3 Integral of derivative

Multidimensional integral of derivative is much more interesting than one-dimensional.

3a Fundamental theorem of integral calculus: introduction

The fundamental theorem of integral calculus (FTIC),[1] in dimension one, states that

(3a1) $$\int_a^b f'(x)\,\mathrm{d}x = f(b-) - f(a+)$$

whenever f is differentiable on (a, b) and f' is Riemann integrable. In particular,

(3a2) $$\int_{\mathbb{R}} f'(x)\,\mathrm{d}x = 0 \quad \text{if } f \in C^1(\mathbb{R}) \text{ has a bounded support.}$$

Here is a multidimensional generalization:

(3a3) $$\int_{\mathbb{R}^n} \det Df = 0 \quad \text{if } f \in C^1(\mathbb{R}^n \to \mathbb{R}^n) \text{ has a bounded support;}$$

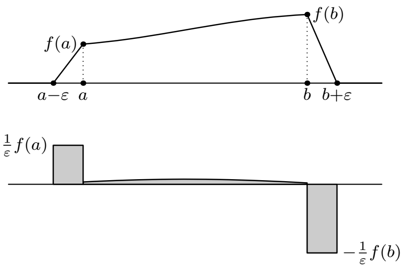
its proof, given later (in Sect. 5c), is considerably harder than the proof of (3a2). And the ultimate form of FTIC is Stokes' theorem (formulated and proved later, in Sect. 6).

The rest of this course is devoted to multidimensional forms of FTIC (including the divergence theorem) and their applications.

[1]The (second) fundamental theorem of (integral) calculus. The terminology is a bit controversial.

3b Integrating the gradient

Interestingly, (3a1) may be thought of as (3a2) applied to the function $f \cdot 1\!\!1_{[a,b]}$. Yes, this function is not differentiable (and moreover, not continuous), but let us approximate it:

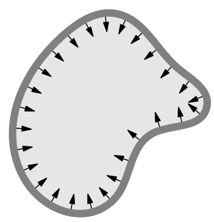


$$f(a) + \int_a^b f'(x)\,\mathrm{d}x - f(b) = 0\,.$$

This simple idea becomes much more interesting in higher dimensions. The equality (3a1) has no evident n-dimensional counterpart, but (3a2) has:

(3b1) $$\int_{\mathbb{R}^n} \nabla f = 0 \quad \text{if } f \in C^1(\mathbb{R}^n) \text{ has a bounded support.}$$

Given a geometric body $E \subset \mathbb{R}^n$, we approximate its indicator $1\!\!1_E$ and take the gradient. In the boundary layer of thickness ε we get the inwards normal vector of length $1/\varepsilon$. In the limit (for $\varepsilon \to 0+$) we see that the integral over ∂E of the (inwards) normal unit vector must vanish. This is indeed true and useful. However, the limiting procedure, helpful for intuition, is less helpful for the proof (which happens often in mathematics).



Now we finish this informal prelude and start the formal theory.

We take the case $n = N - 1$; that is, we consider an n-dimensional manifold M in $\mathbb{R}^{n+1}$, often called a *hypersurface*. In this case, two sides of M will be defined locally (but not globally; think about the Möbius strip), after some preparation. $n=N-1$

Given $x_0 \in M$, we consider germs[1] $[\sigma]$ (at x_0) of functions $\sigma : \mathbb{R}^N \setminus M \to \mathbb{R}$ that are continuous near x_0 and satisfy $\sigma(\cdot) = \pm 1$ near x_0.

3b2 Lemma. There exist exactly 4 such germs; two are constant; the other two are not, and these two are mutually opposite ($[\sigma]$ and $[-\sigma]$).

Proof. The conditions on σ (and M) are invariant under local homeomorphisms. By 2b4(c), WLOG we assume that M is the hyperplane $\mathbb{R}^n \times \{0\}$

[1]Recall Sect. 0e.

near x_0. We take σ of the germ and $\varepsilon > 0$ such that σ is continuous and equal ± 1 on the set $\{x \in \mathbb{R}^N \setminus M : |x - x_0| < \varepsilon\}$, and note that this set has exactly two connected components. $\square$

From now on, $[\sigma]$ stands for one of the two non-constant germs; let us call it *side indicator.*

3b3 Definition. A function $f : \mathbb{R}^N \setminus \overline{M} \to \mathbb{R}$ is *continuous up to M*, if it is continuous (on $\mathbb{R}^N \setminus \overline{M}$) and for every $x_0 \in M$ the limits

$$f_-(x) = \lim_{y \to x, \sigma(y) = -1} f(y) \quad \text{and} \quad f_+(x) = \lim_{y \to x, \sigma(y) = +1} f(y)$$

exist for all $x \in M$ near x_0.

In this case the germs $[f_-], [f_+]$ (of functions on M) are well-defined and continuous. The difference $f_+(x_0) - f_-(x_0)$ of these "one-sided limits" at x_0 is the *jump* of f at x_0. Its sign depends on the sign of σ.

The same applies when M is an n-dimensional manifold in an $(n+1)$-dimensional vector[1] space. In contrast, the unit normal vector and the singular gradient, defined below, require Euclidean metric.

The tangent space $T_x M$, being a hyperplane in $\mathbb{R}^N$, is

$$T_x M = \{h : \langle h, \mathbf{n}_x \rangle = 0\}$$

for some unit vector $\mathbf{n}_x \in \mathbb{R}^N$, the so-called *unit normal vector.* It is well-defined up to the sign. When using together $\mathbf{n}_x$ and the side indicator we always assume that they conform:

$$\sigma(x + \lambda \mathbf{n}_x) = \begin{cases} -1 & \text{for small } \lambda < 0, \\ +1 & \text{for small } \lambda > 0. \end{cases}$$

Thus we have a germ of a mapping $x \mapsto \mathbf{n}_x$. It is continuous due to an explicit formula given in the following exercise. (However, a continuous mapping $x \mapsto \mathbf{n}_x$ on the whole M exists if and only if M is orientable; we'll return to this point in Sect. 4d.)

$n = N - 1$

3b4 Exercise. Let M be locally the graph,

$$\{(x_1, \dots, x_N) : x_N = g(x_1, \dots, x_n)\},$$

of a continuously differentiable function g. Then the formula

$$\sigma(x_1, \dots, x_N) = \begin{cases} -1 & \text{for } x_N < g(x_1, \dots, x_n), \\ +1 & \text{for } x_N > g(x_1, \dots, x_n) \end{cases}$$

[1] Or affine.

defines a side indicator, and the formula

$$\mathbf{n}_x = \frac{1}{\sqrt{1+|\nabla g|^2}}\big(-(D_1 g),\dots,-(D_n g),1\big)$$

defines the corresponding unit normal vector.

Prove it.[1]

Here is a more convenient notation for the one-sided limits:

$$f(x-0\mathbf{n}_x) = f_-(x) \quad \text{and} \quad f(x+0\mathbf{n}_x) = f_+(x)\,.$$

3b5 Definition. The *singular gradient*[2] $\nabla_{\rm sng} f(x)$ at $x \in M$ of a function $f : \mathbb{R}^N \setminus \overline{M} \to \mathbb{R}$ continuous up to M is the vector

$$\nabla_{\rm sng} f(x) = \big(f(x+0\mathbf{n}_x) - f(x-0\mathbf{n}_x)\big)\mathbf{n}_x\,.$$

Note that the singular gradient does not depend on the sign of σ (and $\mathbf{n}_x$). It is a continuous mapping $\nabla_{\rm sng} f : M \to \mathbb{R}^N$. (Think, what happens if M is the Möbius strip.)

3b6 Lemma. Assume that $f : \mathbb{R}^N \setminus \overline{M} \to \mathbb{R}$ is continuously differentiable, and ∇f is bounded (on $\mathbb{R}^N \setminus \overline{M}$). Then f is continuous up to M.

Proof. Let $x_0 \in M$; we have to prove existence of the two one-sided limits. We reuse the argument of the proof of 3b2: the conditions on f (and M) are invariant under local diffeomorphisms.[3] By 2b4(c), WLOG we assume that M is the hyperplane $\mathbb{R}^n \times \{0\}$ in the ε-neighborhood of x_0;[4] and $|\nabla f| \le C$ on this neighborhood. The set $\{x \in \mathbb{R}^N \setminus M : |x-x_0| < \varepsilon\}$ consists of two open half-balls. We have $|f(y)-f(z)| \le C|y-z|$ whenever y,z belong to the same half-ball (think, why). If $y_k \to x_0$ and all y_k belong to the same half-ball, then $f(y_k)$ are a Cauchy sequence, therefore, converge. And $|f(y_k)-f(z_k)| \to 0$ whenever z_k are another such sequence. □

A compact subset of an n-manifold in $\mathbb{R}^N$ is of (N-dimensional) volume zero[5] (recall 2d11(a)). However, this may fail for a bounded subset. When a manifold M is not a closed set,[6] it may be rather wild near a point of $\overline{M} \setminus M$, see the example below. Note that the set $\overline{M} \setminus M$ is closed (which was seen in the proof of 3b6).

[1] Hint: $\frac{\rm d}{{\rm d}\lambda}\big|_{\lambda=0}\varphi(x+\lambda\mathbf{n}_x)$ where $\varphi(x_1,\dots,x_N) = x_N - g(x_1,\dots,x_n)$.

[2] Not a standard terminology.

[3] The bound for ∇f need not be invariant, but boundedness is invariant.

[4] This neighborhood cannot contain a point of $\overline{M} \setminus M$; think, why.

[5] Except for the case $n = N$, of course.

[6] Be warned: "The notion of closed manifold is unrelated with that of a closed set." Wikipedia:Closed manifold#Contrasting terms

3b7 Example. A bounded 1-manifold in $\mathbb{R}^2$ need not be a set of area zero.

We start with a sequence of pairwise disjoint closed intervals $[s_1, t_1], [s_2, t_2], \ldots \subset (0, 1)$ such that $\sum_k (t_k - s_k) = a < 1$ and the open set $G = (s_1, t_1) \cup (s_2, t_2) \cup \ldots$ is dense in $(0, 1)$.[1] The set $M_0 = \{\frac{s_k + t_k}{2} : k = 1, 2, \ldots\}$ of the centers of these intervals is a 0-manifold in $\mathbb{R}$ (a discrete set). Its closure contains $[0, 1] \setminus G$; thus, $v^*(M_0) = v^*(\overline{M}_0) = 1 - a > 0$.

The set $M_1 = M_0 \times (0, 1)$ is a 1-manifold in $\mathbb{R}^2$ (recall 2b9), not of area zero.

$n = N - 1$

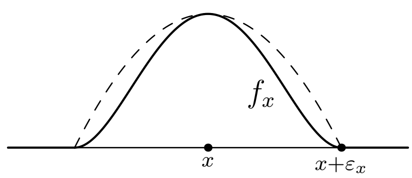
3b8 Theorem. Let $M \subset \mathbb{R}^N$ be an n-manifold, $K \subset M$ a compact subset, and $f : \mathbb{R}^N \setminus K \to \mathbb{R}$ a continuously differentiable function with a bounded support and bounded gradient ∇f (on $\mathbb{R}^N \setminus K$). Then

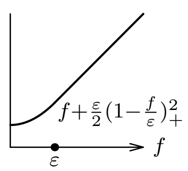
$$\int_{\mathbb{R}^N \setminus K} \nabla f + \int_M \nabla_{\mathrm{sng}} f = 0 \, .$$

3b9 Remark. First, continuity up to M is ensured by Lemma 3b6. Second, both integrands being vector-valued, both integrals are treated as in (2d25)–(2d26). Third, K is of volume zero, and $\nabla f \cdot 1\!\!1_{\mathbb{R}^{n+1} \setminus K}$ is integrable (think, why).[2] Fourth, $\nabla_{\mathrm{sng}} f$ is continuous and compactly supported (by K) on M (think, why).

3b10 Lemma. Let $(U_1, \ldots, U_\ell)$ be an open covering of a compact set $K \subset \mathbb{R}^N$. Then there exist functions $\rho_1, \ldots, \rho_i \in C^1(\mathbb{R}^N \to [0, 1])$ such that $\rho_1 + \cdots + \rho_i = 1$ on K and each ρ_j has a compact support within some U_m.

Proof. Similar to the proof of Lemma 2d3, with the following modifications. First, the sets $U_1, \ldots, U_\ell$ are used instead of charts. Second, functions $f_x : y \mapsto (\varepsilon_x^2 - |y - x|^2)_+^2$ are used instead of $y \mapsto (\varepsilon_x - |y - x|)_+$, in order to ensure continuous differentiability.





Third (for the same reason), $f(x) + \frac{\varepsilon}{2}\big(1 - \frac{f(x)}{\varepsilon}\big)_+^2$ is used in the denominator of ρ_k instead of $\max(f(x), \varepsilon)$. $\square$

Still, $N = n + 1$, and M, K, f are as in Theorem 3b8.

[1] Its complement $[0, 1] \setminus G$ is sometimes called a fat Cantor set.

[2] Lebesgue's criterion applies. But really, a much simpler argument suffices, since "volume zero" is much stronger than "Lebesgue measure zero".

3b11 Lemma. Let $h \in \mathbb{R}^N$ be such that

$$\forall x \in K \;\; h \notin T_x M\,.$$

Then

$$\Big\langle h, \int_{\mathbb{R}^N \setminus K} \nabla f + \int_M \nabla_{\rm sng} f \Big\rangle = 0\,.$$

$n=N-1$

Proof. WLOG, $h = (0, \dots, 0, 1)$ is the last vector of the usual basis of $\mathbb{R}^N$ (otherwise, use another orthonormal basis).

For every $x \in K$ we take a co-chart (U, φ) of M around x. By 2b19(c), $(D\varphi)_x h \ne 0$, that is, $(D_N \varphi)_x \ne 0$. The implicit function theorem gives us an open set $U \subset \mathbb{R}^n$ and an open interval $V \subset \mathbb{R}$ such that $x \in U \times V$ and $M \cap (U \times V)$ is the graph of a function $U \to V$ (of class C^1). Such $U \times V$ are an open covering of K. We take a finite subcovering $W_1, \dots, W_\ell$, add one more open set $W_0 = \mathbb{R}^N \setminus K$, and get a finite open covering of a closed ball B that supports f. Lemma 3b10 gives $\rho_1, \dots, \rho_i \in C^1(\mathbb{R}^N)$ such that $\rho_1 + \dots + \rho_i = 1$ on B and each ρ_j has a compact support within some W_m. Taking into account linearity of integrals and gradients we reduce the claim for f to the same claim for $\rho_1 f, \dots, \rho_i f$. Thus, WLOG, we may assume that f has a compact support either within $\mathbb{R}^N \setminus K$ or within some $U \times V$.

If f has a compact support within $\mathbb{R}^N \setminus K$ then we extend f to K as 0 and get $f \in C^1(\mathbb{R}^N)$, $\nabla_{\rm sng} f = 0$, $\int_{\mathbb{R}^N \setminus K} \nabla f = \int_{\mathbb{R}^N} \nabla f = 0$ by (3b1).

It remains to consider f that has a compact support within some $U \times V$, $V = (a, b)$, such that $M \cap (U \times V)$ is the graph of some $g \in C^1(U)$, $g : U \to (a, b)$. On one hand, taking into account that ∇f is integrable,

$$\begin{aligned}\int_{\mathbb{R}^N \setminus K} \langle h, \nabla f \rangle &= \int_{U \times (a,b) \setminus K} D_N f = \\ &= \int_U \mathrm{d}u_1 \dots \mathrm{d}u_n \bigg(\int_a^{g(u)} + \int_{g(u)}^b \bigg) \mathrm{d}t \frac{\partial}{\partial t} f(u_1, \dots, u_n, t) = \\ &= \int_U \big(f(u, g(u)-) - f(u, g(u)+) \big) \, \mathrm{d}u\,.\end{aligned}$$

On the other hand, using the side indicator

$$\sigma(u,t) = \begin{cases} -1 & \text{for } t < g(u), \\ +1 & \text{for } t > g(u) \end{cases} \quad \text{for } u \in U \text{ and } t \in (a,b)\,,$$

we have for $u \in U$ and $x = (u, g(u))$

$$\mathbf{n}_x = \frac{1}{\sqrt{1 + |\nabla g(u)|^2}} \big(-(D_1 g)_u, \dots, -(D_n g)_u, 1 \big)$$

(recall 3b4); thus,

$$\begin{gathered}
\langle h, \mathbf{n}_x \rangle = \frac{1}{\sqrt{1+|\nabla g(u)|^2}} \,; \\
f(x-0\mathbf{n}_x) = f\big(u, g(u)-\big) \,, \quad f(x+0\mathbf{n}_x) = f\big(u, g(u)+\big) \,; \\
\langle h, \nabla_{\mathrm{sng}} f(x) \rangle = \frac{f\big(u, g(u)+\big) - f\big(u, g(u)-\big)}{\sqrt{1+|\nabla g(u)|^2}} \,;
\end{gathered}$$

and finally, using 2c20,

$$\begin{aligned}
\int_M \langle h, \nabla_{\mathrm{sng}} f \rangle = \int_U \frac{f\big(u, g(u)+\big) - f\big(u, g(u)-\big)}{\sqrt{1+|\nabla g(u)|^2}} \sqrt{1+|\nabla g(u)|^2} = \\
= \int_U \big(f(u, g(u)+) - f(u, g(u)-) \big) \, \mathrm{d}u \,.
\end{aligned}$$

□

***Proof** of Theorem 3b8.* Every point of M has a neighborhood U such that[1]

$$|\langle \mathbf{n}_x, \mathbf{n}_y \rangle| \ge \frac{1}{2} \quad \text{for all } x, y \in M \cap U$$

(since $y \mapsto \mathbf{n}_y$ is continuous near x). A partition of unity (used similarly to the proof of 3b11) reduces the claim for f to the same claim for ρf where $\rho \in C^1(\mathbb{R}^N)$ has a compact support either within $\mathbb{R}^N \setminus K$ or within some U. The former case is trivial (as before); consider the latter case: ρ has a compact support within U. We introduce $\tilde K = K \cap \overline{U}$, extend ρf to $\mathbb{R}^N \setminus \tilde K$ as 0 on $K \setminus \tilde K$, and observe that $\tilde K$ and ρf satisfy the conditions of Theorem 3b8. Thus (renaming $\tilde K$ and ρf into K and f), WLOG,

$$|\langle \mathbf{n}_x, \mathbf{n}_y \rangle| \ge \frac{1}{2} \quad \text{for all } x, y \in K \,.$$

We choose $x_0 \in K$ and note that every $h \in \mathbb{R}^N$ such that $|h - \mathbf{n}_{x_0}| < 1/2$ satisfies the condition of Lemma 3b11, since

$$h \in T_x M \implies \langle h, \mathbf{n}_x \rangle = 0 \implies |\langle \mathbf{n}_{x_0}, \mathbf{n}_x \rangle| = |\langle \mathbf{n}_{x_0} - h, \mathbf{n}_x \rangle| < \frac{1}{2} \implies x \notin K \,.$$

By 3b11, the vector $\int_{\mathbb{R}^N \setminus M} \nabla f + \int_M \nabla_{\mathrm{sng}} f$ is orthogonal to all these h, and therefore, equal to zero. □

[1] Any number of $(0,1)$ may be used instead of $1/2$.

Often $f = uv$ where u, v both satisfy the conditions of Theorem 3b8. Then $\nabla f = u\nabla v + v\nabla u$ (by the product rule), thus,

$$\int_{\mathbb{R}^N \setminus K} u\nabla v = -\int_{\mathbb{R}^N \setminus K} v\nabla u - \int_M \nabla_{\text{sng}}(uv)\,; \tag{3b12}$$

this is a kind of multidimensional integration by parts. And, of course,

$$\int_{\mathbb{R}^N} u\nabla v = -\int_{\mathbb{R}^N} v\nabla u \tag{3b13}$$

for $u, v \in C^1(\mathbb{R}^N)$ such that uv is compactly supported.

Often, a hypersurface M is a boundary $\partial G = \overline{G} \setminus G$ of an open set $G \subset \mathbb{R}^N$. It may seem that in this case M must be orientable, with two sides (globally), inner and outer; but this is generally wrong. A manifold M (not just a hypersurface) is a boundary of some G if and only if M is a closed set. Here is why. On one hand, ∂G is always closed. On the other hand, given a closed M, we may take $G = \mathbb{R}^N \setminus M$ and get $\partial G = M$ (even for the non-orientable compact M of 2b13). Boundedness of G does not help; if $G = B \setminus M$ where $B \supset M$ is an open ball, then ∂G consists of M and the sphere ∂B.

In fact, if a hypersurface is a closed set, then it is orientable;[1] but even in this case both sides may be inner for a given G (try $G = \mathbb{R}^N \setminus M$ or $G = B \setminus M$ again).

An open set $G \subset \mathbb{R}^N$ is called *regular,* if $(\overline{G})^\circ = G$; that is, the interior of the closure of G is equal to G. (Generally it cannot be less than G, but can be more than G; a simple example: $G = \mathbb{R} \setminus \{0\}$.) Equivalently, G is regular if (and only if) $\partial G = \partial(\mathbb{R}^N \setminus \overline{G})$; that is, the boundary of the exterior of G is equal to the boundary of G.

3b14 Definition. A bounded regular open set $G \subset \mathbb{R}^N$ whose boundary ∂G is a (necessarily compact) hypersurface (that is, n-manifold for $n = N - 1$) will be called a *smooth set.*[2]

From now on (till the end of 3b), $G \subset \mathbb{R}^N$ is a smooth set.

By 2d11(a), ∂G is of volume zero; thus, G is Jordan measurable. The function

$$\sigma(x) = \begin{cases} -1 & \text{for } x \in G, \\ +1 & \text{for } x \notin \overline{G} \end{cases}$$

[1] See for instance "Orientability of hypersurfaces in $\mathbb{R}^n$" by H. Samelson, Proc. Amer. Math. Soc. **22**:1 (1969) 301–302.

[2] Not a standard terminology.

is a side indicator on the whole M. The corresponding *outward* unit normal vector $\mathbf{n}_x$ satisfies

$$\begin{aligned} &x + \lambda \mathbf{n}_x \in G \quad \text{for small } \lambda < 0\,, \\ &x + \lambda \mathbf{n}_x \notin G \quad \text{for small } \lambda > 0\,. \end{aligned}$$

Let $f \in C^1(G)$, with ∇f bounded (on G). Then the function $\tilde{f} : \mathbb{R}^N \setminus M \to \mathbb{R}$ defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in G, \\ 0 & \text{for } x \notin \overline{G} \end{cases}$$

is continuous up to M by Lemma 3b6. We extend f to $\overline{G}$ by continuity and get

$$\tilde{f}(x - 0\mathbf{n}_x) = f(x)\,, \quad \tilde{f}(x + 0\mathbf{n}_x) = 0\,;$$
$$\nabla_{\text{sng}} \tilde{f}(x) = -f(x)\mathbf{n}_x\,.$$

By Theorem 3b8 (applied to $\tilde{f}$ and $K = M$),

$$\int_G \nabla f = \int_M f\mathbf{n}\,. \tag{3b15}$$

In particular, for $f(\cdot) = 1$,

$$\int_M \mathbf{n} = 0\,; \tag{3b16}$$

and for a linear function $f : x \mapsto \langle h, x \rangle$,

$$\int_M \langle h, \cdot \rangle \mathbf{n} = v(G)h \quad \text{for } h \in \mathbb{R}^N\,. \tag{3b17}$$

Interestingly, (3b17) for $N = 3$ is basically Archimedes' principle: the upward buoyant force that is exerted on a body immersed in a fluid, is equal to the weight of the fluid that the body displaces.[1] Here is why. At a point $(x, y, z) \in \mathbb{R}^2 \times (-\infty, 0) \subset \mathbb{R}^3$, the depth below the surface of water being $(-z)$, the hydrostatic pressure is $\rho g(-z)$, where ρ is the water density, and $g \approx 9.8\,\text{m/s}^2$ is the gravitational acceleration. Infinitesimally, the force per unit area is $\rho g(-z)(-\mathbf{n}_{(x,y,z)}) = \langle h, (x, y, z) \rangle \mathbf{n}_{(x,y,z)}$ where $h = \rho g(0, 0, 1)$. The total force is $\int_M \langle h, \cdot \rangle \mathbf{n} = v(G)h = \rho g v(G)(0, 0, 1)$, the weight of the mass $\rho v(G)$ of the displaced water, directed upwards.

[1] Wikipedia:Archimedes' principle.

3c Curvilinear iterated integral

The iterated integral approach decomposes an integral over the plane into integrals over parallel lines. It also decomposes an integral over 3-dimensional space into integrals over parallel planes.[1] We want to understand, whether or not a 2-dimensional integral decomposes into integrals over curves $\varphi(\cdot) =$ const; and what happens in dimension 3 (and more). Surprisingly, Theorem 3b8 helps to prove the following.

$n = N - 1$

3c1 Theorem. Let $G \subset \mathbb{R}^N$ be an open set, $\varphi \in C^1(G)$, $\forall x \in G \;\; \nabla\varphi(x) \ne 0$, and $f \in C(G)$ compactly supported. Then for every $c \in \varphi(G)$ the set $M_c = \{x \in G : \varphi(x) = c\}$ is an n-manifold in $\mathbb{R}^N$, the function $c \mapsto \int_{M_c} f$ on $\varphi(G)$ is continuous and compactly supported, and

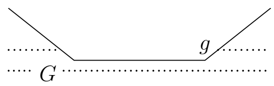
$$\int_{\varphi(G)} \mathrm{d}c \int_{M_c} f = \int_G f|\nabla\varphi| \, .$$

3c2 Remark. The open sets $G \subset \mathbb{R}^N$ and $\varphi(G) \subset \mathbb{R}$ need not be Jordan measurable, but still, the integrals are well-defined, since f is supported by some compact $K \subset G$, and the function $c \mapsto \int_{M_c} f$ is supported by the compact $\varphi(K) \subset \varphi(G)$.

The new factor $|\nabla\varphi|$ shows roughly, how many hypersurfaces M_c intersect an infinitesimal neighborhood of a point.

The set M_c is an n-manifold, just because $(G, \varphi(\cdot) - c)$ is a co-chart of the whole M_c. But continuity of the function $c \mapsto \int_{M_c} f$ is a harder matter.

3c3 Remark. A function $c \mapsto v(M_c)$ need not be continuous on $\varphi(G)$. For a counterexample try $G = \{(x, y) : y < g(x)\} \subset \mathbb{R}^2$ and $\varphi(x, y) = y$.



3c4 Lemma. The function $c \mapsto \int_{M_c} f$ on $\varphi(G)$ is continuous.

Proof. If $x \in G$ satisfies $(D_N\varphi)_x \ne 0$, then the mapping $h : (\tilde{x}_1, \dots, \tilde{x}_N) \mapsto \big(\tilde{x}_1, \dots, \tilde{x}_n, \varphi(\tilde{x}_1, \dots, \tilde{x}_N)\big)$ is a local diffeomorphism near x (recall the proof of the implicit function theorem), and we may take open neighborhoods U of x, V of $(x_1, \dots, x_n)$ and W of $\varphi(x)$ such that h is a diffeomorphism between U and $V \times W$.

If $(D_N\varphi)_x = 0$, we just use another coordinate in the same way.

Using a partition of unity (similarly to the proof of 3b11) we see that, WLOG, f is supported by U such that $h : U \to V \times W$ is a diffeomorphism.

[1] Or alternatively, parallel lines. In this course we restrict ourselves to dimension $n+1$; for dimension $n+m$ see the "Coarea formula" in Encyclopedia of Math.

Now, $M_c \cap U$ is the graph of the function g_c defined by $h^{-1}(x_1, \dots, x_n, c) = \big(x_1, \dots, x_n, g_c(x_1, \dots, x_n)\big)$. Using 2c20,

$$\int_{M_c} f = \int_V f\big(x_1, \dots, x_n, g_c(x_1, \dots, x_n)\big) \sqrt{1 + |\nabla g_c(x_1, \dots, x_n)|^2} \, \mathrm{d}x_1 \dots \mathrm{d}x_n \, .$$

The integrand is uniformly continuous on the relevant compact set, therefore the integral is continuous. □

3c5 Lemma.

$$\int_{\varphi(G)} \mathrm{d}c \int_{M_c} \frac{f}{|\nabla\varphi|} \nabla\varphi = \int_G f \nabla\varphi$$

for G, φ, f and M_c as in Theorem 3c1.

Proof. Let $K \subset G$ be a compact set that supports f. Clearly, φ is bounded on K. WLOG, there exists $C > 0$ such that $0 < \varphi(\cdot) < C$ on K (since we may add a large constant to φ). WLOG, $f \in C^1(G)$, since it can be approximated uniformly by functions of class C^1 supported by a small neighborhood of K (and the volume of the relevant part of M_c is bounded in c).

Given $c \in (0, C) \cap \varphi(G)$, we introduce $G_c = \{x \in G : \varphi(x) > c\}$, $K_c = K \cap M_c$ (empty, if $c \notin \varphi(K)$), and define $f_c : \mathbb{R}^N \setminus K_c \to \mathbb{R}$ by

$$f_c(x) = \begin{cases} f(x) & \text{if } x \in G_c, \\ 0 & \text{otherwise} \end{cases}$$

for $x \in \mathbb{R}^N \setminus K_c$. Clearly, f_c is continuously differentiable (on $\mathbb{R}^N \setminus K_c$), with bounded gradient, and

$$\nabla f_c(x) = \begin{cases} \nabla f(x) & \text{if } x \in G_c, \\ 0 & \text{otherwise} \end{cases}$$

for $x \in \mathbb{R}^N \setminus K_c$.

Using the side indicator

$$\sigma(x) = \begin{cases} -1 & \text{if } \varphi(x) < c, \\ +1 & \text{if } \varphi(x) > c \end{cases}$$

and the unit normal vector

$$\mathbf{n}_x = \frac{1}{|\nabla\varphi(x)|} \nabla\varphi(x) \, ,$$

we see that f_c is continuous up to M_c,

$$f_c(x - 0\mathbf{n}_x) = 0\,, \quad f_c(x + 0\mathbf{n}_x) = f(x)\,;$$
$$\nabla_{\mathrm{sng}} f_c(x) = f(x)\mathbf{n}_x = \frac{f(x)}{|\nabla\varphi(x)|}\nabla\varphi(x)\,.$$

By Theorem 3b8,

$$\int_{\mathbb{R}^N\setminus K_c} \nabla f_c + \int_{M_c} \nabla_{\mathrm{sng}} f_c = 0\,,$$

that is,

$$\int_{G_c} \nabla f + \int_{M_c} \frac{f}{|\nabla\varphi|}\nabla\varphi = 0\,.$$

Now we have to integrate it in c. We apply the iterated integral to the function $G \times (0, C) \to \mathbb{R}^N$, $(x, c) \mapsto \mathbb{1}_{G_c}(x)\nabla f(x)$, integrable since it is discontinuous only on the set $\{(x, \varphi(x)) : x \in K\}$ of volume zero; we get

$$\int_0^C \mathrm{d}c \int_{G_c} \mathrm{d}x\, \nabla f(x) = \int_G \mathrm{d}x\, \nabla f(x) \underbrace{\int_0^C \mathrm{d}c\, \mathbb{1}_{G_c}(x)}_{\varphi(x)} = \int_G \varphi\nabla f\,.$$

By (3b13), $\int_G \varphi\nabla f = -\int_G f\nabla\varphi$. It remains to note that $\int_{G_c} \nabla f = 0$ for $c \in (0, C) \setminus \varphi(G)$, since in this case $f_c \in C^1(G)$. $\square$

***Proof** of Theorem 3c1.* Using a partition of unity (similarly to the proof of 3b11) we see that, WLOG, there exists $h \in \mathbb{R}^N$ such that $|h| = 1$ and $D_h\varphi > 0$ on a compact $K \subset G$ that supports f. Applying Lemma 3c5 to the function $\frac{f|\nabla\varphi|}{\langle h, \nabla\varphi\rangle}$ we get

$$\int_{\varphi(G)} \mathrm{d}c \int_{M_c} \frac{f|\nabla\varphi|}{\langle h, \nabla\varphi\rangle}\frac{\nabla\varphi}{|\nabla\varphi|} = \int_G \frac{f|\nabla\varphi|}{\langle h, \nabla\varphi\rangle}\nabla\varphi\,.$$

It remains to take the scalar product by h. $\square$

3c6 Exercise. Apply Theorem 3c1 to $G = \mathbb{R}^2$, $\varphi(x, y) = y - \sin x$. Is it true that $\iint f(x, y - \sin x)\,\mathrm{d}x\mathrm{d}y = \iint f(x, y)\,\mathrm{d}x\mathrm{d}y$?

3c7 Exercise. (a) Apply Theorem 3c1 to $G = \mathbb{R}^2 \setminus \{0\}$, $\varphi(x) = |x|$; compare the result with integration in polar coordinates. Do they agree?

(b) The same for spherical coordinates.

3c8 Exercise. (a) $\displaystyle\int_0^\infty \mathrm{d}r \int_{|\cdot|=r} f = \int_{|\cdot|>0} f$
for all compactly supported $f \in C(\mathbb{R}^N \setminus \{0\})$.

(b) Generalize it (formulate accurately, and prove) for all integrable f on $\mathbb{R}^N$.[1]

Taking $f(x) = 1$ for $|x| < 1$, otherwise 0, we get $\int_0^1 v(S_r)\,\mathrm{d}r = v(B_1) = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ (recall Sect. 0g), where $S_r = \{x : |x| = r\}$ is a sphere, and $B_1 = \{x : |x| < 1\}$ a ball. By 2d15, $v(S_r) = r^{N-1}v(S_1)$. Thus, $v(B_1) = \int_0^1 r^{N-1}v(S_1)\,\mathrm{d}r = \frac{1}{N}v(S_1)$;

$$v(S_1) = \frac{2\pi^{N/2}}{\Gamma(N/2)}\,. \tag{3c9}$$

3c10 Exercise. Find the $(N-1)$-dimensional volume of the simplex $M = \{x \in (0,\infty)^N : x_1 + \dots + x_N = 1\}$ in $\mathbb{R}^N$.[2]

3c11 Exercise. Integrate the function $x \mapsto x_1^{p_1} \dots x_N^{p_N}$ over the hypersurface $S_+ = \{x \in (0,\infty)^N : |x| = 1\}$ (the positive part of the sphere) in $\mathbb{R}^N$ for $p_1, \dots, p_N \in (-1, \infty)$.[3]

3c12 Exercise. Find $\displaystyle\int_{\mathbb{R}^n} \frac{\mathrm{d}x}{(1+|x|^2)^p}$ for $p \in \big(\frac{n}{2}, \infty\big)$.[4]

3d Divergence: introduction

Here is a straightforward generalization of (3a2):

$$\int_{\mathbb{R}^n} Df = 0 \quad \text{if } f \in C^1(\mathbb{R}^n \to \mathbb{R}^m) \text{ has a bounded support;} \tag{3d1}$$

but this is rather trivial. Indeed, $(Df)_x$ may be thought of as a matrix whose rows are gradients of the coordinate functions $f_1, \dots, f_m \in C^1(\mathbb{R}^n)$ of f, and (3d1) is just (3b1) applied rowwise. We cannot derive (3a3) from (3d1), since

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[1] Hint: $\varphi(x) = |x|$.

[2] Answer: $\sqrt{N}/(N-1)!$. Hint: similar to (3c9); use the multidimensional beta integral of Dirichlet (Sect. 0g) for $p_1 = \dots = p_N = 1$.

[3] Answer: $\frac{\Gamma(\frac{p_1+1}{2})\dots\Gamma(\frac{p_N+1}{2})}{2^{N-1}\Gamma(\frac{p_1+\dots+p_N+N}{2})}$. Hint: $\int_{(0,\infty)^N} \mathrm{e}^{-|x|^2} x_1^{p_1} \dots x_N^{p_N}\,\mathrm{d}x$.

[4] Answer: $\pi^{n/2}\frac{\Gamma(p-\frac{n}{2})}{\Gamma(p)}$. Hint: $\int_0^{\pi/2} \cos^{\alpha-1}\theta \sin^{\beta-1}\theta\,\mathrm{d}\theta = \frac{1}{2}\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta}{2})}$ for $\alpha, \beta \in (0,\infty)$. Alternatively you can do it without manifolds, similarly to $\int f(\|x\|)\,\mathrm{d}x$ in Sect. 0g.

the determinant is a nonlinear function of a matrix. A useful linear function of a matrix is the trace. It follows from (3d1) (when $m = n$) that

(3d2) $$\int_{\mathbb{R}^n} \operatorname{tr}(Df) = 0 \quad \text{if } f \in C^1(\mathbb{R}^n \to \mathbb{R}^n) \text{ has a bounded support.}$$

Now the question is, what is $\operatorname{tr}(Df)$ good for?

Consider a one-parameter family of diffeomorphisms $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ given for $t \in \mathbb{R}$; we assume that the mapping $(x, t) \mapsto \varphi_t(x)$ belongs to $C^2(\mathbb{R}^{n+1} \to \mathbb{R}^n)$, and $\varphi_0(x) = x$ for all $x \in \mathbb{R}^n$. Then $(D\varphi_0)_0 = I$ and $(D\varphi_t)_0 = I + tA + o(t)$ where $A = \frac{d}{dt}\big|_{t=0}(D\varphi_t)_0$; thus, $\det(D\varphi_t)_0 = 1 + t \operatorname{tr} A + o(t)$ for small t (recall Sect. 0f). If $\operatorname{tr} A > 0$, then $\det(D\varphi_t)_0 > 1$ for small $t > 0$, which means that $v\big(\varphi_t(U)\big) > v(U)$ for a small enough neighborhood U of 0 in $\mathbb{R}^n$. Moreover, $v\big(\varphi_t(U)\big) \approx (1 + t \operatorname{tr} A) v(U)$.

In mechanics, a flowing matter may be described this way; every point x flows to another point $\varphi_t(x)$ during the time interval $(0, t)$. A small drop of the flowing matter inflates if $\operatorname{tr} A > 0$ and deflates if $\operatorname{tr} A < 0$. The rate of this inflation/deflation is $\operatorname{tr} A$.

The vector $F(x) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\varphi_t(x)$ is the velocity of the flow at a point x and the instant 0. This mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called the *velocity field* of the flow. We have

$$A = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(D\varphi_t)_0 = \Big(D\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\varphi_t\Big)\Big)_0 = (DF)_0\,,$$

thus, the inflation/deflation rate at the origin is $\operatorname{tr} A = \operatorname{tr}(DF)_0$, and similarly, at a point x it is $\operatorname{tr}(DF)_x$.

The velocity field is a *vector field.* The word “field” in “vector field” is not related to the algebraic notion of a field. Rather, it is related to the physical notion of a force field (gravitational, for example), or the velocity field of a moving matter (usually liquid or gas). Mathematically, a vector field formally is just a mapping $\mathbb{R}^n \to \mathbb{R}^n$; less formally, a vector is attached to each point.

Note that the determinant is well-defined in a (finite-dimensional) vector space; metric is irrelevant. The same holds for the trace.

3d3 Definition. The *divergence* of a mapping (“vector field”) $F \in C^1(\mathbb{R}^n \to \mathbb{R}^n)$ is the function (“scalar field”) $\operatorname{div} F \in C(\mathbb{R}^n)$,

$$\operatorname{div} F = \operatorname{tr}(DF)\,.$$

That is, for $F(x) = \big(F_1(x), \dots, F_n(x)\big)$ we have

$$\operatorname{div} F = D_1F_1 + \dots + D_nF_n = (\nabla F_1)_1 + \dots + (\nabla F_n)_n\,;$$

$$\operatorname{div} F(x_1, \dots, x_n) = \frac{\partial}{\partial x_1}F_1(x_1, \dots, x_n) + \dots + \frac{\partial}{\partial x_n}F_n(x_1, \dots, x_n)\,.$$

Once again: if F is a velocity field, then $\operatorname{div} F$ is the inflation/deflation rate.

For a vector field $F \in C^1(V \to V)$ on an n-dimensional vector space V, still, $\operatorname{div} F = \operatorname{tr}(DF)$; here $(DF)_x : V \to V$.

By (3d2),

$$\int_{\mathbb{R}^n} \operatorname{div} F = 0 \quad \text{if } F \text{ has a bounded support.} \tag{3d4}$$

Similarly to the singular gradient (treated in Sect. 3b), we want to introduce singular divergence; and then, similarly to Theorem 3b8, we want to generalize (3d4) to a vector field continuous up to a surface.

3e Integrating the divergence

$n = N - 1$

Similarly to Sect. 3b we consider a hypersurface, that is, an n-dimensional manifold M in $\mathbb{R}^N$, $N = n + 1$. Similarly to 3b3, for a vector field $F : \mathbb{R}^N \setminus \overline{M} \to \mathbb{R}^N$ we define the notion "continuous up to M". Clearly, $F = (F_1, \dots, F_N)$ is continuous up to M if and only if $F_1, \dots, F_N$ are continuous up to M (as defined by 3b3). The one-sided limits F_-, F_+ are now vector-valued, and the jump $F_+(x_0) - F_-(x_0)$ is a vector; its sign depends on the side indicator. Recall the unit normal vector $\mathbf{n}_x \in \mathbb{R}^N$; its sign also depends on the side indicator. Here is a definition similar to 3b5. As before, we denote $F(x - 0\mathbf{n}_x) = F_-(x)$ and $F(x + 0\mathbf{n}_x) = F_+(x)$.

3e1 Definition. The *singular divergence*[1] $\operatorname{div}_{\text{sng}} F(x)$ at $x \in M$ of a mapping $F : \mathbb{R}^N \setminus \overline{M} \to \mathbb{R}^N$ continuous up to M is the number

$$\operatorname{div}_{\text{sng}} F(x) = \langle F(x + 0\mathbf{n}_x) - F(x - 0\mathbf{n}_x), \mathbf{n}_x \rangle \,.$$

As before, the singular divergence does not depend on the side indicator (and $\mathbf{n}_x$). It is a continuous function $\operatorname{div}_{\text{sng}} F : M \to \mathbb{R}$.

Less formally, the singular divergence is the jump of the normal component of the vector field.

Here is the singular counterpart of the formula

$$\operatorname{div} F = \sum_k (\nabla F_k)_k \,.$$

3e2 Lemma.

$$\operatorname{div}_{\text{sng}} F = \sum_{k=1}^N \big(\nabla_{\text{sng}} F_k\big)_k \,.$$

[1]Not a standard terminology.

Proof.

$$\sum_k \big(\nabla_{\text{sng}} F_k(x)\big)_k = \sum_k \Big(\big(F_k(x+0\mathbf{n}_x) - F_k(x-0\mathbf{n}_x)\big)\mathbf{n}_x\Big)_k =$$
$$= \sum_k \big(F(x+0\mathbf{n}_x) - F(x-0\mathbf{n}_x)\big)_k (\mathbf{n}_x)_k =$$
$$= \langle F(x+0\mathbf{n}_x) - F(x-0\mathbf{n}_x), \mathbf{n}_x \rangle = \operatorname{div}_{\text{sng}} F(x)\,.$$

□

A theorem, similar to 3b8, follows easily.

$n = N-1$

3e3 Theorem. Let $M \subset \mathbb{R}^N$ be an n-manifold, $K \subset M$ a compact subset, and $F : \mathbb{R}^N \setminus K \to \mathbb{R}^N$ a continuously differentiable mapping with a bounded support and bounded derivative (on $\mathbb{R}^N \setminus K$). Then

$$\int_{\mathbb{R}^N \setminus K} \operatorname{div} F + \int_M \operatorname{div}_{\text{sng}} f = 0\,.$$

Proof. We have $F(x) = \big(F_1(x), \dots, F_N(x)\big)$, and Theorem 3b8 applies to each F_k, giving

$$\int_{\mathbb{R}^N \setminus K} \nabla F_k + \int_M \nabla_{\text{sng}} F_k = 0\,.$$

It remains to take the k-th coordinate, and sum up over k. □

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