4 More on Brownian rotations

4a Moment method on the circle

We return to the process $Y_t = \exp(i\sigma B_t + ivt)$, $\sigma, v \in \mathbb{R}$ (as in 3a23). The distribution of Y_t is easy to describe explicitly by a density: for $0 < u < v < 2\pi$,

(4a1)
$$\mathbb{P}\left(Y_t \in e^{i[u,v]} \right) = \int_u^v p_t(x)\, dx\,,$$
$$p_t(x) = \frac{1}{\sqrt{2\pi}|\sigma|\sqrt{t}} \sum_{k=-\infty}^{+\infty} \exp\left(-\frac{(x - vt + 2\pi k)^2}{2\sigma^2 t} \right)$$

(unless $\sigma = 0$, of course); here $e^{i[u,v]}$ denotes the arc $\{e^{ix} : u \le x \le v\}$. This approach, however, does not work for more general situations of 3c, when Y_t may depend on the past of (X_s). This is why we turn to the moment method.

For any random variable $U : \Omega \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the distribution μ of U is uniquely determined by its moments

$$\mathbb{E} U^k = \int_{\mathbb{T}} u^k \,\mu(du) \in \mathbb{C}\,, \qquad k = 1, 2, \dots$$

since functions $1, \cos kx, \sin kx$ span the space of continuous 2π-periodic functions.

For any $k = 1, 2, \dots$ the process $(Y_t^k)_t$ is another Brownian motion in $\mathbb{T}$, and

$$\mathbb{E} Y_t^k = \mathbb{E} \exp(ik\sigma B_t + ikvt) = \exp\left(-\frac{1}{2} k^2 \sigma^2 t \right) e^{ikvt} = \exp\left(\left(-\tfrac{1}{2} k^2 \sigma^2 + ikv\right) t \right).$$

By the way, for $t \to \infty$ we get $\mathbb{E} Y_t^k \to 0$ (unless $\sigma = 0$), which means that the distribution converges (weakly) to the uniform distribution on $\mathbb{T}$. (Do you see it via (4a1)?)

4b Tensor moments of random matrices

4b1 Exercise. The distribution of a random matrix $U : \Omega \to \mathrm{SO}(n)$, in general, is not determined uniquely by the matrices $\mathbb{E} U^k$, $k = 1, 2, \dots$

Prove it (by finding a counterexample).

Hint: restrict yourself to diagonal matrices.

4b2 Definition. For any matrices $A \in \mathrm{M}_m(\mathbb{R})$, $B \in \mathrm{M}_n(\mathbb{R})$, their *tensor product* $A \otimes B \in \mathrm{M}_{mn}(\mathbb{R})$ is

$$(A \otimes B)^{\gamma,\delta}_{\alpha,\beta} = A^\gamma_\alpha B^\delta_\beta \quad \text{for } \alpha, \gamma \in \{1, \dots, m\} \text{ and } \beta, \delta \in \{1, \dots, n\}\,.$$

You see, rows and columns of $A \otimes B$ are numbered by $\{1, \dots, m\} \times \{1, \dots, n\}$ rather than $\{1, \dots, mn\}$. The freedom of enumerating the pairs does not influence algebraic relations between matrices. An example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

(which enumeration is used here?).

Tensor product of several matrices is defined similarly. In particular,

$$A^{\otimes k} = \underbrace{A \otimes \dots \otimes A}_{k} \, .$$

4b3 Exercise. For any random matrix $U : \Omega \to \mathrm{SO}(n)$, the distribution μ of U is uniquely determined by its tensor moments

$$\mathbb{E} U^{\otimes k} = \int_{\mathrm{SO}(n)} u^{\otimes k} \, \mu(du) \in \mathrm{M}_{n^k}(\mathbb{R}) \, .$$

Prove it.

Hint: polynomials are dense among continuous functions on $\mathrm{SO}(n)$.

4b4 Exercise. (a) $(A \otimes B)(x \otimes y) = (Ax) \otimes (By)$ for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $A \in \mathrm{M}_m(\mathbb{R})$, $B \in \mathrm{M}_n(\mathbb{R})$. Here $x \otimes y \in \mathbb{R}^{mn}$ is defined by

$$(x \otimes y)_{\alpha,\beta} = x_\alpha y_\beta \quad \text{for } \alpha \in \{1, \dots, m\}, \ \beta \in \{1, \dots, n\}$$

(up to enumeration...). Similarly, $(x \otimes y)(A \otimes B) = (xA) \otimes (yB)$.

(b) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ for all $A, C \in \mathrm{M}_m(\mathbb{R})$, $B, D \in \mathrm{M}_n(\mathbb{R})$.

Prove it.

Note that factorizable vectors $x \otimes y$ are not the whole $\mathbb{R}^{mn}$, but span $\mathbb{R}^{mn}$.

4b5 Exercise. $\mathrm{SO}(m) \otimes \mathrm{SO}(n) \subset \mathrm{SO}(mn)$.

Prove it.

Hint: $\langle (x_1 \otimes y_1)(U \otimes V), (x_2 \otimes y_2)(U \otimes V) \rangle = \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle$.

4b6 Exercise. (a) If $(U_t)_{t \in [0,\infty)}$ is a one-parameter semigroup in $\mathrm{M}_n(\mathbb{R})$, then $(U_t \otimes U_t)_{t \in [0,\infty)}$ is a one-parameter semigroup in $\mathrm{M}_{n^2}(\mathbb{R})$.

(b) If A is the generator of $(U_t)_{t \in [0,\infty)}$ then

$$(A \otimes \mathbf{1}) + (\mathbf{1} \otimes A)$$

is the generator of $(U_t \otimes U_t)_t$.

Prove it. Generalize it to $U_t^{\otimes k}$.

Hint. (a): use 4b4(b); (b): $(\mathbf{1} + At + o(t)) \otimes (\mathbf{1} + At + o(t)) = \dots$

In other words,

(4b7) $$\exp(A \otimes \mathbf{1} + \mathbf{1} \otimes A) = \exp(A) \otimes \exp(A) \, .$$

In fact, $\exp(A \otimes \mathbf{1}) = \exp(A) \otimes \mathbf{1}$ and $\exp(\mathbf{1} \otimes B) = \mathbf{1} \otimes \exp(B)$, thus $\exp(A \otimes \mathbf{1} + \mathbf{1} \otimes B) = \exp(A) \otimes \exp(B)$. Operators $A \otimes \mathbf{1}$ form an algebra isomorphic to $\mathrm{M}_n(\mathbb{R})$; operators $\mathbf{1} \otimes B$ form another algebra isomorphic to $\mathrm{M}_n(\mathbb{R})$; these are two *commuting* copies of $\mathrm{M}_n(\mathbb{R})$ in $\mathrm{M}_{n^2}(\mathbb{R})$.

4c Tensor powers of Brownian rotations

4c1 Exercise. (a) If $(Y_t)_t$ is a Brownian motion in $\mathrm{SO}(n)$ then $(Y_t^{\otimes k})_t$ is a Brownian motion in $\mathrm{SO}(n^k)$, for any $k = 1, 2, \dots$

(b) If $(B_t, Y_t)_t$ is a morphism of the standard Brownian motion $(B_t)_t$ in $\mathbb{R}$ to a Brownian motion in $\mathrm{SO}(n)$, then $(B_t, Y_t^{\otimes k})_t$ is a morphism of $(B_t)_t$ to a Brownian motion in $\mathrm{SO}(n^k)$.

Prove it.

Hint: use 4b4(b) and 4b5.

In fact, the product of *commuting* Brownian rotations is a Brownian rotation. The same holds for morphisms.

4c2 Exercise. If $(Y_t)_t$ is a Brownian motion in $\mathrm{SO}(n)$ then $(\mathbb{E}\, Y_t)_t$ is a (continuous) one-parameter semigroup in $\mathrm{M}_n(\mathbb{R})$.

Prove it.

Hint: $\mathbb{E}\,(AB) = (\mathbb{E}\, A)(\mathbb{E}\, B)$ for independent random matrices A, B.

We have

$$\mathbb{E}\,(Y_t^{\otimes k}) = \exp(A_k t) \qquad \text{for } t \in [0, \infty),\ k = 1, 2, \dots \tag{4c3}$$

where $A_k \in \mathrm{M}_{n^k}(\mathbb{R})$ is the generator of the semigroup.

4c4 Example. The isomorphism $e^{i\alpha} \mapsto \left(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right)$ between $\mathbb{T}$ and $\mathrm{SO}(2)$ (mentioned before 3b8) turns e^{iB_t} into

$$Y_t = \begin{pmatrix} \cos B_t & \sin B_t \\ -\sin B_t & \cos B_t \end{pmatrix} \in \mathrm{SO}(2)\,.$$

We have

$$\mathbb{E}\, Y_t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \exp(A_1 t)\,, \qquad A_1 = -\frac{1}{2} \cdot \mathbf{1}\,,$$

since $\mathbb{E}\, e^{iB_t} = e^{-t/2}$. Further,

$$Y_t \otimes Y_t = \begin{pmatrix} \cos^2 B_t & \cos B_t \sin B_t & \sin B_t \cos B_t & \sin^2 B_t \\ -\cos B_t \sin B_t & \cos^2 B_t & -\sin^2 B_t & \sin B_t \cos B_t \\ -\sin B_t \cos B_t & -\sin^2 B_t & \cos^2 B_t & \cos B_t \sin B_t \\ \sin^2 B_t & -\sin B_t \cos B_t & -\cos B_t \sin B_t & \cos^2 B_t \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 1 + \cos 2B_t & \sin 2B_t & \sin 2B_t & 1 - \cos 2B_t \\ -\sin 2B_t & 1 + \cos 2B_t & -1 + \cos 2B_t & \sin 2B_t \\ -\sin 2B_t & -1 + \cos 2B_t & 1 + \cos 2B_t & \sin 2B_t \\ 1 - \cos 2B_t & -\sin 2B_t & -\sin 2B_t & 1 + \cos 2B_t \end{pmatrix};$$

$$\mathbb{E}\,(Y_t \otimes Y_t) = \frac{1}{2} \begin{pmatrix} 1 + e^{-2t} & 0 & 0 & 1 - e^{-2t} \\ 0 & 1 + e^{-2t} & -1 + e^{-2t} & 0 \\ 0 & -1 + e^{-2t} & 1 + e^{-2t} & 0 \\ 1 - e^{-2t} & 0 & 0 & 1 + e^{-2t} \end{pmatrix}$$

(it must be a semigroup). For small t,

$$\mathbb{E}\,(Y_t \otimes Y_t) = \begin{pmatrix} 1-t & 0 & 0 & t \\ 0 & 1-t & -t & 0 \\ 0 & -t & 1-t & 0 \\ t & 0 & 0 & 1-t \end{pmatrix} + o(t) = 1 + A_2 t + o(t)\,,$$

$$A_2 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},$$

thus $\mathbb{E}\,(Y_t \otimes Y_t) = \exp(A_2 t)$.

4c5 Exercise. Let

$$Y_t = \operatorname{Texp}\left(i \int_0^t (\sigma\, dB_s + v\, ds) \right)$$

for some $\sigma, v \in i\mathrm{M}_n(\mathbb{R})$, $\sigma^* = \sigma$, $v^* = v$. Then

$$Y_t \otimes Y_t = \operatorname{Texp}\left(i \int_0^t (\sigma \otimes \mathbf{1} + \mathbf{1} \otimes \sigma)\, dB_s + (v \otimes \mathbf{1} + \mathbf{1} \otimes v)\, ds \right).$$

Prove it. Generalize it for $Y_t^{\otimes k}$.

Hint. According to 4c1(b) and 3b, it must be $Y_t \otimes Y_t = \operatorname{Texp}\big(i \int_0^t (\sigma_2\, dB_s + v_2\, ds) \big)$ for some σ_2, v_2; for small t we get $\exp(i\sigma_2 B_t + iv_2 t) = \exp(i\sigma B_t + ivt) \otimes \exp(i\sigma B_t + ivt) + o(t)$; recall (4b7).

In fact,

$$\left(\operatorname{Texp}\left(i \int_0^t dX \right) \right) \left(\operatorname{Texp}\left(i \int_0^t dY \right) \right) = \operatorname{Texp}\left(i \int_0^t d(X+Y) \right)$$

whenever X, Y commute.

4c6 Exercise. Let

$$Y_t = \operatorname{Texp}\left(i \int_0^t (\sigma\, dB_s + v\, ds) \right)$$

for some $\sigma, v \in i\mathrm{M}_n(\mathbb{R})$, $\sigma^* = \sigma$, $v^* = v$. Then

$$A_1 = -\frac{1}{2}\sigma^2 + iv\,,$$

$$\begin{aligned} A_2 &= -\frac{1}{2}(\sigma \otimes \mathbf{1} + \mathbf{1} \otimes \sigma)^2 + i(v \otimes \mathbf{1} + \mathbf{1} \otimes v) = \\ &= -\frac{1}{2}(\sigma^2 \otimes \mathbf{1} + \mathbf{1} \otimes \sigma^2) - \sigma \otimes \sigma + i(v \otimes \mathbf{1} + \mathbf{1} \otimes v)\,, \end{aligned}$$

where A_k are defined by (4c3).

Prove it.

Hint: first, find A_1 by using the asymptotics of Y_t for small t; second, apply the formula for A_1 to $Y_t \otimes Y_t$ using 4c5.

4c7 Example. Let Y_t be as in 4c4, then $i\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the image of $i \in \mathbb{C}$ under the embedding $z \mapsto \begin{pmatrix} \operatorname{Re} z & \operatorname{Im} z \\ -\operatorname{Im} z & \operatorname{Re} z \end{pmatrix}$ (mentioned before 3b8), and $v = 0$. Using 4c6,

$$(i\sigma)^2 = -\mathbf{1}\,; \qquad A_1 = -\frac{1}{2} \cdot \mathbf{1}\,;$$

$$\sigma^2 \otimes \mathbf{1} = \mathbf{1} \otimes \mathbf{1} = \mathbf{1}\,; \quad \mathbf{1} \otimes \sigma^2 = \mathbf{1}\,; \qquad \sigma \otimes \sigma = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\,;$$

$$A_2 = -\mathbf{1} - \sigma \otimes \sigma = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\,;$$

the result conforms to 4c4.

Similarly,

$$A_3 = -\frac{1}{2}(\sigma \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \sigma)^2 + i(v \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes v \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes v)$$

and so on. Having all A_k we know (in principle) all tensor moments of Y_t, therefore, the distribution of Y_t (for any t), according to 4b3.

However, $\operatorname{Texp}\big(i \int (\sigma \, dB_s + v \, ds)\big)$ is only a special case of

$$Y_t = \operatorname{Texp}\left(i \int_0^t (\sigma_1 \, dB_1(s) + \dots + \sigma_m \, dB_m(s) + v \, ds) \right) \tag{4c8}$$

(recall 3c). By a straightforward generalization of 4c1(b), 4c5, 4c6 we get

$$Y_t \otimes Y_t = \operatorname{Texp}\Bigg(i \int_0^t \big((\sigma_1 \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_1) \, dB_1(s) + \dots + (\sigma_m \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_m) \, dB_m(s) + \\ + (v \otimes \mathbf{1} + \mathbf{1} \otimes v) \, ds \big) \Bigg)\,;$$

(4c9)

$$A_1 = -\frac{1}{2}(\sigma_1^2 + \dots + \sigma_m^2) + iv\,;$$

$$A_2 = -\frac{1}{2} \sum_{k=1}^m (\sigma_k \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_k)^2 + i(v \otimes \mathbf{1} + \mathbf{1} \otimes v)\,;$$

$$A_3 = -\frac{1}{2} \sum_{k=1}^m (\sigma_k \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_k \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \sigma_k)^2 + i(v \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes v \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes v)\,;$$

and so on.

4d Not just morphisms

We have a satisfactory theory of morphisms. What about a theory of Brownian rotations? Two questions arise naturally.

4d1. Whether *every* Brownian motion in $\mathrm{SO}(n)$ is of the form (4c8), or not?

4d2. Given two morphisms of the form (4c8), how to decide, whether they give two *identically distributed* Brownian motions in $\mathrm{SO}(n)$, or not?

No doubt that two morphisms can represent the same Brownian rotation (I mean, the same distribution). For example,

$$\begin{aligned} Y_1(t) &= \operatorname{Texp}\Big(i \int_0^t (\sigma \, dB_s + v \, ds) \Big) \, , \\ Y_2(t) &= \operatorname{Texp}\Big(i \int_0^t (-\sigma \, dB_s + v \, ds) \Big) \, . \end{aligned}$$

These two morphisms $B \to Y_1$, $B \to Y_2$ are connected by an automorphism of B, that is, an isomorphism (invertible morphism) $B \to B$; namely, $(B_t, -B_t)_t$. For the two-dimensional $\big(B_1(t), B_2(t)\big)_t$ we may use the automorphism

$$\Big(\big(B_1(t), B_2(t)\big), \big(B_1(t)\cos\alpha + B_2(t)\sin\alpha, \, -B_1(t)\sin\alpha + B_2(t)\cos\alpha\big) \Big)_t$$

where α is a parameter; we get a continuum of morphisms

$$Y_\alpha(t) = \operatorname{Texp}\Big(i \int_0^t \big((\sigma_1 \cos\alpha - \sigma_2 \sin\alpha) \, dB_1(s) + (\sigma_1 \sin\alpha + \sigma_2 \cos\alpha) \, dB_2(s) + v \, ds \big) \Big)$$

such that the distribution of Y_α does not depend on the parameter α. More generally, every rotation of $\mathbb{R}^m$ (namely, every element of $\mathrm{O}(\mathbb{R}^m)$) gives us an automorphism of the standard Brownian motion $\big(B_1(t), \dots, B_m(t)\big)_t$ in $\mathbb{R}^m$.

There exist also morphisms of (B_1, B_2) to B_1; here are two examples:

$$\begin{aligned} &\Big(\big(B_1(t), B_2(t)\big), B_1(t) \Big)_t \, ; \\ &\Big(\big(B_1(t), B_2(t)\big), \frac{B_1(t) + B_2(t)}{\sqrt{2}} \Big)_t \, . \end{aligned}$$

Accordingly, the Brownian rotations

$$\begin{aligned} Y_1(t) &= \operatorname{Texp}\Big(i \int_0^t \Big(\frac{\sigma}{\sqrt{2}} \, dB_1(s) + \frac{\sigma}{\sqrt{2}} \, dB_2(s) + v \, ds \Big) \Big) \, , \\ Y_2(t) &= \operatorname{Texp}\Big(i \int_0^t (\sigma \, dB(s) + v \, ds) \Big) \end{aligned}$$

are identically distributed.

4e Uniqueness theorem

Generators A_k (introduced by (4c3)) depend on the distribution of Y (not on a morphism), and determine uniquely the distribution by 4b3. It is an answer to 4d2: given two morphisms, calculate their $A_1, A_2, \dots$ by (4c9) and compare them.

Fortunately, it is enough to compare A_1, A_2 only! They determine uniquely $A_3, A_4, \dots$ Indeed, knowing $A_1 = -\frac{1}{2}(\sigma_1^2 + \dots + \sigma_m^2) + iv$ we know both v (since $A_1 - A_1^* = 2iv$), and $\sum \sigma_k^2$. Knowing also A_2 we know $\sum (\sigma_k \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_k)^2 = (\sum \sigma_k^2) \otimes \mathbf{1} + \mathbf{1} \otimes (\sum \sigma_k^2) + 2 \sum \sigma_k \otimes \sigma_k$, thus, we know $\sum \sigma_k \otimes \sigma_k$. Now, in order to find A_3 we need $\sum (\sigma_k \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_k \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \sigma_k)^2 = (\sum \sigma_k^2) \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes (\sum \sigma_k^2) \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes (\sum \sigma_k^2) + 2 \sum \sigma_k \otimes \sigma_k \otimes \mathbf{1} + 2 \sum \sigma_k \otimes \mathbf{1} \otimes \sigma_k + 2 \sum \mathbf{1} \otimes \sigma_k \otimes \sigma_k$, and it is enough to find the three sums $\sum \sigma_k \otimes \sigma_k \otimes \mathbf{1}$, $\sum \sigma_k \otimes \mathbf{1} \otimes \sigma_k$ and $\sum \mathbf{1} \otimes \sigma_k \otimes \sigma_k$. No problems with the first and the third sum,

$$\sum \sigma_k \otimes \sigma_k \otimes \mathbf{1} = \Big(\sum \sigma_k \otimes \sigma_k \Big) \otimes \mathbf{1} \,,$$
$$\sum \mathbf{1} \otimes \sigma_k \otimes \sigma_k = \mathbf{1} \otimes \Big(\sum \sigma_k \otimes \sigma_k \Big) \,.$$

The second sum $\sum \sigma_k \otimes \mathbf{1} \otimes \sigma_k$ looks worse, but results from $\sum \sigma_k \otimes \sigma_k$, too:

$$\big(\sum \sigma_k \otimes \mathbf{1} \otimes \sigma_k \big)_{\alpha\beta\gamma}^{\delta\varepsilon\zeta} = \sum (\sigma_k)_\alpha^\delta (\mathbf{1})_\beta^\varepsilon (\sigma_k)_\gamma^\zeta = (\mathbf{1})_\beta^\varepsilon \big(\sum \sigma_k \otimes \sigma_k \big)_{\alpha\gamma}^{\delta\zeta} \,.$$

We see that A_1, A_2 determine A_3 uniquely. The same holds for $A_4, A_5, \dots$ and we get a good answer to 4d2.

4e1 Theorem. (a) The distribution of the Brownian motion $(Y_t)_t$ in $\mathrm{SO}(n)$ given by

$$Y_t = \operatorname{Texp} \bigg(i \int_0^t \big(\sigma_1 \, dB_1(s) + \dots + \sigma_m \, dB_m(s) + v \, ds \big) \bigg)$$

uniquely determines generators $A_1 \in \mathrm{M}_n(\mathbb{R})$, $A_2 \in \mathrm{M}_{n^2}(\mathbb{R})$ of the semigroups $(\mathbb{E}\, Y_t)_t$, $\big(\mathbb{E}\,(Y_t \otimes Y_t)\big)_t$ and is uniquely determined by A_1, A_2.

(b) The same for the two matrices

$$iv \in \mathrm{M}_n(\mathbb{R}) \,, \qquad \sum_{k=1}^m \sigma_k \otimes \sigma_k \in \mathrm{M}_{n^2}(\mathbb{R}) \,.$$

4f Differential operators on the rotation group

The two matrices mentioned in 4e1(b) have an important meaning in terms of differential operators on $\mathrm{SO}(n)$.

A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ has at 0 its gradient (first differential) vector $\nabla f(0) \in \mathbb{R}^n$, namely $\big(\nabla f(0)\big)_\alpha = \frac{\partial}{\partial x_\alpha}\big|_{x=0} f(x)$, and its matrix of second derivatives $\nabla^2 f(0) \in \mathrm{M}_n(\mathbb{R})$, namely $\big(\nabla f(0)\big)_\alpha^\beta = \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\big|_{x=0} f(x)$. The situation is somewhat more complicated for a smooth function $f : \mathrm{SO}(n) \to \mathbb{R}$ on the smooth manifold $\mathrm{SO}(n)$ of dimension n^2. Given such f, we define $\nabla f(\mathbf{1}) \in \mathrm{M}_n(\mathbb{R})$ by

(4f1) $$\big(\nabla f(\mathbf{1})\big)_\alpha^\beta = \frac{\partial}{\partial A_\alpha^\beta}\Big|_{A=0} f(\exp A) \,;$$

the matrix $A \in \mathrm{M}_n(\mathbb{R})$ may be treated as n^2 variables, and A^β_α is one of these variables. Similarly we define $\nabla^2 f(\mathbf{1}) \in \mathrm{M}_{n^2}(\mathbb{R})$ by

(4f2)
$$\big(\nabla^2 f(\mathbf{1})\big)^{\gamma,\delta}_{\alpha,\beta} = \frac{\partial^2}{\partial A^\gamma_\alpha \partial A^\delta_\beta}\Big|_{A=0} f(\exp A)\,.$$

4f3 Exercise.

$$f(\exp A) = f(\mathbf{1}) + \langle \nabla f(\mathbf{1}), A \rangle + \frac12 \langle \nabla^2 f(\mathbf{1}), A \otimes A \rangle + o(\|A\|^2)$$

for $\|A\| \to 0$; here $\langle \cdot, \cdot \rangle$ means[1]

$$\langle A, B \rangle = \sum_{\alpha,\beta} A^\beta_\alpha B^\beta_\alpha \quad \text{for } A, B \in \mathrm{M}_n(\mathbb{R})\,,$$
$$\langle A, B \rangle = \sum_{\alpha,\beta,\gamma,\delta} A^{\gamma,\delta}_{\alpha,\beta} B^{\gamma,\delta}_{\alpha,\beta} \quad \text{for } A, B \in \mathrm{M}_{n^2}(\mathbb{R})\,.$$

Prove it.

The following fact generalizes equalities

$$f(B_t) = f(0) + f'(0) B_t + \frac12 f''(0) B_t^2 + o(t)\,,$$
$$\mathbb{E}\, f(B_t) = f(0) + \frac12 f''(0) t + o(t)$$

from $f : \mathbb{R} \to \mathbb{R}$ to $f : \mathrm{SO}(n) \to \mathbb{R}$. A matrix $D \in \mathrm{M}_{n^2}(\mathbb{R})$ defined by

(4f4)
$$D = -\sum_k \sigma_k \otimes \sigma_k$$

will be very useful.

4f5 Exercise. Let $(Y_t)_t$ be given by (4c8); then

$$f(Y_t) = f(\mathbf{1}) + \sum \langle \nabla f(\mathbf{1}), i\sigma_k \rangle B_k(t) + \langle \nabla f(\mathbf{1}), iv \rangle t - \frac12 \sum \langle \nabla^2 f(\mathbf{1}), \sigma_k \otimes \sigma_k \rangle B_k^2(t) + o(t)\,,$$
$$\mathbb{E}\, f(Y_t) = f(\mathbf{1}) + \Big(\langle \nabla f(\mathbf{1}), iv \rangle + \frac12 \langle \nabla^2 f(\mathbf{1}), D \rangle \Big) t + o(t)$$

for $t \to 0$.
Prove it.

We generalize (4f1), (4f2) as follows: for $U \in \mathrm{SO}(n)$,

(4f6)
$$\big(\nabla f(U)\big)^\beta_\alpha = \frac{\partial}{\partial A^\beta_\alpha}\Big|_{A=0} f(U \exp A)\,;$$
$$\big(\nabla^2 f(U)\big)^{\gamma,\delta}_{\alpha,\beta} = \frac{\partial^2}{\partial A^\gamma_\alpha \partial A^\delta_\beta}\Big|_{A=0} f(U \exp A)\,.$$

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[1]In other words, $\langle A, B \rangle = \mathrm{tr}(AB^*)$.

In other words,

$$\nabla f(U) = \nabla g(\mathbf{1}) \,, \qquad \nabla^2 f(U) = \nabla^2 g(\mathbf{1}) \,, \quad \text{where } g(V) = f(UV) \text{ for all } V \,. \tag{4f7}$$

We consider the convolution semigroup $(\mu_t)_t$ corresponding to $(Y_t)_t$; that is, $Y_t \sim \mu_t$. We define the convolution of a function and a measure (on $\mathrm{SO}(n)$) by

$$(f * \mu)(U) = \int f(UV) \, \mu(dV) \,,$$

then $\mathbb{E} f(Y_t) = \int f \, d\mu_t = (f * \mu_t)(\mathbf{1})$, and 4f5 becomes

$$(f * \mu_t)(\mathbf{1}) = f(\mathbf{1}) + \big(\langle \nabla f(\mathbf{1}), iv \rangle + \frac{1}{2} \langle \nabla^2 f(\mathbf{1}), D \rangle \big) t + o(t) \,.$$

4f8 Exercise. For any $U \in \mathrm{SO}(n)$,

$$(f * \mu_t)(U) = f(U) + \big(\langle \nabla f(U), iv \rangle + \frac{1}{2} \langle \nabla^2 f(U), D \rangle \big) t + o(t)$$

for $t \to 0$.

Prove it.

Hint: use (4f7) and apply 4f5 to g.

4f9 Exercise. Denote $f_t = f * \mu_t$, then

$$f_{t+\Delta t}(U) = f_t(U) + \big(\langle \nabla f_t(U), iv \rangle + \frac{1}{2} \langle \nabla^2 f_t(U), D \rangle \big) \Delta t + o(\Delta t)$$

for $\Delta t \to 0+$ (and $t = \text{const}$).

Prove it.

Hint: apply 4f8 to f_t.

The simplest functions are linear functions,

$$f_B(U) = \langle B, U \rangle \qquad \text{for } U \in \mathrm{SO}(n) \,, \tag{4f10}$$

$B \in \mathrm{M}_n(\mathbb{R})$ being a parameter. We have

$$(f_B * \mu_t)(U) = \int f_B(UV) \, \mu_t(dV) = \int \langle B, UV \rangle \, \mu_t(dV) = \\ = \langle B, U \int V \, \mu_t(dV) \rangle = \langle B, U \exp(A_1 t) \rangle = \langle B \exp(A_1^* t), U \rangle = f_{B \exp(A_1^* t)}(U) \,,$$

that is,

$$f_B * \mu_t = f_{B \exp(A_1^* t)} \,. \tag{4f11}$$

Clearly, $f_t(U) = (f_B * \mu_t)(U) = \langle B \exp(A_1^* t), U \rangle$ is a smooth function of $(t,U) \in [0,\infty) \times \mathrm{SO}(n)$; by 4f9 it satisfies the PDE (partial differential equation)

$$\frac{\partial}{\partial t} f_t(U) = \langle \nabla f_t(U), iv \rangle + \frac{1}{2} \langle \nabla^2 f_t(U), D \rangle \,. \tag{4f12}$$

Quadratic functions are of the form

$$f_B(U) = \langle B, U \otimes U \rangle \qquad \text{for } U \in \mathrm{SO}(n)\,; \tag{4f13}$$

this time, $B \in \mathrm{M}_{n^2}(\mathbb{R})$.

4f14 Exercise. For all $B \in \mathrm{M}_{n^2}(\mathbb{R})$ and $t \in [0,\infty)$,

$$f_B * \mu_t = f_{B \exp(A_2^* t)} \,.$$

Prove it.

The PDE (4f12) holds for quadratic f_0 as well. Similarly, it holds for all polynomials f_0. By approximation (in $C^2(\mathrm{SO}(n))$) it holds for all f_0 of class C^2.

4g Existence theorem

Here is a positive answer to 4d1.[2]

4g1 Theorem. For every Brownian motion $(Y_t)_t$ in $\mathrm{SO}(n)$ there exists a morphism of the standard Brownian motion in $\mathbb{R}^m$ (for some m) to $(Y_t)_t$.

The theorem follows from three lemmas, 4g9, 4g12, 4g15.

Similarly to the variance

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \mathbb{E}(X - \mathbb{E}X)^2$$

of a random variable $X \in L_2(\Omega, \mathbb{R})$, we may introduce the *tensor variance*

$$\mathrm{Var}(U) = \mathbb{E}(U \otimes U) - (\mathbb{E}U) \otimes (\mathbb{E}U) = \mathbb{E}\big((U - \mathbb{E}U) \otimes (U - \mathbb{E}U)\big) \in \mathrm{M}_{n^2}(\mathbb{R}) \tag{4g2}$$

of a random matrix $U \in L_2(\Omega, \mathrm{M}_n(\mathbb{R}))$. Clearly,

$$(\mathrm{Var}\, U)^{\gamma,\delta}_{\alpha,\beta} = \mathrm{Cov}(U^\gamma_\alpha, U^\delta_\beta)\,; \quad (\mathrm{Var}\, U)^{\beta,\beta}_{\alpha,\alpha} = \mathrm{Var}(U^\beta_\alpha)\,.$$

For any Brownian motion $(Y_t)_t$ in $\mathrm{SO}(n)$,

$$\begin{aligned}&\mathrm{Var}(Y_t) = \exp(tA_2) - \exp(tA_1) \otimes \exp(tA_1) = \\ &= \big(\mathbf{1} + tA_2 + o(t)\big) - \big(\mathbf{1} + tA_1 + o(t)\big) \otimes \big(\mathbf{1} + tA_1 + o(t)\big) = t(A_2 - A_1 \otimes \mathbf{1} - \mathbf{1} \otimes A_1) + o(t)\end{aligned}$$

[2]See also: K. Yosida, On Brownian motion in a homogeneous Riemannian space. Pacific J. Math. **2**, 263–270 (1952).

for $t \to 0$. Introducing

$$D = A_2 - A_1 \otimes \mathbf{1} - \mathbf{1} \otimes A_1 \tag{4g3}$$

we get

$$\operatorname{Var}(Y_t) = tD + o(t) \qquad \text{for } t \to 0\,. \tag{4g4}$$

Especially, if A_1, A_2 are given by (4c9), then (4g3) conforms with (4f4):

$$D = -\sum_k \sigma_k \otimes \sigma_k = \sum_k i\sigma_k \otimes i\sigma_k\,. \tag{4g5}$$

4g6 Example. Similarly to 4c4, $e^{i\sigma B_t + ivt}$ turns into

$$Y_t = \begin{pmatrix} \cos(\sigma B_t + vt) & \sin(\sigma B_t + vt) \\ -\sin(\sigma B_t + vt) & \cos(\sigma B_t + vt) \end{pmatrix} \in \mathrm{SO}(2)\,.$$

We have

$$\mathbb{E}\, Y_t = e^{-\sigma^2 t/2} \cdot \begin{pmatrix} \cos vt & \sin vt \\ -\sin vt & \cos vt \end{pmatrix}\,; \qquad A_1 = -\frac{\sigma^2}{2} \cdot \mathbf{1} + v \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,;$$

$$\mathbb{E}\,(Y_t \otimes Y_t) = \frac12 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac12 e^{-2\sigma^2 t} \cos 2vt \cdot \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} +$$

$$+ \frac12 e^{-2\sigma^2 t} \sin 2vt \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}\,;$$

$$A_2 = -\sigma^2 \cdot \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} + v \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}\,;$$

$$A_1 \otimes \mathbf{1} + \mathbf{1} \otimes A_1 = -\sigma^2 \cdot \mathbf{1} + v \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}\,;$$

$$D = \sigma^2 \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \sigma^2 \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,.$$

4g7 Exercise. $\mathbb{E}\big((Y_t - \mathbf{1}) \otimes (Y_t - \mathbf{1}) \big) = tD + o(t)$ for $t \to 0$.

Prove it.

Hint: $\mathbb{E}\big((Y_t - \mathbf{1}) \otimes (Y_t - \mathbf{1}) \big) - \operatorname{Var}(Y_t) = (\mathbf{1} - \mathbb{E}\, Y_t) \otimes (\mathbf{1} - \mathbb{E}\, Y_t)$.

One more formula for D will be given by (4g13).

4g8 Exercise. $\mathbb{E}\,|Y_t - \mathbf{1}|^2 = O(t)$ for $t \to 0$.

Prove it.

Hint: use 4g7.

Here and henceforth $|\dots|$ means not only the absolute value of a real or complex number, but also a norm on $\mathrm{M}_n(\mathbb{R})$. The choice of a norm influences only constants.

Our first lemma is just a linear algebra (rather than probability).

4g9 Lemma. Matrices $A_1 \in \mathrm{M}_n(\mathbb{R})$, $A_2 \in \mathrm{M}_{n^2}(\mathbb{R})$ are of the form (4c9) if and only if they satisfy the following two conditions (where D is defined by (4g3)):

(a) $D^{\gamma,\delta}_{\alpha,\beta} = D^{\delta,\gamma}_{\beta,\alpha}$, $D^{\gamma,\delta}_{\alpha,\beta} = -D^{\alpha,\delta}_{\gamma,\beta}$ for all $\alpha,\beta,\gamma,\delta$, and

$$\sum_{\alpha,\beta,\gamma,\delta} D^{\gamma,\delta}_{\alpha,\beta} Z^\gamma_\alpha Z^\delta_\beta \ge 0 \qquad \text{for all } Z \in \mathrm{M}_n(\mathbb{R})\,;$$

(b) $\sum_\gamma D^{\beta,\gamma}_{\gamma,\alpha} = (A_1)^\beta_\alpha + (A_1)^\alpha_\beta$ for all α,β.[3]

Proof. Let A_1, A_2 be of the form (4c9). By (4g5), $D = \sum_k i\sigma_k \otimes i\sigma_k$; it satisfies $D^{\gamma,\delta}_{\alpha,\beta} = D^{\delta,\gamma}_{\beta,\alpha}$ and $D^{\gamma,\delta}_{\alpha,\beta} = -D^{\alpha,\delta}_{\gamma,\beta}$ since $(i\sigma_k)^* = -i\sigma_k$. Also,

$$\sum_{\alpha,\beta,\gamma,\delta} D^{\gamma,\delta}_{\alpha,\beta} Z^\gamma_\alpha Z^\delta_\beta = \sum_k \Big(\sum_{\alpha,\gamma} (i\sigma_k)^\gamma_\alpha Z^\gamma_\alpha \Big) \Big(\sum_{\beta,\delta} (i\sigma_k)^\delta_\beta Z^\delta_\beta \Big) = \sum_k \Big(\sum_{\alpha,\gamma} (i\sigma_k)^\gamma_\alpha Z^\gamma_\alpha \Big)^2 \ge 0\,,$$

thus (a) holds. Further,[4]

$$\sum_\gamma D^{\beta,\gamma}_{\gamma,\alpha} = \sum_k \sum_\gamma (i\sigma_k)^\beta_\gamma (i\sigma_k)^\gamma_\alpha = \sum_k (i\sigma_k \cdot i\sigma_k)^\beta_\alpha = \Big(-\sum_k \sigma_k^2 \Big)^\beta_\alpha\,;$$

but also

$$A_1 + A_1^* = -\sum_k \sigma_k^2\,,$$

thus (b) holds.

Now assume that (a), (b) hold. A positive quadratic form is a sum of squared linear forms:

$$\sum_{\alpha,\beta,\gamma,\delta} D^{\gamma,\delta}_{\alpha,\beta} Z^\gamma_\alpha Z^\delta_\beta = \sum_k \Big(\sum_{\alpha,\beta} (i\sigma_k)^\beta_\alpha Z^\beta_\alpha \Big)^2 \quad \text{for all } Z \in \mathrm{M}_n(\mathbb{R})$$

for some $i\sigma_k \in \mathrm{M}_n(\mathbb{R})$. That is, $D = \sum_k (i\sigma_k) \otimes (i\sigma_k) = -\sum_k \sigma_k \otimes \sigma_k$. If $Z^* = Z$ then

$$\sum_{\alpha,\beta,\gamma,\delta} D^{\gamma,\delta}_{\alpha,\beta} Z^\gamma_\alpha Z^\delta_\beta = -\sum_{\alpha,\beta,\gamma,\delta} D^{\alpha,\delta}_{\gamma,\beta} Z^\gamma_\alpha Z^\delta_\beta = -\sum_{\alpha,\beta,\gamma,\delta} D^{\alpha,\delta}_{\gamma,\beta} Z^\alpha_\gamma Z^\delta_\beta = -\sum_{\alpha,\beta,\gamma,\delta} D^{\gamma,\delta}_{\alpha,\beta} Z^\gamma_\alpha Z^\delta_\beta$$

[3] See also (4g14).

[4] The upper index of a matrix is the row number, the lower index is the column number.

must vanish, thus $\sum_{\alpha,\beta}(i\sigma_k)^\beta_\alpha Z^\beta_\alpha$ must vanish, which means that $(i\sigma_k)^\alpha_\beta = -(i\sigma_k)^\beta_\alpha$, that is, $(i\sigma_k)^* = -i\sigma_k$ and $\sigma_k^* = \sigma_k$.

We have

$$(A_1 + A_1^*)^\beta_\alpha = \sum_\gamma D^{\beta,\gamma}_{\gamma,\alpha} = -\sum_k \sum_\gamma (\sigma_k)^\beta_\gamma (\sigma_k)^\gamma_\alpha = -\sum_k (\sigma_k^2)^\beta_\alpha \,,$$

thus $A_1 + A_1^* = -\sum_k \sigma_k^2$. Introducing $iv \in \mathrm{M}_n(\mathbb{R})$ by

$$iv = \frac12 (A_1 - A_1^*)$$

we get

$$A_1 = \frac12 (A_1 + A_1^*) + \frac12 (A_1 - A_1^*) = -\frac12 \sum_k \sigma_k^2 + iv \,.$$

Finally,

$$\begin{aligned} A_2 = D + A_1 \otimes \mathbf{1} + \mathbf{1} \otimes A_1 = \\ = -\sum_k \sigma_k \otimes \sigma_k - \frac12 \sum_k (\sigma_k^2 \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_k^2) + iv \otimes \mathbf{1} + \mathbf{1} \otimes iv = \\ = -\frac12 \sum_k (\sigma_k \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_k)^2 + i(v \otimes \mathbf{1} + \mathbf{1} \otimes v) \,; \end{aligned}$$

we see that A_1, A_2 are of the form (4c9). □

4g10 Exercise. Let random variables $M_t \ge 0$, $0 < t < 1$, satisfy (for $t \to 0$)

- $\|M_t\|_{L_\infty} = O(1)$;
- $\|M_t\|_{L_2} = O(\sqrt{t})$;
- $\mathbb{P}\big(M_t \ge \varepsilon \big) = o(t)$ for every $\varepsilon > 0$.

Then $\mathbb{E}\, M_t^3 = o(t)$.

Prove it.

Hint: $\big(\min(M_t, \varepsilon)\big)^3 \le \varepsilon M_t^2$.

4g11 Exercise. Let $(Y_t)_t$ be a Brownian motion in $\mathrm{SO}(n)$. Then $\mathbb{E}\,|Y_t - \mathbf{1}|^3 = o(t)$ for $t \to 0$.

Prove it.

Hint: apply 4g10 to $M_t = |Y_t - \mathbf{1}|$, taking into account 4g8 and 1e1; $\mathrm{SO}(n)$ is not $\mathbb{R}$, but still has an invariant metric.

4g12 Lemma. For every Brownian motion $(Y_t)_t$ in $\mathrm{SO}(n)$ the matrices A_1, A_2 defined by (4c3) satisfy conditions 4g9(a,b).

Proof. Clearly, $D^{\gamma,\delta}_{\alpha,\beta} = D^{\delta,\gamma}_{\beta,\alpha}$. Also,

$$\begin{aligned} \sum_{\alpha,\beta,\gamma,\delta} D^{\gamma,\delta}_{\alpha,\beta} Z^\gamma_\alpha Z^\delta_\beta = \frac{d}{dt}\Big|_{t=0} \sum_{\alpha,\beta,\gamma,\delta} \big(\mathbb{E} Y_t \otimes Y_t - (\mathbb{E} Y_t) \otimes (\mathbb{E} Y_t) \big)^{\gamma,\delta}_{\alpha,\beta} Z^\gamma_\alpha Z^\delta_\beta = \\ = \frac{d}{dt}\Big|_{t=0} \operatorname{Var} \sum_{\alpha,\beta} (Y_t)^\beta_\alpha Z^\beta_\alpha \ge 0 \,. \end{aligned}$$

Every $U \in \mathrm{SO}(n)$ satisfies $\mathbf{1} = UU^* = \big(\mathbf{1} + (U - \mathbf{1})\big)\big(\mathbf{1} + (U^* - \mathbf{1})\big) = \mathbf{1} + (U - \mathbf{1}) + (U^* - \mathbf{1}) + O(|U - \mathbf{1}|^2)$, thus

$$\frac{1}{2}(U + U^*) = \mathbf{1} + O(|U - \mathbf{1}|^2)\,,$$

and

$$U = \frac{1}{2}(U + U^*) + \frac{1}{2}(U - U^*) = \mathbf{1} + \frac{1}{2}(U - U^*) + O(|U - \mathbf{1}|^2)\,.$$

We have

$$(Y_t - \mathbf{1}) \otimes (Y_t - \mathbf{1}) = \big(\tfrac{1}{2}(Y_t - Y_t^*) + O(|Y_t - \mathbf{1}|^2)\big) \otimes \big(\tfrac{1}{2}(Y_t - Y_t^*) + O(|Y_t - \mathbf{1}|^2)\big) = \\ = \frac{1}{4}(Y_t - Y_t^*) \otimes (Y_t - Y_t^*) + O(|Y_t - \mathbf{1}|^3)$$

(with an absolute constant in $O(\dots)$); by 4g11,

$$\mathbb{E}\big((Y_t - \mathbf{1}) \otimes (Y_t - \mathbf{1})\big) = \frac{1}{4}\mathbb{E}\big((Y_t - Y_t^*) \otimes (Y_t - Y_t^*)\big) + o(t)\,.$$

Using 4g7,

$$D = \frac{d}{dt}\bigg|_{t=0} \frac{1}{4}\mathbb{E}\big((Y_t - Y_t^*) \otimes (Y_t - Y_t^*)\big)\,, \tag{4g13}$$

which ensures $D_{\alpha,\beta}^{\gamma,\delta} = -D_{\gamma,\beta}^{\alpha,\delta}$ and finishes the proof of (a).

For proving (b) we start with the equality $\mathbb{E}Y_tY_t^* = 1$;

$$\mathbf{1}_\alpha^\beta = \mathbb{E}\sum_\gamma (Y_t)_\gamma^\beta (Y_t^*)_\alpha^\gamma = \sum_\gamma \big(\mathbb{E}Y_t \otimes Y_t\big)_{\gamma,\gamma}^{\beta,\alpha} = \\ = \sum_\gamma \big(\mathbf{1} + tA_2 + o(t)\big)_{\gamma,\gamma}^{\beta,\alpha} = \mathbf{1}_\alpha^\beta + t\sum_\gamma (A_2)_{\gamma,\gamma}^{\beta,\alpha} + o(t)\,,$$

which means that

$$\sum_\gamma (A_2)_{\gamma,\gamma}^{\beta,\alpha} = 0 \quad \text{for all } \alpha, \beta\,. \tag{4g14}$$

Therefore

$$\sum_\gamma D_{\gamma,\alpha}^{\beta,\gamma} = -\sum_\gamma D_{\gamma,\gamma}^{\beta,\alpha} = -\sum_\gamma (A_2 - A_1 \otimes \mathbf{1} - \mathbf{1} \otimes A_1)_{\gamma,\gamma}^{\beta,\alpha} = \\ = -\sum_\gamma (A_2)_{\gamma,\gamma}^{\beta,\alpha} + \sum_\gamma (A_1)_\gamma^\beta \mathbf{1}_\gamma^\alpha + \sum_\gamma \mathbf{1}_\gamma^\beta (A_1)_\gamma^\alpha = 0 + (A_1)_\alpha^\beta + (A_1)_\beta^\alpha\,.$$

□

Our third lemma is stronger than the uniqueness theorem 4e1.

4g15 Lemma. A Brownian motion in $\mathrm{SO}(n)$ is uniquely determined by generators A_1, A_2 (defined by (4c3)).

Proof. Similarly to 4e it is sufficient to prove that the higher tensor moment generators $A_3, A_4, \dots$ are uniquely determined by A_1, A_2. (In fact we will see that the relations found in 4e hold in general.) Denoting for convenience $Y_t - \mathbf{1} = Z_t$ we have

$$\begin{aligned} e^{tA_3} &= \mathbb{E}\,(Y_t \otimes Y_t \otimes Y_t) = \mathbb{E}\big((\mathbf{1}+Z_t)\otimes(\mathbf{1}+Z_t)\otimes(\mathbf{1}+Z_t)\big) = \\ &= \mathbf{1} + \mathbb{E}\big(Z_t \otimes \mathbf{1} \otimes \mathbf{1} + \text{two such terms}\big) + \mathbb{E}\big(Z_t \otimes Z_t \otimes \mathbf{1} + \text{two such terms}\big) + \mathbb{E}\,(Z_t \otimes Z_t \otimes Z_t) = \\ &= \mathbf{1} + (e^{tA_1} - \mathbf{1}) \otimes \mathbf{1} \otimes \mathbf{1} + \text{two such terms} + \\ &\quad + (e^{tA_2} - e^{tA_1} \otimes \mathbf{1} - \mathbf{1} \otimes e^{tA_1} + \mathbf{1}) \otimes \mathbf{1} + \text{two such terms} + o(t) \end{aligned}$$

by 4g11. That is,

$$\begin{gathered} \mathbf{1} + tA_3 + o(t) = \mathbf{1} + t(A_1 \otimes \mathbf{1} \otimes \mathbf{1} + \text{two such terms}) + \\ + t\big(\underbrace{(A_2 - A_1 \otimes \mathbf{1} - \mathbf{1} \otimes A_1)}_{=D} \otimes \mathbf{1} + \text{two such terms}\big) + o(t)\,; \\ A_3 = (A_1 \otimes \mathbf{1} \otimes \mathbf{1} + \text{two such terms}) + (D \otimes \mathbf{1} + \text{two such terms})\,; \end{gathered}$$

namely,

$$(A_3)^{\delta,\varepsilon,\zeta}_{\alpha,\beta,\gamma} = (A_1)^\delta_\alpha \mathbf{1}^\varepsilon_\beta \mathbf{1}^\zeta_\gamma + \mathbf{1}^\delta_\alpha (A_1)^\varepsilon_\beta \mathbf{1}^\zeta_\gamma + \mathbf{1}^\delta_\alpha \mathbf{1}^\varepsilon_\beta (A_1)^\zeta_\gamma + D^{\delta,\varepsilon}_{\alpha,\beta} \mathbf{1}^\zeta_\gamma + D^{\delta,\zeta}_{\alpha,\gamma} \mathbf{1}^\varepsilon_\beta + D^{\varepsilon,\zeta}_{\beta,\gamma} \mathbf{1}^\delta_\alpha\,.$$

The same argument works for $A_4, A_5, \dots$ □