Solutions to selected exercises

2b24 Exercise (SURFACE OF REVOLUTION OR BODY OF REVOLUTION). Let M_1 be an n-manifold in $\mathbb{R}^3$ (here $n = 1$ or $n = 2$) such that

$$\forall (x, y, z) \in M_1 \;\; (0, -z, y) \notin T_{(x,y,z)} M_1 \,.$$

Consider the set

$$M = \{ (x, cy - sz, sy + cz) : (x, y, z) \in M_1, (c, s) \in S \}$$

where $S = \{ (c, s) \in \mathbb{R}^2 : c^2 + s^2 = 1 \}$ (the circle). Assume that the mapping $\big((x, y, z), (c, s) \big) \mapsto (x, cy - sz, sy + cz)$ is a homeomorphism $M_1 \times S \to M$. Then

(a) M is an $(n + 1)$-manifold in $\mathbb{R}^3$;

(b) if (G_1, ψ_1) is a chart of M_1 and (G_2, ψ_2) is a chart of S, then $(G_1 \times G_2, \psi)$ is a chart of M; here $\psi(u_1, u_2) = (x, cy - sz, sy + cz)$ whenever $\psi_1(u_1) = (x, y, z)$ and $\psi_2(u_2) = (c, s)$.

Prove it.

Solution. Item (a) follows from (b); here is (b).

We have $\psi = F \circ (\psi_1, \psi_2)$ where $F : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$, $F\big((x, y, z), (c, s) \big) = (x, cy - sz, sy + cz)$ and $(\psi_1, \psi_2) : G_1 \times G_2 \to M_1 \times S \subset \mathbb{R}^3 \times \mathbb{R}^2$, $(\psi_1, \psi_2)(u_1, u_2) = \big(\psi_1(u_1), \psi_2(u_2) \big)$.

Similarly to 2b13 it is sufficient to consider a chart of S around the point $(1, 0)$ only, since the rotation $(x, y, z) \mapsto (x, cy - sz, sy + cz)$ of $\mathbb{R}^3$ sends $F\big((x, y, z), (1, 0) \big)$ to $F\big((x, y, z), (c, s) \big)$.

By 2b9, $M_1 \times S \subset \mathbb{R}^3 \times \mathbb{R}^2$ is an $(n + 1)$-manifold; and (from the solution of 2b9), $\big(G_1 \times G_2, (\psi_1, \psi_2) \big)$ is a chart of $M_1 \times S$. Thus, (ψ_1, ψ_2) is a homeomorphism from $G_1 \times G_2$ onto the relatively open (in $M_1 \times S$) set $\psi_1(G_1) \times \psi_2(G_2)$. It is given that $F|_{M_1 \times S}$ is a homeomorphism $M_1 \times S \to M$. Thus, $\psi = F \circ (\psi_1, \psi_2)$ is a homeomorphism from $G_1 \times G_2$ onto the relatively open (in M) set $\psi(G_1 \times G_2)$. Also, $\psi \in C^1(G_1 \times G_2 \to \mathbb{R}^3)$, since $\psi_1 \in C^1(G_1 \to \mathbb{R}^3)$, $\psi_2 \in C^1(G_2 \to \mathbb{R}^2)$, and $F \in C^1(\mathbb{R}^5 \to \mathbb{R}^3)$.

It remains to check that the linear operator $(D\psi)_{(u_1,u_2)} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^3$ is one-to-one, assuming that $\psi_2(u_2) = (1, 0)$. By the chain rule, $(D\psi)_{(u_1,u_2)} = (DF)_{((x,y,z),(1,0))} \circ \big(D(\psi_1, \psi_2) \big)_{(u_1,u_2)}$ where $(x, y, z) = \psi_1(u_1)$. Also, $(D\psi_1)_{u_1} : \mathbb{R}^n \to T_{(x,y,z)} M_1$ is one-to-one, and $(D\psi_2)_{u_2} : \mathbb{R} \to T_{(1,0)} S$ is one-to-one. It remains to check that

$$(DF)_{((x,y,z),(1,0))}(h, k) = 0 \quad \Longrightarrow \quad (h = 0, k = 0)$$

for $h \in T_{(x,y,z)} M_1$, $k \in T_{(1,0)} S$.

We note that

$$\begin{gathered}F\big((x,y,z),(1,0)\big)=(x,y,z)\,;\\ \frac{\partial}{\partial c}F\big((x,y,z),(c,s)\big)=(0,y,z)\,;\\ \frac{\partial}{\partial s}F\big((x,y,z),(c,s)\big)=(0,-z,y)\,.\end{gathered}$$

Also, $T_{(1,0)}S=\{(0,\lambda):\lambda\in\mathbb{R}\}$. Thus,

$$(DF)_{((x,y,z),(1,0))}\big(h,(0,\lambda)\big)=h+\lambda(0,-z,y)\,;$$

if $\lambda\ne 0$ then $h+\lambda(0,-z,y)\ne 0$ since $(0,-z,y)\notin T_{(x,y,z)}M_1$.

Unfortunalety, the formulation of Exercise 2c33 is erroneous: the factor J_{ψ_2} is missing in the formula for J_ψ. I am sorry. Here is the corrected formulation.

2c33 Exercise (SURFACE OF REVOLUTION OR BODY OF REVOLUTION)**.** Let $M_1,n,M,S,(G_1,\psi_1),(G_2,\psi_2),(G_1\times G_2,\psi)$ be as in 2b24(b). Then

$$J_\psi(u_1,u_2)=J_{\psi_1}(u_1)J_{\psi_2}(u_2)\operatorname{dist}((0,-z,y),T_{(x,y,z)}M_1)\quad\text{where }(x,y,z)=\psi_1(u_1)\,.$$

In particular, if $M_1\subset\mathbb{R}^2\times\{0\}$, then also $T_{(x,y,z)}M_1\subset\mathbb{R}^2\times\{0\}$; $(0,-z,y)=(0,0,y)\perp\mathbb{R}^2\times\{0\}$; thus,

$$J_\psi(u_1,u_2)=|y|J_{\psi_1}(u_1)J_{\psi_2}(u_2)\quad\text{where }(x,y,0)=\psi_1(u_1)\,.$$

Prove it.

Solution. Once again, it is sufficient to consider a chart of S around the point $(1,0)$ only. Thus we assume that $\psi_2(0)=(1,0)$ and calculate $J_\psi(u_1,0)$.

Still, $\psi=F\circ(\psi_1,\psi_2)$ and $(DF)_{((x,y,z),(1,0))}\big(h,(0,\lambda)\big)=h+\lambda(0,-z,y)$.

Also, $\psi_2'(0)=(0,\lambda)$ for some λ (since $T_{(1,0)}S=\{(0,\lambda):\lambda\in\mathbb{R}\}$, still); and $J_{\psi_2}(0)=|\lambda|$.

By the chain rule (of Analysis-3),

$$\begin{aligned}(D\psi)_{(u_1,0)}&=(DF)_{((x,y,z),(1,0))}\circ\big(D(\psi_1,\psi_2)\big)_{(u_1,0)}:(h,k)\mapsto\\ &\mapsto(DF)_{((x,y,z),(1,0))}\big((D\psi_1)_{u_1}h,(D\psi_2)_0k\big)=(D\psi_1)_{u_1}h+\lambda k(0,-z,y)\end{aligned}$$

for $h\in\mathbb{R}^n$ and $k\in\mathbb{R}$; here $(x,y,z)=\psi_1(u_1)$. In particular, $(D_k\psi)_{(u_1,0)}=(D_k\psi_1)_{u_1}$ for $1\le k\le n$, and $(D_{n+1}\psi)_{(u_1,0)}=\lambda(0,-z,y)$.

The $(n+1)$-dimensional volume of the parallelotope spanned by these $n+1$ vectors $(D_1\psi)_{(u_1,0)},\dots,(D_{n+1}\psi)_{(u_1,0)}$ is $J_\psi(u_1,0)$, while the n-dimensional volume of its base, the parallelotope spanned by the first n vectors, is

$J_{\psi_1}(u_1)$. The latter parallelotope (the base) spans $T_{(x,y,z)}M_1$; thus, the height of the $(n+1)$-dimensional parallelotope is $|\lambda| \operatorname{dist}((0,-z,y), T_{(x,y,z)}M_1)$. Taking into account that $J_{\psi_2}(0) = |\lambda|$ we get

$$J_\psi(u_1,u_2) = J_{\psi_1}(u_1) J_{\psi_2}(u_2) \operatorname{dist}((0,-z,y), T_{(x,y,z)}M_1) \,.$$