2 Markov and strong Markov

2a Restart at a nonrandom time

Let X be any one of the three processes introduced in Sect. 1 (the Brownian motion, the Cauchy process, the special Lévy process) on a probability space $(\Omega, \mathcal{F}, P)$. We construct a random function Y on the product $\Omega^2 = \Omega \times \Omega$ (that is, $(\Omega, \mathcal{F}, P) \times (\Omega, \mathcal{F}, P)$) by glueing together two independent sample functions as follows:

(2a1)
$$Y(t)(\omega_1, \omega_2) = \begin{cases} X(t)(\omega_1) & \text{for } t \le 1, \\ X(1)(\omega_1) + X(t-1)(\omega_2) & \text{for } t \ge 1. \end{cases}$$

Clearly, sample functions of Y are right continuous.

2a2 Exercise. Y is distributed like X.[1]
Prove it.

This is the *Markov property:* at the instant 1 the process X forgets its past and retains only a single point, $X(1)$.[2] Of course, the Markov property holds at every instant $t \in (0, \infty)$, not just 1.

We turn to the Brownian motion, B. Given $x \in (0, \infty)$, we define the *hitting time* $T_x : \Omega \to [0, \infty]$ by

(2a3)
$$T_x = \inf\{t : B(t) = x\}$$

(as usual, $\inf \emptyset = \infty$).

[1]Recall 1c, especially 1c3. See also 2f4.

[2]By the way, a process with differentiable sample functions cannot be Markov (unless it is nonrandom); it have to retain $X'(1)$.

2a4 Exercise. (a) T_x is measurable (in ω, for a fixed x); (b) the distribution of T_x is uniquely determined, that is, does not depend on the choice of $(\Omega, \mathcal{F}, P)$ and B as far as B is a Brownian motion.[1]

Prove it.

Such statements should be made every time we construct a random variable out of the Brownian motion;[2] however, they will be usually omitted.

2a5 Exercise. T_x is distributed like $x^2 T_1$.

Prove it.

We introduce the random variable[3]

$$L = \max\{t \in [0,1] : B(t) = 0\}\,, \tag{2a6}$$

and want to calculate its distribution,

$$\mathbb{P}\big(L < t\big) = \mathbb{P}\big(\forall s \in [t,1] \;\; B(s) \ne 0\big) = ?$$

Given $B(t) = x > 0$, the conditional probability of this event should be equal to

$$\mathbb{P}\big(\forall s \in [0,1-t] \;\; B(s) \ne x\big) = \mathbb{P}\big(T_x > 1-t\big) = \mathbb{P}\Big(T_1 > \frac{1-t}{x^2}\Big)$$

(think, why); for $x < 0$ the situation is similar. We guess that

$$\mathbb{P}\big(L < t\big) = \int_{-\infty}^{\infty} p_t(x) \mathbb{P}\Big(T_1 > \frac{1-t}{x^2}\Big)\,\mathrm{d}x\,, \tag{2a7}$$

where $p_t(x) = (2\pi t)^{-1/2} \exp\big(-\frac{x^2}{2t}\big)$.

The proof combines the Markov property of the Brownian motion with the Fubini theorem. We use ω_1 on $[0,t]$, swich to ω_2 on $[t,1]$, substitute this combination for B into L and get

$$\begin{aligned}\mathbb{P}\big(L < t\big) &= \mathbb{P}\big(\forall s \in [t,1] \;\; B(s) \ne 0\big) = \\ &= (P \times P)\{(\omega_1,\omega_2) : \forall s \in [t,1] \;\; B(t)(\omega_1) + B(s-t)(\omega_2) \ne 0\} = \\ &= \int_\Omega f\big(B(t)(\omega_1)\big)\,P(\mathrm{d}\omega_1) = \mathbb{E}\,f\big(B(t)\big) = \int_{\mathbb{R}} p_t(x) f(x)\,\mathrm{d}x\,,\end{aligned}$$

[1] See also 2f3, 2f4.

[2] For instance, L and R, see (2a6), (2a9).

[3] 'L is for left or last' [1, Sect. 7.2, Exer. 2.2].

where

$$f(x) = P\{\omega_2 : \forall s \in [t,1] \;\; x + B(s-t)(\omega_2) \ne 0\} =$$
$$= \mathbb{P}\big(\forall s \in [0, 1-t] \;\; B(s) \ne -x\big) = \mathbb{P}\big(T_{|x|} > 1-t\big) = \mathbb{P}\Big(T_1 > \frac{1-t}{x^2}\Big)\,;$$

(2a7) follows.

2a8 Exercise. Let[1]

(2a9)
$$R = \inf\{t \in [1,\infty) : B(t) = 0\}$$

(possibly, ∞).[2] Then

(2a10)
$$\mathbb{P}\big(R > 1+t\big) = \int_{-\infty}^{\infty} p_1(x)\mathbb{P}\Big(T_1 > \frac{t}{x^2}\Big)\,\mathrm{d}x\,.$$

Prove it.

2b Hit and restart

Similarly to (2a1) we let (recall (2a3))

(2b1)
$$Y(t)(\omega_1,\omega_2) = \begin{cases} B(t)(\omega_1) & \text{for } t \le T_1(\omega_1), \\ 1 + B(t - T_1(\omega_1))(\omega_2) & \text{for } t \ge T_1(\omega_1). \end{cases}$$

2b2 Proposition. Y is distributed like B.

The proof will be given in 2c, but do not hesitate to use 2b2 now.
This is a special case of *strong Markov property.*[3]
You see, the process B forgets the past when hitting the level 1. Of course, the same happens when hitting x, for every $x \in \mathbb{R}$, not just 1.

2b3 Exercise. Prove that

$$\mathbb{P}\Big(\max_{[0,t]} B(\cdot) \ge 1\Big) = 2\,\mathbb{P}\big(B(t) \ge 1\big)\,.$$

Similarly, $\mathbb{P}\big(\max_{[0,t]} B(\cdot) \ge x\big) = 2\,\mathbb{P}\big(B(t) \ge x\big)$ for all $x \in [0,\infty)$. Thus,

(2b4)
$$\max_{[0,t]} B(\cdot) \quad \text{is distributed like} \quad |B(t)|\,.$$

---

[1] 'R is for right or return' [1, Sect. 7.2, Exer. 2.1].
[2] But see 2b5.
[3] See also 2f8.

The distribution of T_x is therefore

$$\mathbb{P}\big(T_x \le t\big) = \mathbb{P}\Big(\max_{[0,t]} B(\cdot) \ge x\Big) = 2\,\mathbb{P}\big(B(t) \ge x\big) = \\ = 2\,\mathbb{P}\Big(B(1) \ge \frac{x}{\sqrt{t}}\Big) = 2\int_{x/\sqrt{t}}^{\infty} p_1(y)\,\mathrm{d}y\,.$$

2b5 Exercise. Prove that

$$\inf_{[0,\infty)} B(\cdot) = -\infty\,, \quad \sup_{[0,\infty)} B(\cdot) = \infty \quad \text{a.s.}$$

2b6 Exercise. Almost surely,

$$\forall \varepsilon > 0 \ \Big(\min_{[0,\varepsilon]} B(\cdot) < 0 \text{ and } \max_{[0,\varepsilon]} B(\cdot) > 0\Big)\,.$$

Prove it.

2b7 Exercise. B does not restart at the random time L (defined by (2a6)). Prove it.

Now we are in position to finalize the calculation of the distribution of L and R started in (2a7), (2a10); the integrals need some effort, and give

(2b8) $$\mathbb{P}\big(L \le t\big) = \frac{2}{\pi}\arcsin\sqrt{t} \quad \text{for } 0 \le t \le 1\,,$$

(2b9) $$\mathbb{P}\big(R \le t\big) = \frac{2}{\pi}\arctan\sqrt{t-1} \quad \text{for } 1 \le t < \infty\,,$$

see [1, Sect. 7.4, Example 4.4].

Let us calculate the density:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\big(T_x \le t\big) = 2\frac{\mathrm{d}}{\mathrm{d}t}\int_{x/\sqrt{t}}^{\infty} p_1(y)\,\mathrm{d}y = \\ = -2p_1\Big(\frac{x}{\sqrt{t}}\Big)\cdot x \cdot \Big(-\frac{1}{2}t^{-3/2}\Big) = \frac{x}{t^{3/2}}p_1\Big(\frac{x}{\sqrt{t}}\Big) = \\ = \frac{x}{t}p_t(x) = \frac{1}{\sqrt{2\pi}}\frac{x}{t^{3/2}}\exp\Big(-\frac{x^2}{2t}\Big)\,;$$

the derivative is continuous on $[0.\infty)$ (in spite of t in the denominator); we got the density (of the distribution) of T_x. Note that $\mathbb{E}\,T_x = \infty$. Interestingly, T_x is distributed like the special Lévy process at *time* x.

2b10 Exercise. For all $x, y \in (0, \infty)$,

$$T_{x+y} - T_x \text{ is independent of } T_x \text{ and distributed like } T_y \, .$$

Prove it.

The formula $p_{s+t} = p_s * p_t$ for $p_t(x) = \frac{t}{\sqrt{2\pi} x^{3/2}} \exp\bigl(-\frac{t^2}{2x}\bigr)$, claimed in 1a without proof, follows from 2b10!

Similarly to 2b10, the process $(T_x)_{x\in[0,\infty)}$ has stationary independent increments. Also, its sample functions are continuous from the left (think, why).

The random function $(T_x)_{x\in[0,\infty)}$ is distributed as the left-continuous modification of the special Lévy process.

See also [1, Sect. 7.4].

But wait, we did not prove 2b2 yet. . . [1]

2c Delayed restart

An important step toward the proof of Prop. 2b2 is made here. Instead of the random time T_1 taking on a continuum of values we introduce (for a given n) a random time τ_n with a finite number of values,

(2c1)
$$\tau_n = \frac{k}{2^n} \quad \text{whenever } \frac{k-1}{2^n} < T_1 \le \frac{k}{2^n} \quad \text{for } k = 1, 2, \dots, 2^{2n} \, ;$$
$$\tau_n = \infty \quad \text{whenever } T_1 > 2^n \, .$$

Clearly, $\tau_n \downarrow T_1$ a.s., as $n \to \infty$.

Similarly to (2b1) we restart at τ_n,

(2c2)
$$Y_n(t)(\omega_1, \omega_2) = \begin{cases} B(t)(\omega_1) & \text{for } t \le \tau_n(\omega_1), \\ B\bigl(\tau_n(\omega_1)\bigr)(\omega_1) + B\bigl(t - \tau_n(\omega_1)\bigr)(\omega_2) & \text{for } t \ge \tau_n(\omega_1). \end{cases}$$

Similarly to 2b2 we claim the following.

2c3 Lemma. For every n the random function Y_n is distributed like B.

The proof will be given in 2e. Now we'll deduce 2b2 from 2c3.

[1] "It may be difficult for the novice to appreciate the fact that twenty five years ago a formal proof of the strong Markov property was a major event." Kai Lai Chung, John B. Walsh, "Markov processes, Brownian motion, and time symmetry", second edition, Springer (1982 and) 2005; see page 73.

Proof of 2b2 (assuming 2c3). The random function Y defined by (2b1) is evidently continuous. In order to prove that Y is distributed like B it is sufficient to check that $\big(Y(t_1),\dots,Y(t_j)\big) \sim \big(B(t_1),\dots,B(t_j)\big)$ for all j and $t_1,\dots,t_j \in (0,\infty)$. To this end it is sufficient to check that

$$\mathbb{E}\,\varphi\big(Y(t_1),\dots,Y(t_j)\big) = \mathbb{E}\,\varphi\big(B(t_1),\dots,B(t_j)\big) \tag{2c4}$$

for every j and every bounded continuous $\varphi : \mathbb{R}^j \to \mathbb{R}$.

By 2c3,

$$\mathbb{E}\,\varphi\big(Y_n(t_1),\dots,Y_n(t_j)\big) = \mathbb{E}\,\varphi\big(B(t_1),\dots,B(t_j)\big) \tag{2c5}$$

for all n. As $n \to \infty$, we have (for almost all ω_1, ω_2)

$$\begin{gathered}
\tau_n(\omega_1) \downarrow T_1(\omega_1)\,; \\
B\big(\tau_n(\omega_1)\big)(\omega_1) \to B\big(T_1(\omega_1)\big)(\omega_1) = 1\,; \\
t - \tau_n(\omega_1) \to t - T_1(\omega_1)\,; \\
B\big(t - \tau_n(\omega_1)\big)(\omega_2) \to B\big(t - T_1(\omega_1)\big)(\omega_2)\,; \\
Y_n(t)(\omega_1,\omega_2) \to Y(t)(\omega_1,\omega_2)
\end{gathered}$$

for $t \ge T_1(\omega_1)$. And clearly $Y_n(t)(\omega_1,\omega_2) = B(t)(\omega_1) = Y(t)(\omega_1,\omega_2)$ for $t < T_1(\omega_1)$, if n is large enough. Thus, $Y_n(t) \to Y(t)$ a.s. (for each t); therefore

$$\mathbb{E}\,\varphi\big(Y_n(t_1),\dots,Y_n(t_j)\big) \to \mathbb{E}\,\varphi\big(Y(t_1),\dots,Y(t_j)\big)$$

by the bounded convergence theorem. In combination with (2c5) it gives (2c4). $\square$

2d Maybe restart, maybe not

Here we prove Lemma 2c3 for the simplest case, $n = 0$. (Be careful, mind 2b7!) By (2c1),

$$\tau_0 = \begin{cases} 1 & \text{if } T_1 \le 1, \\ \infty & \text{if } T_1 > 1. \end{cases}$$

By (2c2),[1]

$$Y_0(t)(\omega_1,\omega_3) = \begin{cases} B(t)(\omega_1) & \text{if } t \le 1, \\ B(t)(\omega_1) & \text{if } t \ge 1 \text{ and } \max_{[0,1]} B(\cdot)(\omega_1) < 1, \\ B(1)(\omega_1) + B(t-1)(\omega_3) & \text{if } t \ge 1 \text{ and } \max_{[0,1]} B(\cdot)(\omega_1) \ge 1. \end{cases}$$

[1] Why ω_3? Wait a little...

We want to prove that $Y_0 \sim B$. The distribution of Y_0 does not change if we replace B with another process X distributed like B. We choose (recall (2a1))

$$X(t)(\omega_1, \omega_2) = \begin{cases} B(t)(\omega_1) & \text{for } t \le 1, \\ B(1)(\omega_1) + B(t-1)(\omega_2) & \text{for } t \ge 1 \end{cases}$$

and consider

$$\begin{aligned} & Y(t)(\omega_1, \omega_2, \omega_3) = \\ & = \begin{cases} X(t)(\omega_1, \omega_2) & \text{if } t \le 1, \\ X(t)(\omega_1, \omega_2) & \text{if } t \ge 1 \text{ and } \max_{[0,1]} X(\cdot)(\omega_1, \omega_2) < 1, \\ X(1)(\omega_1, \omega_2) + B(t-1)(\omega_3) & \text{if } t \ge 1 \text{ and } \max_{[0,1]} X(\cdot)(\omega_1, \omega_2) \ge 1. \end{cases} \end{aligned}$$

Similarly to 2a4,[1] Y is distributed like Y_0. We have

$$Y(t)(\omega_1, \omega_2, \omega_3) = \begin{cases} B(t)(\omega_1) & \text{if } t \le 1, \\ B(1)(\omega_1) + B(t-1)(\omega_2) & \text{if } t \ge 1 \text{ and } \omega_1 \in A, \\ B(1)(\omega_1) + B(t-1)(\omega_3) & \text{if } t \ge 1 \text{ and } \omega_1 \notin A, \end{cases}$$

where $A = \{\omega_1 : \max_{[0,1]} B(\cdot)(\omega_1) < 1\}$.

2d1 Exercise. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $A \subset \Omega$ a measurable set, $f : \Omega^2 \to \mathbb{R}$ a bounded measurable function. Define $g : \Omega^3 \to \mathbb{R}$ by

$$g(\omega_1, \omega_2, \omega_3) = \begin{cases} f(\omega_1, \omega_2) & \text{if } \omega_1 \in A, \\ f(\omega_1, \omega_3) & \text{if } \omega_1 \notin A. \end{cases}$$

Then

$$\iiint_{\Omega^3} g \, \mathrm{d}(P \times P \times P) = \iint_{\Omega^2} f \, \mathrm{d}(P \times P) \, .$$

Prove it.

It follows that Y is distributed like X, therefore, like B, which proves Lemma 2c3 for $n = 0$.

2e The proof, at last

If two non-overlapping changes are separately harmless, then they are jointly harmless in the following sense.

[1] See also 2f4.

2e1 Exercise. (a) Let X, Y_1, Y_2 be identically distributed random variables (on a probability space) such that $\mathbb{P}\big(Y_1 \ne X \text{ and } Y_2 \ne X\big) = 0$. Then the random variable Z defined by

$$Z = \begin{cases} X & \text{if } Y_1 = X \text{ and } Y_2 = X, \\ Y_1 & \text{if } Y_1 \ne X \text{ and } Y_2 = X, \\ Y_2 & \text{if } Y_1 = X \text{ and } Y_2 \ne X \end{cases}$$

is distributed like X.

(b) The same holds for random vectors and random continuous functions.

Prove it.

The same holds for any finite (or countable) collection of pairwise non-overlapping changes.

Proof of 2c3. We consider random continuous functions

$$Y_{n,k}(t)(\omega_1, \omega_2) = \\ = \begin{cases} B(t)(\omega_1) & \text{if } t \le k \cdot 2^{-n}, \\ B(t)(\omega_1) & \text{if } t \ge k \cdot 2^{-n} \text{ and } \tau_n(\omega_1) \ne k \cdot 2^{-n}, \\ B(k \cdot 2^{-n})(\omega_1) + B(t - k \cdot 2^{-n})(\omega_2) & \text{if } t \ge k \cdot 2^{-n} \text{ and } \tau_n(\omega_1) = k \cdot 2^{-n}. \end{cases}$$

Each $Y_{n,k}$ is distributed like B by the argument of 2d. It remains to apply 2e1. □

2f Technicalities: sigma-fields and stopping times

The Borel σ-field[1] $\mathcal{B}$ on the space $C[0,1]$ of all continuous functions $[0,1] \to \mathbb{R}$ can be defined in many equivalent ways; here is the best one for our purposes:

$$\begin{gathered} \mathcal{B} \text{ is generated by the functions} \\ C[0,1] \ni f \mapsto f(t) \in \mathbb{R} \\ \text{where } t \text{ runs over } [0,1]. \end{gathered} \tag{2f1}$$

2f2 Exercise. Prove that each of the following four sets of functions $C[0,1] \to \mathbb{R}$ generates the Borel σ-field:

(a) $f \mapsto f(t)$ for rational $t \in [0,1]$;

(b) $f \mapsto \max_{[a,b]} f(\cdot)$ for $[a,b] \subset [0,1]$;

(c) $f \mapsto \int_a^b f(x)\,\mathrm{d}x$ for $[a,b] \subset [0,1]$;

(d) $f \mapsto \|f - g\|$ for $g \in C[0,1]$.

Prove it.

[1] In other words, "σ-algebra".

It follows easily from (d) that the Borel σ-field is generated by open (or closed) balls, as well as by open (or closed) sets.

For any $t \in [0, \infty)$ the Borel σ-field on $C[0, t]$ is defined similarly.

Now, for a given $t \in [0, \infty)$ we define a σ-field $\mathcal{B}_t$ on the set $C[0, \infty)$ of all continuous (not necessarily bounded) functions $[0, \infty) \to \mathbb{R}$ as consisting of inverse images of all Borel subsets of $C[0, t]$ under the restriction map

$$C[0, \infty) \ni f \mapsto f|_{[0,t]} \in C[0, t] \, .$$

Clearly, $\mathcal{B}_t$ is generated by the functions

$$C[0, \infty) \ni f \mapsto f(s) \in \mathbb{R}$$

for $s \in [0, t]$.

The σ-field generated by $\cup_t \mathcal{B}_t$ will be denoted by $\mathcal{B}_\infty$ and called the Borel σ-field of $C[0, \infty)$. Clearly, $\mathcal{B}_\infty$ is generated by the functions

$$C[0, \infty) \ni f \mapsto f(t) \in \mathbb{R}$$

for $t \in [0, \infty)$.

Here are two equivalent definitions of a *random continuous function.*

2f3 Exercise. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then the following two conditions on a function $X : \Omega \to C[0, \infty)$ are equivalent:

(a) for each $t \in [0, \infty)$ the function

$$\Omega \ni \omega \mapsto X(t)(\omega)$$

is $\mathcal{F}$-measurable;

(b) for each $\mathcal{B}_\infty$-measurable function $\varphi : C[0, \infty) \to \mathbb{R}$, the function

$$\Omega \ni \omega \mapsto \varphi\big(X(\cdot)(\omega)\big)$$

is $\mathcal{F}$-measurable.

Prove it.

For the next exercise you need something like the monotone class theorem or Dynkin's $\pi - \lambda$ theorem; see [1, Appendix A2, (2.1) and (2.2)].

Here are two equivalent definitions of *identically distributed* random continuous functions.

2f4 Exercise. The following two conditions on random continuous functions[1] X, Y are equivalent:

[1] Maybe, on different probability spaces.

(a) for every n and $t_1, \dots, t_n \in [0, \infty)$ the random vectors $\big(X(t_1), \dots, X(t_n)\big)$ and $\big(Y(t_1), \dots, Y(t_n)\big)$ are identically distributed;

(b) for every $\mathcal{B}_\infty$-measurable function $\varphi : C[0, \infty) \to \mathbb{R}$ the random variables $\varphi\big(X(\cdot)\big)$ and $\varphi\big(Y(\cdot)\big)$ are identically distributed.

Prove it.

2f5 Definition. A *stopping time* is a function $T : C[0, \infty) \to [0, \infty]$ such that

$$\{f \in C[0, \infty) : T(f) \le t\} \in \mathcal{B}_t$$

for all $t \in [0, \infty)$.

2f6 Exercise. The hitting time T_1 defined by

$$T_1(f) = \inf\{t : f(t) = 1\}$$

(∞, if the set is empty) is a stopping time.

Prove it.

2f7 Exercise. The function L defined by

$$L(f) = \sup\{t \in [0, 1] : f(t) = 0\}$$

(0, if the set is empty) is not a stopping time.

Prove it.

Here is the strong Markov property of the Brownian motion.

2f8 Theorem. If T is a stopping time then the random function

$$Y(t)(\omega_1, \omega_2) = \begin{cases} B(t)(\omega_1) & \text{for } t \le T(\omega_1), \\ B\big(T(\omega_1)\big)(\omega_1) + B\big(t - T(\omega_1)\big)(\omega_2) & \text{for } t \ge T(\omega_1) \end{cases}$$

on $\Omega \times \Omega$ is distributed like the Brownian motion B.

The proof is quite similar to the proof of 2b2.

2f9 Remark. A weaker (than 2f5) assumption

$$\{f \in C[0, \infty) : T(f) < t\} \in \mathcal{B}_t \quad \text{for all } t \in [0, \infty)$$

is still sufficient for Theorem 2f8 to hold.

(Anyway, a delay is stipulated by the proof, recall 2c). Such T is called a stopping time of the (right-continuous) filtration $(\mathcal{B}_{t+})_t$, where $\mathcal{B}_{t+} = \cap_{\varepsilon>0}\mathcal{B}_{t+\varepsilon}$. In contrast, 2f5 defines a stopping time of the filtration $(\mathcal{B}_t)_t$. (Generally, a filtration is defined as an increasing family of σ-fields.)

Here is an example of a stopping time of $(\mathcal{B}_{t+})_t$ but not $(\mathcal{B}_t)_t$:

$$T_{1+} = \inf\{t : B(t) > 1\}\,.$$

Note that $T_t \downarrow T_{1+}$ as $t \downarrow 1$. Similarly, T_{x+} are introduced for all $x \in [0,\infty)$. Due to 2f9, all said in 2b about the process $(T_x)_{x\in[0,\infty)}$ holds also for $(T_{x+})_{x\in[0,\infty)}$, except for the left continuity; this time we get right continuity.

The random function $(T_{x+})_{x\in[0,\infty)}$ is distributed as the special Lévy process.

2f10 Exercise. $\mathbb{P}\big(T_x = T_{x+}\big) = 1$ for each $x \in [0,\infty)$.

Prove it.

2g Hints to exercises

2a2: Calculate the joint distribution of $Y(t_1), Y(t_2) - Y(t_1), \dots, Y(t_n) - Y(t_{n-1})$ assuming that $t_1 < \dots < t_n$ and $1 \in \{t_1, \dots, t_n\}$.

2a4: $\{\omega : T_x > t\} = \{\omega : \sup_{[0,t]} B(\cdot) < x\}$.

2a5: Use 1c2.

2b3: $\mathbb{P}\big(\max_{[0,t]} B(\cdot) \ge 1\big) = \mathbb{P}\big(T_1 \le t\big)$; use 2b2.

2b5: $\mathbb{P}\big(T_x < \infty\big) = 1$.

2b6: $\lim_{x\to0+} \mathbb{P}\big(T_x < \varepsilon\big) = ?$

2b7: use 2b6.

2b10: Use 2b2.

2d1: Fubini theorem.

2f2: (a), (b), (c): if φ_n are measurable (w.r.t. a given σ-field) and $\varphi_n \to \varphi$ pointwise, then φ is also measurable. (d): take a sequence $(g_n)_n$ dense in $C[0,1]$ and note that $\sup_{n:\|f-g_n\|<1} g_n(t) = f(t) + 1$.

2f3: (a) $\Longrightarrow$(b): all sets $A \subset C[0,\infty)$ such that $X^{-1}(A) \in \mathcal{F}$ are a σ-field.

2f10: T_x and T_{x+} are identically distributed, and $T_x \le T_{x+}$.

References

[1] R. Durrett, *Probability: theory and examples,* 1996.

Index