1 Independent increments

1a Convolution semigroups

The distribution P_{X+Y} of the sum $X+Y$ of two *independent* random variables $X, Y : \Omega \to \mathbb{R}$ is uniquely determined by the distributions P_X, P_Y of X, Y; namely,

$$\int f \, dP_{X+Y} = \iint f(x+y) \, dP_X(x) dP_Y(y)$$

for any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$. It means that $P_{X+Y} = P_X * P_Y$. The *convolution* $\mu * \nu$ of two probability measures μ, ν on $\mathbb{R}$ is defined by

$$\int f \, d(\mu * \nu) = \iint f(x+y) \, d\mu(x) d\nu(y) \, .$$

That is, $\mu * \nu$ results from the measure $\mu \otimes \nu$ on $\mathbb{R}^2$ via the map $\mathbb{R}^2 \ni (x, y) \mapsto x + y \in \mathbb{R}$.

1a1 Definition. A (one-parameter) *convolution semigroup* in $\mathbb{R}$ is a family $(\mu_t)_{t\in[0,\infty)}$ of probability measures μ_t on (the Borel σ-field of) $\mathbb{R}$ such that μ_0 is concentrated at 0, and

$$\mu_{s+t} = \mu_s * \mu_t \qquad \text{for all } s, t \in [0, \infty) \, .$$

1a2 Example. Let $\mathrm{N}(0, 1)$ be the standard normal distribution,

$$d\,\mathrm{N}(0, 1)(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \, ,$$

and $\mathrm{N}(0, \sigma^2)$ be the image of $\mathrm{N}(0, 1)$ under the map $x \mapsto \sigma x$. Then $\mathrm{N}(0, \sigma_1^2) * \mathrm{N}(0, \sigma_2^2) = \mathrm{N}(0, \sigma_1^2 + \sigma_2^2)$. Thus, $\big(\mathrm{N}(0, t)\big)_{t\in[0,\infty)}$ is a convolution semigroup.

1a3 Example. Let $\Pi(\lambda)$ be the Poisson distribution,

$$\int f \, d\Pi(\lambda) = \sum_{n=0}^{\infty} f(n) \frac{\lambda^n}{n!} e^{-\lambda} \, ;$$

here $\lambda \in [0, \infty)$ (and $0^0 = 1$). Then $\Pi(\lambda_1) * \Pi(\lambda_2) = \Pi(\lambda_1 + \lambda_2)$. Thus, $\big(\Pi(t)\big)_{t\in[0,\infty)}$ is a convolution semigroup.

Note that any convolution semigroup satisfies

$$\int e^{iux} \, d\mu_{s+t}(x) = \Big(\int e^{iux} \, d\mu_s(x) \Big) \Big(\int e^{iux} \, d\mu_t(x) \Big) \, .$$

Especially,

$$\int e^{iux} \, d\,\mathrm{N}(0, t)(x) = \int e^{iu\sqrt{t}\,x} \, d\,\mathrm{N}(0, 1)(x) = \exp\Big(-\frac{u^2}{2} t \Big) \, ;$$

$$\int e^{iux} \, d\Pi(t)(x) = \sum_{n=0}^{\infty} e^{iun} \frac{t^n}{n!} e^{-t} = \exp\big((e^{iu} - 1) t \big) \, .$$

1a4 Example. Let μ_1 be the Cauchy distribution,

$$d\mu_1(x) = \frac{1}{\pi}\frac{1}{1+x^2}\,dx = \frac{1}{\pi} d\,\arctan x\,,$$

and μ_t its image under the map $x \mapsto tx$. We have

$$\int e^{iux}\,d\mu_1(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{iux}\Big(\frac{1}{x-i} - \frac{1}{x+i}\Big)\,dx = e^{-|u|}\,,$$

therefore $\int e^{iux}\,d\mu_t(x) = \int e^{iutx}\,d\mu_1(x) = \exp(-|u|t)$, which means that $\big(\mu_t\big)_{t\in[0,\infty)}$ is a convolution semigroup.

Beyond $\mathbb{R}$, we may consider convolution semigroups in $\mathbb{R}^n$, in a torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, in non-commutative groups etc. Here is a general framework.

1a5 Definition. (a) A *measurable semigroup* consists of a set G, an element $e \in G$ ('the unit'), a binary operation $G \times G \ni (x,y) \mapsto xy \in G$ and a σ-field $\mathcal{B}_G$ of subsets of G such that[1]

- $ex = x = xe$ for all $x \in G$;
- $(xy)z = x(yz)$ for all $x,y,z \in G$ (associativity);
- the binary operation is a measurable map from $(G,\mathcal{B}_G) \times (G,\mathcal{B}_G)$ to $(G,\mathcal{B}_G)$.

(b) A (one-parameter) *convolution semigroup* in a measurable semigroup G is a family $(\mu_t)_{t\in[0,\infty)}$ of probability measures μ_t on (the σ-field $\mathcal{B}_G$ of) G such that μ_0 is concentrated at e, and

$$\mu_{s+t} = \mu_s * \mu_t \qquad \text{for all } s,t \in [0,\infty)\,.$$

Of course, the convolution $\mu * \nu$ is the image of $\mu \otimes \nu$ under $(x,y) \mapsto xy$.

1a6 Example. The family

$$\mu_t = \begin{cases} \text{the atom at 0,} & \text{for } t = 0, \\ \text{the uniform distribution on } \mathbb{T}^1, & \text{for } t > 0 \end{cases}$$

is a convolution semigroup in $\mathbb{T}^1$.

Every convolution semigroup in $\mathbb{R}$ induces the corresponding convolution semigroup in the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Of course, the circle with the operation $(x,y) \mapsto x + y$ and the Borel σ-field is a measurable semigroup.

1b Independent increments

Let $(\Omega,\mathcal{F},P)$ be a probability space. (In fact, just $(0,1)$ with Lebesgue measure is enough for everything.) Equivalence classes of measurable functions $\Omega \to \mathbb{R}$ form a linear space

[1]Further restrictions on $(G,\mathcal{B}_G)$ will be imposed in 1b (before 1b8) and 1c (before 1c2).

$L_0 = L_0(\Omega, \mathcal{F}, P)$. Given $X_1, X_2, \dots \in L_0$, we say that $X_n \to 0$ in L_0, if $X_0 \to 0$ in probability, that is,

$$\forall \varepsilon > 0 \;\; \mathbb{P}\big(|X_n| > \varepsilon \big) \xrightarrow[n\to\infty]{} 0 \, .$$

Several compatible metrics (but not norms) on L_0 are well-known, especially,

$$\text{(1b1)} \qquad \begin{gathered} \operatorname{dist}_1(X,Y) = \inf\{\varepsilon > 0 : \mathbb{P}\big(|X-Y| > \varepsilon \big) < \varepsilon\} \, ; \\ \operatorname{dist}_2(X,Y) = \mathbb{E} \, \frac{|X-Y|}{1+|X-Y|} \, . \end{gathered}$$

Thus, L_0 is a metrizable linear topological space. I always assume that L_0 is separable (which is a restriction on $(\Omega, \mathcal{F}, P)$, satisfied by $(0,1)$). In fact, L_0 is complete.

A random process is just a family of random variables, however, what is a random variable? Some authors define it as a measurable function $\Omega \to \mathbb{R}$, but I prefer the following.

1b2 Definition. (a) A *random variable* is an element of L_0.

(b) A *random process* on a set T is a family $(X_t)_{t\in T}$ of random variables, in other words, a map $X : T \to L_0$.

1b3 Definition. A random process X on $[0,\infty)$ has *independent increments,* if $X(0) = 0$ and random variables $X(t_2) - X(t_1), X(t_3) - X(t_2), \dots, X(t_n) - X(t_{n-1})$ are independent whenever $0 \le t_1 \le \dots \le t_n < \infty$. If, in addition, the distribution of $X(s+t) - X(s)$ does not depends on s, then X has *stationary* independent increments.

Note that $X(t_1) - X(s_1), \dots, X(t_n) - X(s_n)$ are independent whenever $0 \le s_1 \le t_1 \le s_2 \le t_2 \le \dots \le s_n \le t_n < \infty$ (provided that X has independent increments).

1b4 Exercise. If X is a random process with stationary independent increments, then distributions μ_t of random variables $X(t)$ form a convolution semigroup $(\mu_t)_{t\in[0,\infty)}$.

Prove it.

For a process on the set $[0,\infty) \cap \mathbb{Q}$ of all nonnegative rational numbers,[2] stationary independent increments are defined evidently.

1b5 Lemma. For every convolution semigroup $(\mu_t)_{t\in[0,\infty)}$ in $\mathbb{R}$, there exists a process X with stationary independent increments, such that X_t is distributed μ_t, for all $t \in [0,\infty) \cap \mathbb{Q}$.

(I mean, there exists X on some probability space; in fact, $\Omega = (0,1)$ fits.) (Of course, we could start with $(\mu_t)_{t\in[0,\infty)\cap\mathbb{Q}}$.)

Extension by continuity from $[0,\infty) \cap \mathbb{Q}$ to $[0,\infty)$ will be carried in 1c.

Beyond $\mathbb{R}$, we may consider $\mathbb{R}^n$, $\mathbb{T}^n$ etc. Spaces $L_0(\Omega, \mathbb{T}^n) = L_0\big((\Omega, \mathcal{F}, P), \mathbb{T}^n\big)$, processes $X : [0,\infty) \to L_0(\Omega, \mathbb{T}^n)$ and independent increments are defined evidently. In a noncommutative group, increments may be defined as $X^{-1}(s)X(t)$ or alternatively $X(t)X^{-1}(s)$. However, what about processes valued in a measurable semigroup? It may happen that $xy = xz$ but $y \ne z$. Instead of $X_{s,t} = X^{-1}(s)X(t)$ we start with $X_{s,t}$ and require $X_{r,s}X_{s,t} = X_{r,t}$ (see below).

[2] Any other countable dense subgroup of $\mathbb{R}$ may be used instead of $\mathbb{Q}$.

For a measurable semigroup G the set $L_0(\Omega, G)$ of all G-valued random variables, that is, equivalence classes of measurable maps $\Omega \to G$ is a semigroup (not equipped with a topology).[3]

1b6 Definition. Given a measurable semigroup G and a probability space $(\Omega, \mathcal{F}, P)$, we define an *abstract stochastic flow*[4] as a family $(X_{s,t})_{s \le t; s,t \in [0,\infty)}$ of G-valued random variables $X_{s,t} \in L_0(\Omega, G)$ such that

- $X_{r,s} X_{s,t} = X_{r,t}$ whenever $0 \le r \le s \le t < \infty$;
- $X_{t_1,t_2}, \dots, X_{t_{n-1},t_n}$ are independent whenever $0 \le t_1 \le \dots \le t_n < \infty$.

The abstract stochastic flow is called *stationary,* if

- the distribution of $X_{s,s+t}$ does not depend on s.

1b7 Exercise. Every stationary abstract stochastic flow leads to a convolution semigroup $(\mu_t)_{t \in [0,\infty)}$; namely, μ_t is the distribution of $X_{s,s+t}$.
Prove it.

In order to generalize 1b5 we assume that the measurable space $(G, \mathcal{B}_G)$ is *universally measurable.*[5] It means existence of a metric ρ on G such that (G, ρ) is a separable metric space, $\mathcal{B}_G = \mathcal{B}_\rho$ is the Borel σ-field of ρ, and every probability measure on $(G, \mathcal{B}_\rho)$ is concentrated on a countable union of compact sets (regularity). In fact, regularity does not depend on the choice of ρ.[6] FROM NOW ON, ALL MEASURABLE SEMIGROUPS ARE ASSUMED TO BE UNIVERSALLY MEASURABLE.

1b8 Lemma. For any convolution semigroup $(\mu_t)_{t \in [0,\infty)}$ in any measurable semigroup G, there exists a stationary abstract stochastic flow $(X_{s,t})_{s \le t; s,t \in [0,\infty) \cap \mathbb{Q}}$ such that $X_{s,s+t}$ is distributed μ_t for all $s, t \in [0, \infty) \cap \mathbb{Q}$.

1c Continuity in probability

Quite often we have a metric ρ on a measurable semigroup G, satisfying[7]

$$\forall x, y \in G \quad \rho(xy, x) \le \rho(y, e)\,; \tag{1c1}$$

here e is the unit of G (that is, $ex = x = xe$ for all $x \in G$). Note that (1c1) evidently holds (with equality) for $G = \mathbb{R}, \mathbb{R}^n, \mathbb{T}^n$. FROM NOW ON, EACH MEASURABLE SEMIGROUP G IS ASSUMED TO BE EQUIPPED WITH A METRIC ρ SATISFYING (1c1) AND MEASURABLE (that is, the map $(G, \mathcal{B}_G) \times (G, \mathcal{B}_G) \ni (x, y) \mapsto \rho(x, y) \in \mathbb{R}$ must be measurable).

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[3]It is equipped with a σ-field.

[4]The reason for this term will be explained later.

[5]See also Sect. 12.1 (especially, Theorem 12.1.2 and Problem 2) in book 'Real Analysis and Probability' by R.M. Dudley.

[6]An equivalent definition (without metrics): $\mathcal{B}_G$ is countably generated, and every probability measure on $(G, \mathcal{B}_G)$ is the image of Lebesgue measure on $(0, 1)$ under some measurable map $(0, 1) \to G$.

[7]Do not think that (1c1) implies continuity of xy in x and y. In fact, $\rho(x_n, e) \to 0$ does not imply $\rho(x_n y, y) \to 0$. Thus, G need not be a topological semigroup. Also, the metric space (G, ρ) need not be separable.

1c2 Definition. A convolution semigroup $(\mu_t)_{t\in[0,\infty)}$ in a measurable semigroup G is *continuous in probability,* if

$$\forall \varepsilon > 0 \;\; \mu_t\bigl(\{x \in G : \rho(e,x) > \varepsilon\}\bigr) \xrightarrow[t\to 0]{} 0 \, .$$

Note that it holds for 1a2, 1a3 and 1a4, but fails for 1a6.

Let $(X_{s,t})_{s\le t; s,t\in[0,\infty)\cap\mathbb{Q}}$ be a stationary abstract stochastic flow corresponding to a convolution semigroup $(\mu_t)_{t\in[0,\infty)\cap\mathbb{Q}}$. The convolution semigroup is continuous in probability if and only if $X_{0,t} \to e$ in probability for $t \to 0$, that is, $\operatorname{dist}(X_{0,t}, e) \to 0$ for $t \to 0$ (here 'dist' is any one out of the two metrics (1b1)). Almost surely,

$$\rho(X_{0,s+t}, X_{0,s}) = \rho(X_{0,s}X_{s,s+t}, X_{0,s}) \le \rho(X_{s,s+t}, e) \, ,$$

therefore

$$\operatorname{dist}(X_{0,s+t}, X_{0,s}) \le \operatorname{dist}(X_{s,s+t}, e) = \operatorname{dist}(X_{0,t}, e) \xrightarrow[t\to 0]{} 0 \, .$$

It means that the function

$$[0,\infty) \cap \mathbb{Q} \ni t \mapsto X_{0,t} \in L_0(\Omega, G)$$

is uniformly continuous. Therefore it can be extended by continuity to $[0,\infty)$. The completion of G appears for a while, but ultimately disappears, since μ_t is concentrated on G.

1c3 Theorem. For every continuous in probability convolution semigroup $(\mu_t)_{t\in[0,\infty)}$ in a measurable semigroup G there exists a stationary abstract stochastic flow $(X_{s,t})_{s\le t; s,t\in[0,\infty)}$ such that for every t the random variable $X_{0,t}$ is distributed μ_t.

If G is a group (not just semigroup), we get a process $(X_t)_{t\in[0,\infty)}$ with stationary independent increments; namely, $X_t = X_{0,t}$; $X_{s,t} = X_s^{-1}X_t$. Especially, 1a2 leads to the standard Brownian motion in $\mathbb{R}$; 1a3 — to the Poisson process (of rate 1); 1a4 — to a so-called stable process (but 1a6 — to nothing).

1d Sample paths: one-sided limits

Having a random process $(X_t)_{t\in[0,\infty)}$ we cannot introduce its sample paths as functions $[0,\infty) \ni t \mapsto X_t(\omega) \in \mathbb{R}$, since X_t is an equivalence class rather than a function. Nevertheless we can do so for $t \in [0,\infty) \cap \mathbb{Q}$. The equivalence class of a measurable map $\Omega \to \mathbb{R}^{[0,\infty)\cap\mathbb{Q}}$ is well-defined.

Continuity in probability does not mean continuous sample paths. For example, the Poisson process is continuous in probability, but its sample paths cannot be continuous. More exactly, its sample paths on $[0,\infty) \cap \mathbb{Q}$ are integer-valued step functions. They are continuous on $[0,\infty) \cap \mathbb{Q}$, but cannot be extended by continuity to $[0,\infty)$; they fail to be locally uniformly continuous.

Here is a non-evident well-known result.[8]

[8]See also Sect. 3.1 (especially, Theorems 5,6,8) and Sects. 5.1, 5.3 (especially, Theorems 3, 19) in book 'Random processes with independent increments' by A.V. Skorokhod.

1d1 Theorem. Let $(X_t)_{t\in[0,\infty)\cap\mathbb{Q}}$, $X_t \in L_0(\Omega,\mathbb{R})$, be a random process with stationary independent increments, continuous in probability. Then almost all ω are such that for all $u \in [0,\infty)$ the limits

$$\lim_{t\to u-} X(t,\omega)\,, \qquad \lim_{t\to u+} X(t,\omega)$$

exist. (Here t runs over $[0,\infty)\cap\mathbb{Q}$.)

The following argument is the key to the proof.

1d2 Exercise. Let $(X_t)_{t\in\{1,\dots,n\}}$ be a process with independent increments. Then for every $\varepsilon > 0$

$$\mathbb{P}\Big(\max_{t=1,\dots,n} |X_t - X_1| > 2\varepsilon \Big) \min_{t=1,\dots,n} \mathbb{P}\big(|X_t - X_n| \le \varepsilon \big) \le \mathbb{P}\big(|X_n - X_1| > \varepsilon \big)\,.$$

Prove it.

(Do not confuse $\mathbb{P}\big(\max\dots\big)$ and $\max\mathbb{P}\big(\dots\big)$.)

Hint. Clearly, $\mathbb{P}\big(|X_t - X_1| > 2\varepsilon \big)\mathbb{P}\big(|X_n - X_t| \le \varepsilon \big) \le \mathbb{P}\big(|X_n - X_1| > \varepsilon,\ |X_t - X_1| > 2\varepsilon \big)$ for each t. However, summation over t fails, since the events $|X_t - X_1| > 2\varepsilon$ are not disjoint. Introduce disjoint events by considering the first (random) t such that $|X_t - X_1| > 2\varepsilon$. (It is the 'stopping time' idea.)

The rest of the proof of Theorem 1d1 is outlined below.

1d3 Exercise. For X as in Theorem 1d1, let $\varepsilon \in (0,\infty)$ and $\delta \in (0,\infty)\cap\mathbb{Q}$ be such that

$$\forall t \in [0,\delta]\cap\mathbb{Q} \quad \mathbb{P}\big(|X_\delta - X_t| > \varepsilon \big) \le \frac{1}{3}\,.$$

Then:

(a) $\mathbb{P}\big(\exists t \in [0,\delta]\cap\mathbb{Q} \quad |X_t - X_0| > 2\varepsilon \big) \le 1/2$;

(b) for each n, existence of $t_1,\dots,t_n \in [0,\delta]\cap\mathbb{Q}$ such that $t_1 < \dots < t_n$ and $|X_{t_2} - X_{t_1}| > 2\varepsilon$, $\dots$, $|X_{t_n} - X_{t_{n-1}}| > 2\varepsilon$ is an event of probability $\le 1/2^n$.

Prove it.

Hint. (a): apply 1d2 to X restricted to an arbitrary finite subset of $[0,\delta]$; (b): first do it for X restricted to a finite subset of $[0,\delta]$; use the 'stopping time' idea.

1d4 Exercise. Prove Theorem 1d1.

Hint. Almost all ω are such that for every $\varepsilon > 0$ there exists n such that no $t_1 < \dots < t_n$ satisfy $|X_{t_2} - X_{t_1}| > 2\varepsilon$, $\dots$, $|X_{t_n} - X_{t_{n-1}}| > 2\varepsilon$. First, prove it (via 1d3) in a neighborhood of an arbitrary point of $[0,\infty)$. Then cover $[0,t]$ by a finite number of such neighborhoods.

1d5 Definition. Let T be a set and $X = (X_t)_{t\in T}$ a random process, $X_t \in L_0(\Omega,\mathbb{R})$. A *modification* of X is an equivalence class of a map $\xi : \Omega \to \mathbb{R}^T$ such that for every $t \in T$ the function $\Omega \ni \omega \mapsto \xi(\omega)(t) \in \mathbb{R}$ belongs to the equivalence class X_t.

You see, a modification chooses a function in (almost) every equivalence class X_t.

1d6 Exercise. Let $(X_t)_{t\in[0,\infty)}$ be a random process with independent increments, continuous in probability.

(a) The following formulas define two modifications of X:

$$\xi_1(\omega,t) = \lim_{\mathbb{Q}\ni s\to t-} X(t,\omega)\,, \quad \xi_2(\omega,t) = \lim_{\mathbb{Q}\ni s\to t+} X(t,\omega)\,.$$

(b) X has one and only one left continuous modification, namely ξ_1. The same for right continuity (and ξ_2).

Prove it.

Hint: in general, if $Y_n \to Z_1$ in probability and $Y_n \to Z_2$ a.s., then $Z_1 = Z_2$ a.s. (think, why).

Note that $\xi_1 \ne \xi_2$ for the Poisson process.

All said above remains true for processes valued in $\mathbb{R}^n$ or $\mathbb{T}^n$. The general case, abstract stochastic flows in a measurable semigroup G, needs more attention. Note for example that $\rho(X_{r,s}, X_{r,t})$ need not be independent of $X_{r,s}$, since it is not the same as $\rho(e, X_{s,t})$.

Denoting for convenience

$$|x| = \rho(x,e) \quad \text{for } x \in G$$

we have

$$|xy| \le |x| + |y|\,, \qquad |x| \le |xy| + |y| \quad \text{and} \quad \rho(x,xy) \le |y|$$

for all $x, y \in G$. Indeed (recall (1c1)),

$$\underbrace{\rho(xy,e)}_{=|xy|} \le \underbrace{\rho(xy,x)}_{\le\rho(y,e)=|y|} + \underbrace{\rho(x,e)}_{=|x|}; \qquad \underbrace{\rho(x,e)}_{=|x|} \le \underbrace{\rho(x,xy)}_{\le\rho(e,y)=|y|} + \underbrace{\rho(xy,e)}_{=|xy|}\,.$$

1d7 Exercise. Let $(X_{s,t})_{s\le t; s,t\in\{1,\dots,n\}}$ be an abstract stochastic flow in a measurable semigroup G. Then for every $\varepsilon > 0$

$$\mathbb{P}\Big(\max_{t=1,\dots,n} |X_{1,t}| > 2\varepsilon \Big) \min_{t=1,\dots,n} \mathbb{P}\Big(|X_{t,n}| \le \varepsilon \Big) \le \mathbb{P}\Big(|X_{1,n}| > \varepsilon \Big)\,.$$

Prove it.

Hint: similar to 1d2; take into account that $|X_{1,t}| \le |X_{1,n}| + |X_{t,n}|$.

1d8 Definition. An abstract stochastic flow $(X_{s,t})_{s\le t; s,t\in[0,\infty)}$ is *continuous in probability* if for every $t \in [0,\infty)$ and $\varepsilon > 0$

$$\mathbb{P}\Big(|X_{t-\delta,t}| > \varepsilon \Big) \xrightarrow[\delta\to 0]{} 0\,, \quad \mathbb{P}\Big(|X_{t,t+\delta}| > \varepsilon \Big) \xrightarrow[\delta\to 0]{} 0\,.$$

Here $|x| = \rho(x,e)$, as before. For a stationary flow it means that $\mathbb{P}\Big(|X_{0,t}| > \varepsilon \Big) \to 0$ for $t \to 0+$.

1d9 Exercise. Let $(X_{s,t})_{s\le t; s,t\in[0,\infty)}$ be an abstract stochastic flow in a measurable semigroup G, and $r,t \in [0,\infty) \cap \mathbb{Q}$, $\varepsilon \in (0,\infty)$ be such that $r < t$ and

$$\forall s \in [r,t] \cap \mathbb{Q} \quad \mathbb{P}\Big(|X_{s,t}| > \varepsilon \Big) \le \frac{1}{3}\,.$$

Then

(a) $\mathbb{P}\big(\exists s \in [r,t]\cap\mathbb{Q}\;\; |X_{r,s}| > 2\varepsilon\big) \le 1/2$;

(b) for each n, existence of $s_1,\dots,s_n \in [r,t]\cap\mathbb{Q}$ such that $s_1 < \dots < s_n$ and $|X_{s_1,s_2}| > 2\varepsilon$, $\dots$, $|X_{s_{n-1}} - X_{s_n}| > 2\varepsilon$ is an event of probability $\le 1/2^n$.

Prove it.

Hint: similar to 1d3.

1d10 Theorem. Let an abstract stochastic flow $(X_{s,t})_{s\le t; s,t\in[0,\infty)}$ be continuous in probability. Then the process $(X_{0,t})_{t\in[0,\infty)}$ has one and only one left continuous modification, and one and only one right continuous modification.

1d11 Exercise. Prove Theorem 1d10.

Hint. First, generalize 1d1, 1d4; (G,ρ) need not be complete, but anyway, a Cauchy sequence having a convergent subsequence must converge. Also, a sequence converging in probability has a subsequence converging a.s. Afterwards, generalize 1d6.

1e Sample paths: continuity

1e1 Theorem. The following two conditions are equivalent for every random process $(X_t)_{t\in[0,\infty)}$, $X_t \in L_0(\Omega,\mathbb{R})$, with stationary independent increments.

(a) The process has a continuous modification (that is, a modification with continuous sample paths).

(b) For every $\varepsilon > 0$,

$$\frac{1}{t}\mathbb{P}\big(|X_t| > \varepsilon\big) \xrightarrow[t\to 0]{} 0\,.$$

Note the coefficient $1/t$ making (b) much stronger than just continuity in probability. The Poisson process is continuous in probability but violates (b).

When proving the theorem we may restrict ourselves to processes continuous in probability, since (b) implies continuity in probability, and (a) implies it as well (indeed, convergence a.s. implies convergence in probability).

By 1d6, the process has the left continuous modification ξ_1 and the right continuous modification ξ_2. Clearly, 1e1(a) is equivalent to $\xi_1 = \xi_2$. The latter may be reformulated as

$$\forall\varepsilon > 0 \quad \sup_{t\in[0,\infty)} |\xi_1(\cdot,t) - \xi_2(\cdot,t)| \le \varepsilon\,.$$

1e2 Exercise. Almost surely,

$$\max_{k=1,\dots,n} |X_{\frac{k}{n}} - X_{\frac{k-1}{n}}| \xrightarrow[n\to\infty]{} \sup_{t\in(0,1)} |\xi_1(\cdot,t) - \xi_2(\cdot,t)|\,.$$

Prove it.

Hint: we may assume that the sample path is continuous at all rational points, since $\mathbb{P}\big(\xi_1(\cdot,t) = \xi_2(\cdot,t)\big) = 1$ for every t. The rest is analysis, not probability.

1e3 Exercise. Prove Theorem 1e1.

Hint:

$$\mathbb{P}\Big(\max_{k=1,\dots,n} |X_{\frac{k}{n}} - X_{\frac{k-1}{n}}| \le \varepsilon\Big) = \Big(\mathbb{P}\big(|X_{\frac{1}{n}}| \le \varepsilon\big)\Big)^n\,.$$

1e4 Exercise. Prove that the standard Brownian motion in $\mathbb{R}$ has a continuous modification.

Hint: apply 1e1 to 1a2.

When a process has a continuous modification (evidently unique), other modifications are of no interest, and we need not distinguish between the modification and the process. Usually we say just "the Brownian motion is (sample) continuous".

For flows we have an implication (rather than equivalence).

1e5 Exercise. Let a convolution semigroup $(\mu_t)_{t\in[0,\infty)}$ in a measurable semigroup G satisfy[9]

$$\frac{1}{t}\mu_t\big(\{x : |x| > \varepsilon\}\big) \xrightarrow[t\to 0]{} 0$$

for all $\varepsilon > 0$. Then the corresponding stationary abstract stochastic flow $(X_{s,t})_{s\le t;s,t\in[0,\infty)}$ is such that the process $(X_{0,t})_{t\in[0,\infty)}$ is sample continuous.[10]

Prove it.

Hint: $\rho\big(X_{0,\frac{k-1}{n}}, X_{0,\frac{k}{n}}\big) \le |X_{\frac{k-1}{n},\frac{k}{n}}|$.

[9] $|x| = \rho(x,e)$, as before.

[10] That is, has a sample continuous modification.