_2 \in [0,\infty)\,.$$

Then nonatomicity of $P_S$ implies nonatomicity of $P_A$.
Prove it.

So, $P_A$ is nonatomic. Any bunch of nonzero probability would give an atom to $P_A$ (due to 3c2). Therefore there is no such bunch, and Theorem 3c4 is applicable:

$$\mathbf{\Pi}^{\max}(s; P_A) = \int_0^s F_S(s')\,ds'$$

for all $s \in [0, \infty)$. Theorem 3c10 is also applicable, and we get the following result.

**3f3. Lemma.** Let $P_S$ be nonatomic. Then a strategy $\mu$ supports a symmetric equilibrium if and only if it is a pure strategy $A = \varphi(S)$ where $\varphi$ is an increasing function such that $P_A$ is nonatomic and equivalent conditions 3c10(a,b) are satisfied.

Condition 3c10(b) may be rewritten using (3c8) as

$$\mathbf{L}(\varphi(s); P_A) = \mathbb{E}\big(\, S \,\big|\, S \le s \,\big) \cdot \mathbb{P}\big(\, S \le s \,\big)\,.$$

Till now, everything is quite similar to 3d. However, the 'second price' payment rule (3f1) gives $\mathbf{L}(\varphi(s); P_A) = \mathbb{E}\big(\, A \,\big|\, A \le \varphi(s) \,\big) \cdot \mathbb{P}\big(\, A \le \varphi(s) \,\big)$, that is,

$$\mathbf{L}(\varphi(s); P_A) = \mathbb{E}\big(\, \varphi(S) \,\big|\, S \le s \,\big) \cdot \mathbb{P}\big(\, S \le s \,\big)$$

for $P_S$-almost all $s$. Thus, 3c10(b) becomes

$$\mathbb{E}\big(\, \varphi(S) \,\big|\, S \le s \,\big) = \mathbb{E}\big(\, S \,\big|\, S \le s \,\big)\,,$$

which is very easy to satisfy: $\varphi(s) = s$, that is, $A = S$. The simplest strategy indeed; just bid your signal! No other $\varphi$ can satisfy the condition, since the equality

$$\int_0^s \big(s' - \varphi(s')\big)\,dP_S(s') = 0 \quad \text{for all } s$$

implies $s - \varphi(s) = 0$ for $P_S$-almost all $s$. And so, Lemma 3f3 gives the next result.

**3f4. Theorem.** If $P_S$ is nonatomic then the second price auction has one and only one symmetric equilibrium, namely,

$$A = S\,.$$

Interestingly, the distribution of $A$ can have gaps, which never happens to the first price auction.

The very simple form of the strategy may suggest that it should follow from an elementary argument. And indeed, the strategy $A = S$ is a best response to *every* strategy (not just to itself) for a simple reason; find it! Such a strategy is called *dominant.*

You see, Lemma 3f3 is an awkward way to an almost evident solution of the second price auction. However, my goal here is illustrating a general method rather than solving a special case.

## 3g Reserve price and entry cost

We return to the first price auction, however, we introduce a reserve price $r \in [0,\infty)$ and an entry cost $c \in [0,\infty)$; it means that, first, the auctioneer does not want to sell the object for $r$ (or cheaper), and second, participation itself has a cost $c$ (irrespective of winning). Now, the (expected) loss is

$$\mathbf{L}(a_1; a_2) = \begin{cases} 0 & \text{if } a_1 = 0, \\ c & \text{if } 0 < a_1 < a_2, \\ c + \frac{1}{2}(r + a_1) & \text{if } 0 < a_1 = a_2, \\ c + r + a_1 & \text{if } a_1 > a_2. \end{cases} \tag{3g1}$$

An action $a > 0$ is now treated as bidding $r+a$.[12] The action 0 is treated as non-participation, see (3e1). Note that the game of 3d is the special case $r = 0$, $c = 0$.

Assume that $P_S$ is nonatomic. Let $\mu$ be a strategy supporting a symmetric equilibrium, and denote by $P_A$ the corresponding action distribution. Corollary 3b3 ensures that $\mu$ is an increasing pure strategy; it is concentrated on the graph of an increasing function, $A = \varphi(S)$. Exercise 3c7 shows that $\mathbf{\Pi}^{\max}(0; P_A) = 0$. The loss function (3g1) satisfies Conditions 3e2(a,b) (check it). Thus, Corollary 3e4 shows that $P_A$ cannot have atoms on $(0,\infty)$.

Till now, everything is as before. However, 3f2 is now inapplicable, and an atom at 0 is quite possible; it happens when there is a bunch of the form $[0, s_0]$. Possible absence of the atom is described as $s_0 = 0$. Though, there is one more possibility: $s_0 = +\infty$, which means that the whole $[0,\infty)$ is a bunch. Note that

$$\begin{aligned} s \le s_0 \quad &\Longrightarrow \quad \mathbf{\Pi}^{\max}(s; P_A) = 0\,, \\ s > s_0 \quad &\Longrightarrow \quad \mathbf{\Pi}^{\max}(s; P_A) > 0\,. \end{aligned} \tag{3g2}$$

Other bunches can exist, but they necessarily are of zero probability. Also, it may happen that $s_0$ lies in a gap of $P_S$; do not think that its place within the gap does not matter!

**3g3. Exercise.**

$$\mathbf{p}^{\text{win}}(s) = \begin{cases} 0 & \text{for } s < s_0, \\ F_S(s) & \text{for } s > s_0. \end{cases}$$

Prove it.
Hint: use Lemma 3c3.

However, $\mathbf{p}^{\text{win}}(s) > 0$ (therefore $F_S(s) > 0$) whenever $s > s_0$ (by (3g2) and (3c1)). It follows (recall 3b1) that

$$\begin{aligned} \varphi(s) = 0 \quad &\text{for } P_S\text{-almost all } s \in (0, s_0)\,, \\ \varphi(s) > 0 \quad &\text{for } P_S\text{-almost all } s \in (s_0, \infty)\,. \end{aligned} \tag{3g4}$$

[12] Of course, we may also treat it as bidding $a$ but consider the action space $\{0\} \cup (r,\infty)$, which leads to the same theory in a slightly different form.

Note that $\mathbf{L}(a_1; a_2) \ge c$ whenever $a_1 > 0$; thus $\mathbf{L}(a_1; P_A) \ge c$ for $a_1 > 0$. For $P_S$-almost all $s \in (s_0, \infty)$ we have $\mathbf{L}(\varphi(s); P_A) \ge c$ and $\mathbf{G}(\varphi(s), s; P_A) = s\mathbf{p}^{\text{win}}(s) = sF_S(s)$, therefore $0 < \mathbf{\Pi}^{\max}(s; P_A) = \mathbf{\Pi}(\varphi(s), s; P_A) = \mathbf{G}(\varphi(s), s; P_A) - \mathbf{L}(\varphi(s); P_A) \le sF_S(s) - c$, that is, $sF_S(s) > c$. It follows that[13]

(3g5) $$\mathbb{P}\big( A = 0 \big) = \mathbb{P}\big( S \le s_0 \big) > 0 \quad \text{if } c > 0\,.$$

Combining 3g3 with (3c1) we get (instead of 3c4)

(3g6) $$\mathbf{\Pi}^{\max}(s; P_A) = \begin{cases} 0 & \text{for } s \le s_0, \\ \int_{s_0}^{s} F_S(s')\, ds' & \text{for } s \ge s_0. \end{cases}$$

That is necessary; in order to support a symmetric equilibrium, $\varphi$ must satisfy (3g6) for some $s_0$. What about sufficiency? If $\varphi$ satisfies (3g6) for *some* $s_0$, does it mean that $\varphi$ supports a symmetric equilibrium? Hopefully, it does not (otherwise we would get a continuum of equilibria). Theorem 3c10 about sufficiency is designed only for nonatomic $P_A$. Reconsidering its proof we see that it fails when $P_A$ has a gap immediately after an atom,[14] or an atom is the last (maximal) point of the support.

The case $F_S(s_0) = 1$, in other words, $\mathbb{P}\big( A = 0 \big) = 1$, must be treated separately (our sufficient condition cannot help, when the atom $a = 0$ is the maximal point of the support of $A$).

**3g7. Exercise.** The trivial strategy $A = 0$ supports a symmetric equilibrium if and only if $\mathbb{P}\big( S > c + r \big) = 0$.

Prove it.

Hint. Assuming $\mathbb{P}\big( S > c + r + \varepsilon \big) > 0$ try the strategy

$$A = \begin{cases} 0 & \text{if } S \le c + r + \varepsilon, \\ \varepsilon & \text{if } S > c + r + \varepsilon. \end{cases}$$

Now assume that $c > 0$ and $\mathbb{P}\big( S > c + r \big) > 0$; then $0 < F_S(s_0) < 1$ by (3g5), 3g7 and (3g4). The argument of Theorem 3c10 fails if $P_A$ has a gap of the form $(0, a)$.

**3g8. Exercise.** $P_A$ cannot have a gap.

Prove it.

Hint: recall the proof of 2b10(b).

Introduce

(3g9) $$s_1 = \inf\{ s : F_S(s) > F_S(s_0) \}\,;$$

note that $s_0 \le s_1$ and $F(s_0) = F(s_1)$. It follows from 3g8 that[15]

$$\begin{aligned} \varphi(s) &> 0 \quad \text{for } s > s_1\,, \\ \varphi(s) &\to 0 \quad \text{for } s \to s_1+\,. \end{aligned}$$

[13] However, it does not follow that $s_0 F_S(s_0) \ge c$; think, why.

[14] In that case an action inside the gap can be better than $\varphi(s)$ for some $s$.

[15] True, the function $\varphi$ may be changed at will on a set of probability zero. However, we may (and will) assume that $\varphi$ is monotone.

We have

$$\mathbf{L}(0+;a_2) = \begin{cases} c & \text{if } a_2 > 0, \\ c+r & \text{if } a_2 = 0; \end{cases}$$

$$\mathbf{L}(0+;P_A) = c + rp_0 \,, \quad \text{where } p_0 = F_S(s_0) = \mathbb{P}\big( A = 0 \big) \,.$$

Let $s_n \to s_1+$ and $a_n = \varphi(s_n)$, then $a_n \to 0+$ and $\mathbf{L}(a_n;P_A) \to c+rp_0$. Also, $\mathbf{G}(a_n,s_n;P_A) = s_n F_S(s_n) \to s_1 F_S(s_1+) = s_1 p_0$. Thus, $\mathbf{\Pi}^{\max}(s_n;P_A) = \mathbf{\Pi}(a_n,s_n;P_A) = \mathbf{G}(a_n,s_n;P_A) - \mathbf{L}(a_n;P_A) \to s_1 p_0 - (c+rp_0) = (s_1 - r)p_0 - c$, that is,

$$\mathbf{\Pi}^{\max}(s_1;P_A) = (s_1 - r)p_0 - c \,.$$

On the other hand, $\mathbf{\Pi}^{\max}(s_0;P_A) = 0$ (just by (3g2)), and by 3g3, $\mathbf{p}^{\text{win}}(s) = F_S(s) = p_0$ for $s \in (s_0,s_1)$.[16] Thus, the equality $\mathbf{\Pi}^{\max}(s_1;P_A) - \mathbf{\Pi}^{\max}(s_0;P_A) = \int_{s_0}^{s_1} \mathbf{p}^{\text{win}}(s)\,ds$ becomes $(s_1 - r)p_0 - c - 0 = \int_{s_0}^{s_1} p_0\,ds = (s_1 - s_0)p_0$, or $(s_0 - r)p_0 = c$, that is,

$$(s_0 - r)F_S(s_0) = c \,. \tag{3g10}$$

Note also that $\mathbf{\Pi}(0+,s;P_A) = sp_0 - (c+rp_0) = (s-r)p_0 - c$, which may be used for deriving (3g10) without considering $c_1$, and also for checking sufficiency in the next result that combines all arguments together.

**3g11. Lemma.** Let $P_S$ be nonatomic, $c > 0$, and $\mathbb{P}\big( S > c + r \big) > 0$. Then a strategy $\mu$ supports a symmetric equilibrium if and only if it is a pure strategy $A = \varphi(S)$ where $\varphi$ is an increasing function such that $P_A$ has no gap of the form $(0,a)$, no atoms on $(0,\infty)$, and there exists $s_0 \in (0,+\infty)$ such that

(a) $\varphi(s) = 0 \quad \text{for } s < s_0 \,;$

(b) $\varphi(s) > 0 \quad \text{for } s > s_0 \,;$

(c) $(s_0 - r)F_S(s_0) = c \,;$

(d) $\mathbf{L}(\varphi(s);P_A) = sF_S(s) - \int_{s_0}^{s} F_S(s')\,ds' \quad \text{for } s > s_0 \,.$

Condition 3g11(c) is an equation for $s_0$. It has one and only one solution, since the left-hand side is a continuous strictly increasing (as far as $F_S(s) > 0$) function on $[r,\infty)$, equal to 0 at $r$ and tending to $+\infty$ at $+\infty$. Note that

$$s_0 > r \,.$$

The condition $\mathbb{P}\big( S > c + r \big) > 0$, that is, $F_S(c+r) < 1$, ensures that $\mathbb{P}\big( S > s_0 \big) > 0$ (think, why).

[16] You see, if $s_0 < s_1$ then $[s_0,s_1]$ is the second bunch; in contrast to the first bunch $[0,s_0]$, the second bunch is of probability zero. No optimal action exists for $s \in (s_0,s_1)$; in some sense, the optimum is located at $a = 0+$.

For $a_1 > 0$ we have $\mathbf{L}(a_1; P_A) = c + (r + a_1)F_A(a_1)$; thus 3g11(d) for $s > s_0$ becomes

$$c + (r + \varphi(s))F_A(\varphi(s)) = sF_S(s) - \int_{s_0}^{s} F_S(s')\,ds'\,;$$

taking into account that $F_A(\varphi(s)) = F_S(s)$ we get

$$\text{(3g12)} \qquad \varphi(s) = -r + \frac{1}{F_S(s)}\Big( -c + sF_S(s) - \int_{s_0}^{s} F_S(s')\,ds' \Big)$$

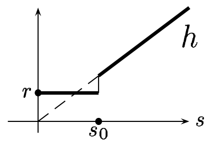
for $s > s_0$. Similarly to the calculation before (3c8),

$$(s - s_0)F_S(s) - \int_{s_0}^{s} F_S(s')\,ds' = \int_{s_0}^{s} \big(F_S(s) - F_S(s')\big)\,ds' = \int_{s_0}^{s} \mathbb{E}\,\mathbf{1}_{[s',s]}(S)\,ds' =$$
$$= \mathbb{E}\big((S - s_0)\mathbf{1}_{[s_0,s]}(S)\big) = \mathbb{E}\big(S\mathbf{1}_{[s_0,s]}(S)\big) - s_0\big(F_S(s) - F_S(s_0)\big)\,;$$

$$-c + sF_S(s) - \int_{s_0}^{s} F_S(s')\,ds' = -c + s_0F_S(s) + (s - s_0)F_S(s) - \int \dots =$$
$$= -c + s_0F_S(s) + \mathbb{E}\big(S\mathbf{1}_{[s_0,s]}(S)\big) = -s_0F_S(s) + s_0F_S(s_0) = -c + s_0F_S(s_0) + \mathbb{E}\big(S\mathbf{1}_{[s_0,s]}(S)\big)\,.$$

However, 3g11(c) gives $-c + s_0F_S(s_0) = rF_S(s_0)$, thus $-c + sF_S(s) - \int \dots = rF_S(s_0) + \mathbb{E}\big(S\mathbf{1}_{[s_0,s]}(S)\big) = \mathbb{E}\big(r\mathbf{1}_{[0,s_0]}(S)\big) + \mathbb{E}\big(S\mathbf{1}_{[s_0,s]}(S)\big) = \mathbb{E}\big(h(S)\mathbf{1}_{[0,s]}(S)\big)$, where

$$h(s) = \begin{cases} r & \text{for } s \in [0, s_0), \\ s & \text{for } s \in (s_0, \infty). \end{cases}$$




Now (3g12) becomes

$$\varphi(s) = -r + \frac{1}{F_S(s)}\mathbb{E}\big(h(S)\mathbf{1}_{[0,s]}(S)\big) = -r + \mathbb{E}\big(\,h(S)\,\big|\,S \le s\,\big)$$

for $s > s_0$. It is easy to see that such $\varphi$ increases strictly on the support of $S$ (intersected with $(s_0, +\infty)$); the conclusion follows.

**3g13. Theorem.** If $P_S$ is nonatomic, $c > 0$ and $\mathbb{P}\big(\,S > c + r\,\big) > 0$, then the first price auction has one and only one symmetric equilibrium, namely, $A = \varphi(S)$, where

$$\varphi(s) = \begin{cases} 0 & \text{for } s < s_0, \\ -r + \mathbb{E}\big(\,h(S)\,\big|\,S \le s\,\big) & \text{for } s > s_0; \end{cases} \qquad h(s) = \begin{cases} r & \text{for } s < s_0, \\ s & \text{for } s > s_0; \end{cases}$$

and the participation threshold $s_0 \in (r, +\infty)$ is the unique solution of the equation

$$(s_0 - r)F_S(s_0) = c\,.$$

Do not forget that the bid is not $\varphi(s)$ but $r+\varphi(s) = \mathbb{E}\left( h(S) \,\middle|\, S \le s \right)$. Compare it with the formula of 3d, $\mathbb{E}\left( S \,\middle|\, S \le s \right)$.

We turn to the case $c = 0$ (no entry cost). We still have (3g1)–(3g7), and $\mathbf{L}(0+; P_A) = rp_0$, where $p_0 = F_S(s_0) = \mathbb{P}\left( A = 0 \right)$; also, $\mathbf{G}(0+, s; P_A) = sp_0$ and $\mathbf{\Pi}(0+, s; P_A) = (s-r)p_0$, which suggests that $s_0 = r$ provided that $F_S(r) > 0$.

No action is profitable when $s < r$; thus $\mathbf{\Pi}^{\max}(s; P_A) = 0$ for $s < r$, therefore $s_0 \ge r$.

If $F_S(r) > 0$ then $p_0 > 0$, therefore $\mathbf{\Pi}^{\max}(s; P_A) \ge \mathbf{\Pi}(0+, s; P_A) = (s-r)p_0 > 0$ for $s > r$, therefore $s_0 \le r$; so,

(3g14) $$\text{if } F_S(r) > 0 \quad \text{then } s_0 = r\,.$$

What happens if $F_S(r) = 0$? Recall that $\mathbf{p}^{\text{win}}(s) = F_S(s)$ for $s > s_0$ (by 3g3), and $\mathbf{p}^{\text{win}}(s) > 0$ for $s > s_0$ (just from general facts); therefore $F_S(s) > 0$ for $s > s_0$, which means $s_0 \ge s_{\min}$, where $s_{\min} = \inf\{s : F_S(s) > 0\}$; note that $r \le s_{\min}$ (since $F_S(r) = 0$). We recall that $p_0 > 0$ implies $s_0 \le r$; now, however, $s_0 \le r$ implies $p_0 = 0$; thus, the case $p_0 > 0$ is now impossible. We have $p_0 = 0$, therefore $s_0 \le s_{\min}$. So,

(3g15) $$\text{if } F_S(r) = 0 \quad \text{then } s_0 = \inf\{s : F_S(s) > 0\}\,.$$

(Observe the distinction: a positive entry cost *always* implies a positive quitting probability, which cannot be said about reserve price.)

If $p_0 > 0$, that is, $P_A$ has an atom at 0, then 3g8 works as before; otherwise we do not need it (for sufficiency).

Lemma 3g11 works with (3g14), (3g15) instead of item (c). And finally, Theorem 3g13 works with (3g14), (3g15) instead of the equation $(s_0 - r)F_S(s_0) = c$. Note that the limit $c \to 0+$ gives the right result.

## 3h An all-pay auction

The payment rule 'pay your bid anyway', belonging to so-called all-pay auctions, means the (expected) loss

(3h1) $$\mathbf{L}(a_1; a_2) = a_1\,.$$

**3h2. Exercise.** If $P_S$ is nonatomic then the 'pay your bid anyway' auction has one and only one symmetric equilibrium, namely,

$$A = \varphi(S)\,, \quad \text{where } \varphi(s) = \mathbb{E}\left( S \,\middle|\, S \le s \right) \cdot \mathbb{P}\left( S \le s \right).$$

Prove it.

Compare the support of $P_A$ with that of the first-price auction (3d).

## 3i Let the loser pay

A bizarre payment rule considered here says, 'loser, pay your bid'; the winner get the object for free; it means the (expected) loss

(3i1)
$$\mathbf{L}(a_1;a_2) = \begin{cases} a_1 & \text{if } a_1 < a_2, \\ \frac12 a_1 & \text{if } a_1 = a_2, \\ 0 & \text{if } a_1 > a_2. \end{cases}$$

Assume that $P_S$ is nonatomic. Let $\mu$ be a strategy supporting a symmetric equilibrium, and denote by $P_A$ the corresponding action distribution. Corollary 3b3 ensures that $\mu$ is an increasing pure strategy; it is concentrated on the graph of an increasing function, $A = \varphi(S)$. Exercise 3c7 shows that $\mathbf{\Pi}^{\max}(0;P_A) = 0$. The loss function (3i1) satisfies Conditions 3e2(a,b) as well as 3f2; therefore $P_A$ is nonatomic. Any bunch of nonzero probability would give an atom to $P_A$ due to 3c2. Therefore there is no such bunch, and Theorem 3c4 is applicable:

$$\mathbf{\Pi}^{\max}(s;P_A) = \int_0^s F_S(s')\,ds'$$

for all $s \in [0,\infty)$. Theorem 3c10 does not work, since $\mathbf{L}(a_1;a_2)$ does not increase in $a_1$. However, the following modification of Theorem 3c10 helps.

**3i2. Exercise.** Let $P_S$ be nonatomic, and $\mu$ be a pure strategy $A = \varphi(S)$ where $\varphi$ is an increasing function such that $P_A$ is nonatomic. Assume that $\mathbf{L}(a;P_A)$ is a continuous function of $a \in [0,\infty)$, increasing on every constancy interval of $F_A$,[17] and $\mathbf{L}(0;P_A) = 0$, and $\mathbf{L}(a;P_A) \ge 0$ for all $a$. Then Conditions 3c10(a,b) are equivalent, and if they are satisfied then $\mu$ supports a symmetric equilibrium.

Prove it.

Hint: having an increasing sequence of $s_n \in E$ such that $a_n = \varphi(s_n) \le a$ and $F_S(s_n) \to F_A(a)$ conclude that $F_A(a_n) \to F_A(a_\infty) = F_A(a)$ and $\mathbf{L}(a_n;P_A) \to \mathbf{L}(a_\infty;P_A) \le \mathbf{L}(a;P_A)$, where $a_\infty = \lim a_n$.

So, necessity and sufficiency are both checked, as follows.

**3i3. Lemma.** Let $P_S$ be nonatomic. Then a strategy $\mu$ supports a symmetric equilibrium if and only if it is a pure strategy $A = \varphi(S)$ where $\varphi$ is an increasing function such that $P_A$ is nonatomic and equivalent conditions 3c10(a,b) are satisfied.

Condition 3c10(b) may be rewritten using (3c8) as

$$\mathbf{L}(\varphi(s);P_A) = \mathbb{E}\left( S \,\middle|\, S \le s \right) \cdot \mathbb{P}\left( S \le s \right).$$

On the other hand, the 'let the loser pay' rule (3i1) gives $\mathbf{L}(\varphi(s);P_A) = \varphi(s)\mathbb{P}\left( A > \varphi(s) \right)$, that is,

$$\mathbf{L}(\varphi(s);P_A) = \varphi(s)\mathbb{P}\left( S > s \right)$$

[17] Not just a gap of $P_A$; the constant value is allowed to be 0 or 1.

for $P_S$-almost all $s$. So, Condition 3c10(b) becomes

$$\varphi(s) = \frac{F_S(s)}{1 - F_S(s)} \mathbb{E}\left( S \,\middle|\, S \le s \right)$$

for $P_S$-almost all $s$. Clearly, such a function increases strictly on the support of $S$. And so, Lemma 3i3 gives the next result.

**3i4. Theorem.** If $P_S$ is nonatomic then the 'let the loser pay' auction has one and only one symmetric equilibrium, namely,

$$A = \varphi(S) , \quad \text{where } \varphi(s) = \frac{F_S(s)}{1 - F_S(s)} \mathbb{E}\left( S \,\middle|\, S \le s \right) .$$

**3i5. Exercise.** Describe the support of $P_A$.

Interestingly, a bounded strategy (that is, such that $\mathbb{P}\left( A \le a \right) = 1$ for $a$ large enough) cannot support a symmetric equilibrium for an evident reason: the competitor bidding the large $a$ always wins and never pays.