2 Stochastic integration: Wiener chaos

2a Discrete model suggests

Consider the standard one-dimensional random walk with time pitch $1/n$ and space pitch $1/\sqrt{n}$; here n is a (large) parameter. The walk is a random process $X = (X_t)_{t\in\{0,\frac1n,\frac2n,\dots\}}$, $X_t \in L_0(\Omega,\mathbb{R})$, with stationary independent increments such that

$$\mathbb{P}\Big(X_{1/n} = -\frac{1}{\sqrt{n}}\Big) = \frac12 = \mathbb{P}\Big(X_{1/n} = +\frac{1}{\sqrt{n}}\Big).$$

Clearly,

$$\mathbb{E}X_t = 0\,, \qquad \operatorname{Var}(X_t) = t$$

for all $t \in \{0,\frac1n,\frac2n,\dots\}$. By a classical limit theorem, the distribution of (say) X_1 is approximately normal, if n is large. We may treat X as a discrete counterpart of the standard Brownian motion $B = (B_t)_{t\in[0,\infty)}$ in $\mathbb{R}$. Increments $X_{s,t} = X_t - X_s$ form a discrete-time stationary abstract stochastic flow;

$$X_{0,1} = X_{0,\frac1n} + X_{\frac1n,\frac2n} + \dots + X_{\frac{n-1}{n},1}\,.$$

The homomorphism $x \mapsto e^{ix}$ of the additive group $\mathbb{R}$ to the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (the circle) transforms X into a random walk (or flow) in $\mathbb{T}$,

$$Y_{s,t} = \exp(iX_{s,t}) = \cos X_{s,t} + i\sin X_{s,t}\,,$$

and B into a Brownian motion (or flow) (e^{iB}) in $\mathbb{T}$. We have

$$Y_{0,\frac1n} = \cos\frac{1}{\sqrt{n}} \pm i\sin\frac{1}{\sqrt{n}} = \cos\frac{1}{\sqrt{n}} + iX_{0,\frac1n}\sqrt{n}\sin\frac{1}{\sqrt{n}}\,;$$

$$Y_{0,1} = \Big(\cos\frac{1}{\sqrt{n}} + iX_{0,\frac1n}\sqrt{n}\sin\frac{1}{\sqrt{n}}\Big)\dots\Big(\cos\frac{1}{\sqrt{n}} + iX_{\frac{n-1}{n},1}\sqrt{n}\sin\frac{1}{\sqrt{n}}\Big) =$$
$$= \sum_{m=0}^n \Big(\cos\frac{1}{\sqrt{n}}\Big)^{n-m}\Big(i\sqrt{n}\sin\frac{1}{\sqrt{n}}\Big)^m I_m\,,$$

where

$$I_m = I_{m,n} = \sum_{0\le\frac{k_1}{n}<\dots<\frac{k_m}{n}<1} \Delta X_{\frac{k_1}{n}}\dots\Delta X_{\frac{k_m}{n}}\,, \qquad \Delta X_{\frac{k}{n}} = X_{\frac{k}{n},\frac{k+1}{n}}\,.$$

Random variables $I_0, I_1, \dots, I_n$ are orthogonal, and

$$\|I_m\|^2_{L_2(\Omega)} = \binom{n}{m}\frac{1}{n^m}\,;$$

$$\sum_{m=0}^n \Big(\cos^2\frac{1}{\sqrt{n}}\Big)^{n-m}\Big(n\sin^2\frac{1}{\sqrt{n}}\Big)^m\binom{n}{m}\frac{1}{n^m} = \Big(\cos^2\frac{1}{\sqrt{n}} + \sin^2\frac{1}{\sqrt{n}}\Big)^n = 1 = \|Y_{0,1}\|^2\,,$$

as it should be. For large n the binomial distribution described by $\Bigl(\cos^2\frac{1}{\sqrt{n}}+\sin^2\frac{1}{\sqrt{n}}\Bigr)^n$ approaches the Poisson distribution (of parameter 1),

$$\binom{n}{m}\Bigl(\cos^2\frac{1}{\sqrt{n}}\Bigr)^{n-m}\Bigl(\sin^2\frac{1}{\sqrt{n}}\Bigr)^m \xrightarrow[n\to\infty]{} \frac{1}{m!}e^{-1}\,.$$

A continuous counterpart is suggested:

$$\exp(iB_{0,1})=\sum_{m=0}^{\infty}e^{-1/2}i^m I_m$$

for some orthogonal $I_m\in L_2(\Omega)$ satisfying

$$\|I_m\|^2=\frac{1}{m!}\,.$$

It should be

$$I_m=\int\cdots\int_{0<t_1<\dots<t_m<1}dB_{t_1}\dots dB_{t_m}\,,$$

but for now it is an equality between two undefined symbols.

2b Hermite polynomials

The same argument as in 2a is applicable to $\exp(i\lambda X_{0,1})$ for all $\lambda\in\mathbb{R}$:

$$\exp(i\lambda X_{0,\frac1n})=\cos\frac{\lambda}{\sqrt{n}}\pm i\sin\frac{\lambda}{\sqrt{n}}=\cos\frac{\lambda}{\sqrt{n}}+iX_{0,\frac1n}\sqrt{n}\sin\frac{\lambda}{\sqrt{n}}\,;$$

$$\exp(i\lambda X_{0,1})=\sum_{m=0}^{n}\Bigl(\cos\frac{\lambda}{\sqrt{n}}\Bigr)^{n-m}\Bigl(i\sqrt{n}\sin\frac{\lambda}{\sqrt{n}}\Bigr)^m I_m\,,$$

I_m being the same as before;

$$\Bigl(\cos\frac{\lambda}{\sqrt{n}}\Bigr)^n\xrightarrow[n\to\infty]{}\exp\Bigl(-\frac{\lambda^2}{2}\Bigr)\,,\qquad \sqrt{n}\sin\frac{\lambda}{\sqrt{n}}\xrightarrow[n\to\infty]{}\lambda\,,$$

and we guess that

(2b1)
$$\exp(i\lambda B_{0,1})=\exp\Bigl(-\frac{\lambda^2}{2}\Bigr)\sum_{m=0}^{\infty}i^m\lambda^m I_m$$

for all λ. Thus,

$$\sum_{m=0}^{\infty}i^m\lambda^m I_m=\exp\Bigl(i\lambda B_{0,1}+\frac{\lambda^2}{2}\Bigr)=$$
$$=1+\Bigl(i\lambda B+\frac12\lambda^2\Bigr)+\frac12\Bigl(i\lambda B+\frac12\lambda^2\Bigr)^2+\frac16\Bigl(i\lambda B+\frac12\lambda^2\Bigr)^3+\dots=$$
$$=1+i\lambda B+\frac12\lambda^2(1-B^2)+\frac16\lambda^3(3iB-iB^3)+\dots$$

(here $B = B_{0,1}$), which means that

$$I_0 = 1\,; \quad I_1 = B_{0,1}\,; \quad I_2 = \frac{1}{2}(B_{0,1}^2 - 1)\,; \quad I_3 = \frac{1}{6}(B_{0,1}^3 - 3B_{0,1})\,,$$

and so on. That is,

$$\begin{gathered} \int_0^1 dB_t = B_{0,1}\,; \\ \iint\limits_{0<s<t<1} dB_s dB_t = \frac{1}{2}(B_{0,1}^2 - 1)\,; \\ \iiint\limits_{0<r<s<t<1} dB_r dB_s dB_t = \frac{1}{6}(B_{0,1}^3 - 3B_{0,1})\,, \end{gathered} \tag{2b2}$$

and so on. Polynomials on the right-hand side are well-defined, and we may use these equalities for *defining* the integrals on the left-hand side!

2b3 Definition. *Hermite polynomials* $H_n(\cdot)$ are defined by the formula

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} H_n(x) = \exp\Big(xy - \frac{y^2}{2} \Big)\,.$$

(Be warned, however: some authors use $H_n(x\sqrt{2})$ or $2^{n/2}H_n(x\sqrt{2})$, call these functions 'Hermite polynomials' and denote them $H_n(x)$.) In other words,[1]

$$H_n(x) = \frac{d^n}{dy^n}\Big|_{y=0} \exp\Big(xy - \frac{y^2}{2} \Big) = \frac{1}{i^n}\frac{d^n}{d\lambda^n}\Big|_{\lambda=0} \exp\Big(i\lambda x + \frac{\lambda^2}{2} \Big)\,. \tag{2b4}$$

Also,

$$\begin{aligned} H_1(x) &= x\,; & x &= H_1(x)\,; \\ H_2(x) &= x^2 - 1\,; & x^2 &= H_2(x) + 1\,; \\ H_3(x) &= x^3 - 3x\,; & x^3 &= H_3(x) + 3H_1(x)\,; \\ H_4(x) &= x^4 - 6x^2 + 3\,; & x^4 &= H_4(x) + 6H_2(x) + 3\,. \end{aligned}$$

In fact,

$$\begin{gathered} \mathbb{E}\, H_m(X) H_n(X) = 0 \quad \text{if } m \ne n \text{ and } X \sim N(0,1)\,; \\ \text{that is, } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2/2}\, dx = 0 \quad \text{for } m \ne n\,; \end{gathered} \tag{2b5}$$

[1] Some classical formulas for Hermite polynomials:

$$\begin{gathered} H_n(x) = (-1)^n \exp\Big(\frac{1}{2}x^2 \Big) \frac{d^n}{dx^n} \exp\Big(-\frac{1}{2}x^2 \Big)\,; \\ \frac{d}{dx} H_n(x) = nH_{n-1}(x)\,; \\ H_{n+1}(x) = xH_n(x) - nH_{n-1}(x) = \Big(x - \frac{d}{dx} \Big) H_n(x)\,. \end{gathered}$$

the classical orthogonality property of Hermite polynomials;

(2b6)
$$\mathbb{E}\, H_n^2(X) = n! \quad \text{if } X \sim N(0,1)\,;$$
$$\text{that is, } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} H_n^2(x) e^{-x^2/2}\, dx = n!\,;$$

a proof will be given later (after 2d4).

Functions $\frac{1}{\sqrt{n!}} H_n(\cdot)$ are orthonormal in $L_2(\mathrm{N}(0,1))$; in other words, functions $x \mapsto \frac{1}{\sqrt{n!}} H_n(\cdot)\sqrt{(2\pi)^{-1/2} e^{-x^2/2}}$ are orthonormal in $L_2(\mathbb{R})$. Whether they are a basis of the whole L_2, or only a subspace? That is, are polynomials dense in $L_2(\mathrm{N}(0,1))$? Of course, every continuous function can be approximated by polynomials on any bounded interval; but these polynomials are large outside the interval.

We have (recall (2b3))

(2b7)
$$\sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} H_n(x) = e^{\lambda^2/2} e^{i\lambda x}\,;$$

the convergence is pointwise (in fact, uniform on bounded intervals). On the other hand, the series converges (to something!) in $L_2(\mathrm{N}(0,1))$, since

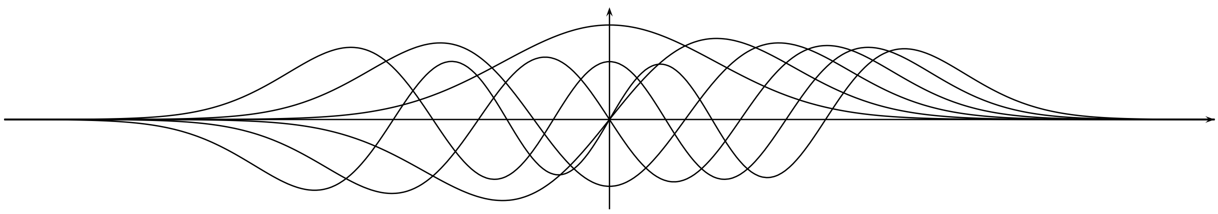
$$\sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} H_n(x) = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{\sqrt{n!}} \cdot \frac{1}{\sqrt{n!}} H_n(x)\,, \quad \text{and} \sum \left| \frac{(i\lambda)^n}{\sqrt{n!}} \right|^2 = e^{\lambda^2} < \infty\,.$$

The two limits must conform (think, why);

(2b8)
$$\sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} H_n(x) = e^{\lambda^2/2} e^{i\lambda x} \quad \text{in } L_2(\mathrm{N}(0,1))\,.$$

We see that the function $x \mapsto e^{i\lambda x}$ belongs to the subspace (spanned by Hermite polynomials), for every $\lambda \in \mathbb{R}$. Their linear combinations are dense in continuous periodic functions[2] And continuous periodic functions (of all periods) are dense in $L_2(\mathrm{N}(0,1))$. So,

(2b9)
$$\text{functions } \frac{1}{\sqrt{n!}} H_n(\cdot) \text{ are an orthonormal basis of } L_2(\mathrm{N}(0,1))\,;$$
$$\text{functions } x \mapsto \frac{1}{\sqrt{n!}} H_n(x) \sqrt{(2\pi)^{-1/2} e^{-x^2/2}} \text{ are an orthonormal basis of } L_2(\mathbb{R})\,.$$



[2] Alternatively, one may use periodic step functions.

2c Intuitive ideas

Comparing (2b1) and 2b3 (or rather (2b8)) we may *define*

(2c1) $$\int\cdots\int_{0<t_1<\dots<t_m<1} dB_{t_1}\dots dB_{t_m} = \frac{1}{m!}H_m(B_{0,1})$$

(see also (2b2)). Of course, it is a tentative definition; the general case will be treated in 2d. We have

$$\exp(i\lambda B_{0,1}) = \exp\Big(-\frac{\lambda^2}{2}\Big)\sum_{m=0}^\infty i^m\lambda^m \int\cdots\int_{0<t_1<\dots<t_m<1} dB_{t_1}\dots dB_{t_m}\,.$$

What about the integral over (say) $0<t_1<\dots<t_m<4$? Taking into account that the processes

$(B_t)_{t\in[0,\infty)}$ and $\big(\frac12 B_{4t}\big)_{t\in[0,\infty)}$ are identically distributed

(think, why), it should be

$$\int\cdots\int_{0<t_1<\dots<t_m<1} \big(\tfrac12 dB_{4t_1}\big)\dots\big(\tfrac12 dB_{4t_m}\big) = \frac{1}{m!}H_m\big(\tfrac12 B_{0,4}\big)\,,$$

that is,

$$\int\cdots\int_{0<t_1<\dots<t_m<4} dB_{t_1}\dots dB_{t_m} = \frac{2^m}{m!}H_m\big(\tfrac12 B_{0,4}\big)\,,$$

and more generally,

(2c2) $$\int\cdots\int_{0<t_1<\dots<t_m<t} dB_{t_1}\dots dB_{t_m} = \frac{t^{m/2}}{m!}H_m\Big(\frac{1}{\sqrt t}B_{0,t}\Big)$$

for any $t\in(0,\infty)$; note that $\frac{1}{\sqrt t}B_{0,t}\sim \mathrm N(0,1)$. The argument above is heuristical; for now, we extend the tentative definition (2c1) to (still tentative) definition (2c2). For example,

$$\iint_{0<r<s<t} dB_r dB_s = \frac t2\Big(\frac1t B_{0,t}^2 - 1\Big) = \frac12(B_{0,t}^2 - t)\,,$$

which is paradoxical; we could expect that

$$\iint_{0<r<s<t} dB_r dB_s = \int_0^t\Big(\int_0^s dB_r\Big)dB_s = \int_0^t B_s\,dB_s = \frac12\int_0^t d(B_s^2) = \frac12 B_{0,t}^2$$

(arguments being heuristic, still). Why "$\dots - t$"? Look closer at the formula $d(B_s^2) = 2B_s\,dB_s$. Denoting for short $B=B_s$, $\Delta B = B_{s,s+\Delta s}$ we have $\Delta(B^2) = (B+\Delta B)^2 - B^2 = 2B\Delta B + (\Delta B)^2$. In the usual (smooth, non-stochastic) analysis one discards $(\Delta B)^2$ since it is much smaller than Δs. However, the Brownian sample path is far from being smooth; in fact, $\mathbb E\,(\Delta B)^2 = \Delta t$. The astonishing formula $\frac12(B_{0,t}^2 - t)$ means that[3]

(2c3) $$(dB_t)^2 = dt\,,$$

[3] In fact, $df(B_t) = f'(B_t)\,dB_t + \frac12 f''(B_t)\,dt$ (for any smooth $f:\mathbb R\to\mathbb R$) by the famous Itô formula.

and indeed,

$$\sum_{k=1}^{n} \big(B_{\frac{k-1}{n},\frac{k}{n}} \big)^2 \xrightarrow[n\to\infty]{} 1 \quad \text{in probability}$$

by the weak law of large numbers.

2c4 Exercise. Prove that

$$\exp(i\lambda B_{0,t}) = \exp\Big(-\frac12 \lambda^2 t \Big) \sum_{m=0}^{\infty} i^m \lambda^m \int \dots \int_{0<t_1<\dots<t_m<t} dB_{t_1} \dots dB_{t_m} \, .$$

(Give a proof, not a heuristic argument!)

Hint.

$$t^{m/2} H_m \Big(\frac{x}{\sqrt t} \Big) = \frac{1}{i^m} \frac{d^m}{d\lambda^m} \Big|_{\lambda=0} \exp \Big(i\lambda x + \frac{\lambda^2 t}{2} \Big) \, . \tag{2c5}$$

Note that

$$\int \dots \int_{0<t_1<\dots<t_m<t} dB_{t_1} \dots dB_{t_m} = \frac{1}{i^m m!} \frac{d^m}{d\lambda^m} \Big|_{\lambda=0} \exp \Big(i\lambda B_{0,t} + \frac{\lambda^2 t}{2} \Big) \, . \tag{2c6}$$

Consider two equalities, say,

$$\exp(3i\lambda B_{0,0.5}) = \exp\Big(-\frac{3^2\lambda^2}{2\cdot 2} \Big) \sum_{n=0}^{\infty} i^n 3^n \lambda^n I_n(0,0.5) \, ,$$

$$\exp(5i\lambda B_{0.5,1}) = \exp\Big(-\frac{5^2\lambda^2}{2\cdot 2} \Big) \sum_{n=0}^{\infty} i^n 5^n \lambda^n I_n(0.5,1) \, ;$$

here $I_n(a,b) = \int \dots \int_{a<t_1<\dots<t_n<b} dB_{t_1} \dots dB_{t_n}$. The former equality is a special case of (2c4), the latter is its natural generalization. (Our arguments are still heuristic.) Multiplying the two equalities we get

$$\exp\big(i\lambda (3B_{0,0.5} + 5B_{0.5,1}) \big) = \exp\Big(-\frac12 \cdot \frac{3^2+5^2}{2} \lambda^2 \Big) \sum_{n=0}^{\infty} i^n \lambda^n \sum_{k=0}^{n} \big(3^k I_k(0,0.5) \cdot 5^{n-k} I_{n-k}(0.5,1) \big) \, ,$$

which suggests that

$$\exp\Big(i\lambda \int_0^1 f(t)\, dB_t \Big) = \exp\Big(-\frac{\lambda^2}{2} \int_0^1 f^2(t)\, dt \Big) \sum_{n=0}^{\infty} i^n \lambda^n I_{n,f} \, ,$$

$$f(t) = \begin{cases} 3 & \text{for } t \in (0,0.5), \\ 5 & \text{for } t \in (0.5,1), \end{cases}$$

$$I_{n,f} = \int \dots \int_{0<t_1<\dots<t_n<1} f(t_1) \dots f(t_n)\, dB_{t_1} \dots dB_{t_n} = \sum_{k=0}^{n} \big(3^k I_k(0,0.5) \cdot 5^{n-k} I_{n-k}(0.5,1) \big) \, .$$

Note that

$$I_{n,f} = \frac{1}{i^n n!} \frac{d^n}{d\lambda^n}\Big|_{\lambda=0} \exp\Big(i\lambda \int_0^1 f(t)\,dB_t + \frac{\lambda^2}{2} \int_0^1 f^2(t)\,dt \Big) .$$

Similarly,

$$\int\!\!\cdots\!\!\int_{0<t_1<\dots<t_n<\infty} f(t_1)\dots f(t_n)\,dB_{t_1}\dots dB_{t_n} = \frac{1}{i^n n!} \frac{d^n}{d\lambda^n}\Big|_{\lambda=0} \exp\Big(i\lambda \int_0^\infty f(t)\,dB_t + \frac{\lambda^2}{2} \int_0^\infty f^2(t)\,dt \Big)$$

for every step function $f : [0,\infty) \to \mathbb{R}$, that is, a *finite* linear combination of indicators $\mathbf{1}_{(a,b)}$, $0 \le a < b < \infty$. Here $\int f(t)\,dB_t$ may be defined by

$$\int_0^\infty f(t)\,dB_t = - \int_0^\infty B_t\,df(t) = - \sum_t B_t\big(f(t+) - f(t-)\big) ,$$

the sum being taken over all discontinuities of f. Note that $\int \mathbf{1}_{(a,b)}(t)\,dB_t = B_{a,b}$.

2d Definitions

Let $(B_t)_{t\in[0,\infty)}$, $B_t \in L_0(\Omega,\mathbb{R})$, be the standard Brownian motion.

2d1 Exercise. There exists one and only one isometric linear operator $T : L_2(0,\infty) \to L_2(\Omega)$ such that

$$T(\mathbf{1}_{(0,t)}) = B_t \quad \text{for } t \in [0,\infty) .$$

Prove it.

Hint: $\langle \mathbf{1}_{(0,s)}, \mathbf{1}_{(0,t)} \rangle = \langle B_s, B_t \rangle$; define T on step functions and extend it by continuity.

2d2 Definition. The *linear stochastic integral*

$$\int_0^\infty f(t)\,dB_t$$

for $f \in L_2(0,\infty)$ is the random variable Tf, where T is as in 2d1.

Note that $\int f\,dB \sim \mathrm{N}(0,\|f\|^2)$ and $\int \mathbf{1}_{(s,t)}\,dB = B_{s,t} = B_t - B_s$.

2d3 Exercise. The (nonlinear) map $\mathcal{E} : L_2(0,\infty) \to L_2(\Omega)$ defined by

$$\mathcal{E}(f) = \exp\Big(i \int f\,dB + \tfrac12 \|f\|^2 \Big)$$

satisfies

$$\langle \mathcal{E}(f), \mathcal{E}(g) \rangle = e^{\langle f,g \rangle} \quad \text{for all } f,g .$$

Prove it.

Hint: if $X \sim \mathrm{N}(0,1)$ then $\mathbb{E}\, e^{i\lambda X} = e^{-\lambda^2/2}$.

2d4 Exercise. The (nonlinear) map $\mathcal{E}_n : L_2(0,\infty) \to L_2(\Omega)$ defined (for a given n) by

$$\mathcal{E}_n(f) = \frac{d^n}{d\lambda^n}\Big|_{\lambda=0} \mathcal{E}(\lambda f)$$

satisfies

$$\langle \mathcal{E}_n(f), \mathcal{E}_n(g) \rangle = n! \langle f, g \rangle^n \quad \text{for all } f, g \, .$$

Prove it.

Hint: differentiate 2d3.

Clearly, $\mathcal{E}_n(\lambda f) = \lambda^n \mathcal{E}_n(f)$ for $\lambda \in \mathbb{R}$. Also, if $\|f\| = 1$ then $\mathcal{E}_n(f) = i^n H_n\big(\int f \, dB\big)$ by (2b4).[4] Now 2d4 gives (2b5), (2b6) and moreover, $\mathbb{E}\big(H_n(X)H_n(Y)\big) = (\mathbb{E}XY)^n$ whenever X, Y are jointly normal, $X \sim \mathrm{N}(0,1)$, $Y \sim \mathrm{N}(0,1)$.

We denote

$$\Delta_n = \{(t_1, \dots, t_n) : 0 < t_1 < \dots < t_n < \infty\}$$

and for every $f \in L_2(0,\infty)$ we define $\mathcal{D}_n(f) \in L_2(\Delta_n)$ by

$$\mathcal{D}_n(f)(t_1, \dots, t_n) = f(t_1) \dots f(t_n) \quad \text{for } (t_1, \dots, t_n) \in \Delta_n \, .$$

2d5 Exercise. $\langle \mathcal{D}_n(f), \mathcal{D}_n(g) \rangle = \frac{1}{n!} \langle f, g \rangle^n$ for all $f, g \in L_2(0,\infty)$. (We equip Δ_n with the Lebesgue measure.)

Prove it.

Hint: the integral over Δ_n is $(1/n!)$ of the integral over the whole $(0,\infty)^n$.

2d6 Exercise. (a) Functions of the form $(t_1, \dots, t_n) \mapsto f_1(t_1) \dots f_n(t_n)$ are linear combinations of functions of the form $\mathcal{D}_n(f)$.

(b) The closed linear subspace of $L_2(\Delta_n)$ spanned by functions of the form $\mathcal{D}_n(f)$ is the whole $L_2(\Delta_n)$.

Prove it.

Hint. (a) For $n = 2$: $ab = \frac{1}{4}(a+b)^2 - \frac{1}{4}(a-b)^2$. In general: $a_1 \dots a_n = \frac{1}{2^n n!} \sum_{\sigma_1, \dots, \sigma_n \in \{-1,+1\}} \sigma_1 \dots \sigma_n (\sigma_1 a_1 + \dots + \sigma_n a_n)^n$.

(b): Consider $\mathbf{1}_{(a_1,b_1)}(t_1) \dots \mathbf{1}_{(a_n,b_n)}(t_n)$.

2d7 Exercise. For each n there exists one and only one isometric linear operator $T_n : L_2(\Delta_n) \to L_2(\Omega)$ such that

$$T_n\big(\mathcal{D}_n(f)\big) = \frac{1}{n!} \mathcal{E}_n(f) \quad \text{for all } f \in L_2(0,\infty) \, .$$

Prove it.

Hint: $\langle \mathcal{E}_n(f), \mathcal{E}_n(g) \rangle = (n!)^2 \langle \mathcal{D}_n(f), \mathcal{D}_n(g) \rangle$; define T on linear combinations of $\mathcal{D}_n(f)$ and extend it by continuity.

[4] Thus, $\mathcal{E}_n(f) = i^n \|f\|^n H_n\big(\frac{1}{\|f\|} \int f \, dB\big)$, which conforms to (2c5).

2d8 Definition. The *multiple stochastic integral*

$$\int\cdots\int_{0<t_1<\dots<t_n<\infty} f(t_1,\dots,t_n)\,dB_{t_1}\dots dB_{t_n}$$

for $f \in L_2(\Delta_n)$ is the random variable $i^{-n}T_n f$, where T_n is as in 2d7.

Summing over all permutations of $t_1,\dots,t_n$ we get the integral over the whole $(0,\infty)^n$ excluding however 'diagonal' hyperplanes $\{t : t_k = t_l\}$. That is, we *define*

(2d9) $$\int\cdots\int_{t_1,\dots,t_n \text{ differ}} f(t_1,\dots,t_n)\,dB_{t_1}\dots dB_{t_n} = \sum_j \int\cdots\int_{0<t_1<\dots<t_n<\infty} f(t_{j_1},\dots,t_{j_n})\,dB_{t_1}\dots dB_{t_n}\,,$$

where j runs over all the $n!$ permutations of $1,\dots,n$.

2e Some properties

2e1 Exercise. For all $f,g \in L_2(0,\infty)$

$$\iint_{s\ne t} f(s)g(t)\,dB_s dB_t = -\frac{\partial^2}{\partial\lambda\partial\mu}\Big|_{\lambda=\mu=0} \mathcal{E}(\lambda f + \mu g)\,.$$

Prove it.

Hint. Both sides are symmetric bilinear forms of f,g. It is sufficient to prove the equality for $f = g$ (recall 2d6).

2e2 Exercise. If $f,g \in L_2(0,\infty)$ are orthogonal then

$$\mathcal{E}(f+g) = \mathcal{E}(f)\mathcal{E}(g)\,.$$

Prove it.

2e3 Exercise. If $f,g \in L_2(0,\infty)$ are orthogonal then

$$\iint_{s\ne t} f(s)g(t)\,dB_s dB_t = \Big(\int f\,dB\Big)\Big(\int g\,dB\Big)\,.$$

Prove it.

Hint: use 2e1, 2e2.

2e4 Exercise. Generalize 2e3 to n orthogonal functions.

Hint. First, generalize 2e1.

2e5 Exercise. Generalize 2e3 to arbitrary (not just orthogonal) f,g. Does the result conforms to the informal relation (2c3)?

2e6 Exercise. Let $0 \le s_0 < s_1 < \dots < s_n < \infty$.

(a)

$$\int \dots \int_{0<t_1<\dots<t_n<\infty} \mathbf{1}_{(s_0,s_1)}(t_1) \dots \mathbf{1}_{(s_{n-1},s_n)}(t_n) \, dB_{t_1} \dots dB_{t_n} = B_{s_0,s_1} \dots B_{s_{n-1},s_n} \, .$$

(b) Moreover, for any $f_1 \in L_2(s_0,s_1), \dots, f_n \in L_2(s_{n-1},s_n)$

$$\int \dots \int_{0<t_1<\dots<t_n<\infty} f_1(t_1) \dots f_n(t_n) \, dB_{t_1} \dots dB_{t_n} = \Big(\int f_1 \, dB \Big) \dots \Big(\int f_n \, dB \Big)$$

(the functions being extended by 0).

Prove it.

Hint: use 2e4.

2e7 Definition. Let $0 \le s < t \le \infty$. The closed linear subspace $H_n(s,t) \subset L_2(\Omega)$ of random variables $\int \dots \int_{0<t_1<\dots<t_n<\infty} f(t_1,\dots,t_n) \, dB_{t_1} \dots dB_{t_n}$ for all $f \in L_2\big(\Delta_n \cap (s,t)^n\big)$ is called the n-th *Wiener chaos* on (s,t).

Note that $\int \dots \int_{t_1,\dots,t_n \text{ differ}} f(t_1,\dots,t_n) \, dB_{t_1} \dots dB_{t_n}$ belongs to $H_n(0,\infty)$.

2e8 Exercise. The chaos spaces $H_n = H_n(0,\infty)$ for $n = 0,1,2,\dots$ (H_0 being the one-dimensional space of constants) are mutually orthogonal.

Prove it.

Hint. Similarly to 2d4 show that $\langle \mathcal{E}_m(f), \mathcal{E}_n(g) \rangle = 0$ for all f,g provided that $m \ne n$. Use 2d6.

Do the chaos spaces span the whole $L_2(\Omega)$?

2e9 Exercise. Let $\mathcal{F}_n \subset \mathcal{F}$ be sub-σ-fields, $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$, and $\mathcal{F}_\infty \subset \mathcal{F}$ be the sub-σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots$. Then $L_2(\mathcal{F}_1) \cup L_2(\mathcal{F}_2) \cup \dots$ is dense in $L_2(\mathcal{F}_\infty)$. (Of course, $L_2(\mathcal{F}_n)$ means $L_2(\Omega, \mathcal{F}_n, P)$.)

Prove it.

Hint. First, consider indicators of sets. Sets $A \in \mathcal{F}$ such that $\mathbf{1}_A$ belongs to the closure of $\{ \mathbf{1}_B : B \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \}$ are a σ-field.

Given $0 \le s < t \le \infty$, we introduce the sub-σ-field $\mathcal{F}^B_{s,t} \subset \mathcal{F}$ generated by random variables $B_{u,v}$ for all u,v such that $s < u < v < t$.

2e10 Exercise. $H_0(s,t) \oplus H_1(s,t) \oplus \dots = L_2(\mathcal{F}^B_{s,t})$.

Prove it.

Hint. By 2e9 it is enough to consider the sub-σ-field generated by a finite number of $B_{u,v}$, say, $B_{u_0,u_1}, \dots, B_{u_{n-1},u_n}$ where $u = u_0 < u_1 < \dots < u_n = v$. Among all functions of $B_{u_0,u_1}, \dots, B_{u_{n-1},u_n}$, linear combinations of functions $\exp(i\lambda_1 B_{u_0,u_1} + \dots + i\lambda_n B_{u_{n-1},u_n})$ are dense (recall the argument before (2b9)).

Here is the general form of a random variable $X \in L_2(\mathcal{F}^B_{0,\infty})$:

$$X = f_0 + \int_0^\infty f_1(t)\,dB_t + \iint\limits_{0<s<t<\infty} f_2(s,t)\,dB_s dB_t + \dots =$$

$$= \sum_{n=0}^\infty \int \dots \int_{0<t_1<\dots<t_n<\infty} f_n(t_1,\dots,t_n)\,dB_{t_1}\dots dB_{t_n} \,;$$

$$f_n \in L_2(\Delta_n)\,;$$

$$\|X\|^2 = \sum_{n=0}^\infty \|f_n\|^2 \,.$$

A wonder! A *measurable* function of the Brownian motion is the sum of a power series.