0 Preliminaries

0a Conventions, notation, terminology etc.

Unless stated otherwise (or even always):

$\mathbb{R}$ .... the real line

$\mathbb{R}^n$ .... $\{(x_1,\dots,x_n) : x_1,\dots,x_n \in \mathbb{R}\}$

Thus, $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ up to canonical isomorphism.[1]

$A \subset B$ .... $\forall x \; (x \in A \implies x \in B)$

Thus, $(A \subset B) \wedge (B \subset A) \iff (A = B)$.[2]

$A \uplus B$ .... just $A \cup B$ when $A \cap B = \emptyset$, otherwise undefined.

$(1,\dots,n)$ or $(x_1,\dots,x_n)$ .... finite sequence

$(1,2,\dots)$ or $(x_1,x_2,\dots)$ .... infinite sequence

$f : A \to B$ .... $f \subset A \times B$ and $\forall x \in A \; \exists!\, y \in B \;\; (x,y) \in f$.[3]

Tx .... the same as $T(x)$ when a mapping T is linear.

$|x|$ (for $x \in \mathbb{R}^n$) .... $\sqrt{x_1^2 + \dots + x_n^2}$

$\overline{A}$, A° (for $A \subset \mathbb{R}^n$) .... the closure and the interior

The derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ at $x \in \mathbb{R}^n$, denoted by $(Df)_x$, is a linear operator $\mathbb{R}^n \to \mathbb{R}^m$ such that $f(x+h) = f(x) + (Df)_x h + o(|h|)$ as $h \to 0$. Thus, Df is a mapping from $\mathbb{R}^n$ to the nm-dimensional space of linear operators. Also, $(Df)_x h = (D_h f)_x$. For $n = 1$, $(Df)_x h = h f'(x)$, $f'(x) \in \mathbb{R}^m$.

Index of terminology and notation is often available at the end of a section.

[1] 'a rule of thumb: there is a canonical isomorphism between X and Y if and only if you would feel comfortable writing "X = Y"' — Reid Barton, see Mathoverflow, What is the definition of "canonical"?

[2] Why "$\subset$" and "$\subsetneqq$" rather than "$\subseteq$" and "$\subset$"? Since I need "$\subset$" several times a day, while "$\subsetneqq$" hardly once a month.

[3] Here B is the codomain, generally not the image of f.

0b Improper integral

You know: For every open $U \subset \mathbb{R}^n$ such that ∂U is a null set, and every $f : U \to \mathbb{R}$ continuous almost everywhere, we define first, assuming $f(\cdot) \ge 0$,

$$\int_U f = \sup \biggl\{ \int_E f : E \text{ Jordan}, \overline{E} \subset U, f \text{ bounded on } E \biggr\} \in [0, \infty] \, ,$$

and then, assuming instead $\int_U |f| < \infty$,

$$\int_U f = \int_U f^+ - \int_U f^- \in (-\infty, +\infty) \, .$$

0b1 Remark. The same applies to arbitrary open U (even if ∂U is not a null set).

0b2 Remark. Equivalently, for $f(\cdot) \ge 0$,

$$\int_U f = \sup \biggl\{ \int_{\mathbb{R}^n} g \biggm| g : \mathbb{R}^n \to \mathbb{R} \text{ Riemann integrable,} \\ 0 \le g \le f \text{ on } U, \; g = 0 \text{ on } \mathbb{R}^n \setminus U \biggr\} \in [0, \infty] \, .$$

0b3 Remark. $\int_U (f_1 + f_2) = \int_U f_1 + \int_U f_2 \in [0, \infty]$ for all $f_1, f_2 \ge 0$ on U, continuous almost everywhere.

0b4 Remark. Equivalently,

$$\int_U (g - h) = \int_U g - \int_U h$$

for all $g, h : U \to [0, \infty)$ continuous almost everywhere, with finite integrals.

If $g_1 - h_1 = g_2 - h_2$, then $\int_U g_1 - \int_U h_1 = \int_U g_2 - \int_U h_2$, since

$$g_1 - h_1 = g_2 - h_2 \implies g_1 + h_2 = g_2 + h_1 \implies \int_G (g_1 + h_2) = \int_G (g_2 + h_1) \implies \\ \implies \int_G g_1 + \int_G h_2 = \int_G g_2 + \int_G h_1 \implies \int_G g_1 - \int_G h_1 = \int_G g_2 - \int_G h_2 \, .$$

0b5 Remark. All functions $f : U \to \mathbb{R}$ that are *improperly integrable,* that is, continuous almost everywhere and such that $\int_U |f| < \infty$, are a vector space, and the improper integral is a linear functional on this space.

0c Change of variable

You know: Let $U, V \subset \mathbb{R}^n$ be open sets, U bounded, and $\varphi : U \to V$ a diffeomorphism (of class C^1). Then for every Jordan set E such that $\overline{E} \subset U$, the set $F = \varphi(E)$ is Jordan, $\overline{F} \subset V$; and for every Riemann integrable $f : F \to \mathbb{R}$ the function $f \circ \varphi$ (or rather, $f \circ \varphi|_E$) on E is Riemann integrable, and

$$\int_F f = \int_E (f \circ \varphi) |\det D\varphi| \,.$$

0c1 Remark. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism (of class C^1), and $f : V \to \mathbb{R}$. Then

(a) f is improperly integrable on V if and only if $(f \circ \varphi)|\det D\varphi|$ is improperly integrable on U; and

(b) in this case

$$\int_V f = \int_U (f \circ \varphi) |\det D\varphi| \,.$$

0d Additive set functions

A Riemann integrable $f : \mathbb{R}^n \to \mathbb{R}$ leads to the *set function* F,

$$F(A) = \int_A f \quad \text{for Jordan } A \subset \mathbb{R}^n \,,$$

and F is *additive:*

$$F(A \uplus B) = F(A) + F(B) \,.$$

Conversely, f is the density of F:

$$f(x) = \lim_{A \to x, v(A) \ne 0} \frac{F(A)}{v(A)} \quad \text{whenever } x \text{ is a point of continuity of } f \,;$$

here "$A \to x$" means that $\sup_{a \in A} |a - x| \to 0$, and v is the Jordan measure.

0e Germs

Two functions on $\mathbb{R}^n$ (or another space) are said to be equal *near* a given point x, if they are equal on some neighborhood of x. Equality near x is an equivalence relation. Its equivalence classes are called *germs* (of functions) at x. The germ of f at x is denoted by $[f]_x$. The same applies to mappings from $\mathbb{R}^n$ to any $\mathbb{R}^m$, as well as from a neighborhood of x to $\mathbb{R}^m$.

Many properties of functions apply readily to germs, according to the pattern

$[f]_x$ is called $\boxed{}$ when f is $\boxed{}$ near x; here $\boxed{}$ may be "linear", "bounded", "continuous", "one-to-one" etc.

By the way, "continuous at x" is not the same as "continuous near x" (think, why).

If $[f_1]_x = [f_2]_x$ then $\lim_{y\to x} f_1(y) = \lim_{y\to x} f_2(y)$ in the following sense: either both limits exist and coincide, or neither limit exists. This way the notion of limit applies to germs; it is a *local* notion.

0f Some linear algebra

$A^{-1} = \dfrac{1}{\det A}\operatorname{adj} A$ for every invertible $n \times n$ matrix A; here $\operatorname{adj} A$, the adjugate matrix, satisfies $(\operatorname{adj} A)_{i,j} = (-1)^{i+j} M_{j,i}$; and $M_{j,i}$ is the minor, that is, the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row j and column i from A.

Thus, $A \operatorname{adj} A = (\det A)I$, whence $\forall i \;\; \sum_j A_{i,j}(\operatorname{adj} A)_{j,i} = \det A$ (Laplace expansion), and $\frac{\partial}{\partial A_{i,j}} \det A = (\operatorname{adj} A)_{j,i}$; it means, $(D_H \det)_A = \operatorname{tr}(H \operatorname{adj} A)$ (Jacobi's formula). In particular, $(D_H \det)_I = \operatorname{tr} H$, that is, $(D\det)_I = \operatorname{tr}$.

0g Some n-dimensional volumes and integrals

The volume of the unit ball $\{x \in \mathbb{R}^n : |x| < 1\}$:

$$V_n = \frac{2\pi^{n/2}}{n\Gamma\big(\frac{n}{2}\big)}\,.$$

More generally, for every norm $\|\cdot\|$ on $\mathbb{R}^n$ the unit ball $\{x \in \mathbb{R}^n : \|x\| < 1\}$ is Jordan measurable; denoting its volume by V we have[1]

$$\int_{\mathbb{R}^n} f(\|x\|)\,\mathrm{d}x = nV \int_0^\infty f(r) r^{n-1}\,\mathrm{d}r \in [0,\infty]$$

for every $f : [0,\infty) \to [0,\infty)$ continuous almost everywhere.

Multidimensional beta integral of Dirichlet:[2]

$$\underset{\substack{x_1,\dots x_n>0,\\ x_1+\dots+x_n<1}}{\int\cdots\int} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n+1)}$$

for all $p_1,\dots p_n > 0$.

[1] Improper integration is used.

[2] Improper integration is used, unless $p_1,\dots p_n \ge 1$.