9 Improper integral

Riemann integral and volume are generalized to unbounded functions and sets.

9a Introduction

The n-dimensional unit ball in the l_p metric,

$$E = \{(x_1,\dots,x_n) : |x_1|^p + \dots + |x_n|^p \le 1\},$$

is an admissible set, and its volume is a Riemann integral,

$$v(E) = \int_{\mathbb{R}^n} 1\!\!1_E ,$$

of a bounded function with bounded support. In Sect. 9f we'll calculate it:

$$v(E) = \frac{2^n \Gamma^n\big(\frac1p\big)}{p^n \Gamma\big(\frac np + 1\big)}$$

where Γ is a function defined by

$$\Gamma(t) = \int_0^\infty x^{t-1} \mathrm{e}^{-x} \, \mathrm{d}x \quad \text{for } t > 0 ;$$

here the integrand has no bounded support; and for $t = \frac1p < 1$ it is also unbounded (near 0). Thus we need a more general, so-called improper integral, even for calculating the volume of a bounded body!

In relatively simple cases the improper integral may be treated via *ad hoc* limiting procedure adapted to the given function; for example,

$$\int_0^\infty x^{t-1} \mathrm{e}^{-x} \, \mathrm{d}x = \lim_{k\to\infty} \int_{1/k}^k x^{t-1} \mathrm{e}^{-x} \, \mathrm{d}x .$$

In more complicated cases it is better to have a theory able to integrate rather general functions on rather general n-dimensional sets. Different functions may tend to infinity on different subsets (points, lines, surfaces), and still, we expect $\int (af + bg) = a\int f + b\int g$ (linearity) to hold, as well as change of variables.[1]

9b Positive integrands

We consider an open set $G \subset \mathbb{R}^n$ and functions $f : G \to [0, \infty)$ continuous almost everywhere.[2] We do not assume that G is bounded. We also do not assume that G is admissible, even if it is bounded.[3] "Continuous almost everywhere"means that the set $A \subset G$ of all discontinuity points of f has measure 0 (recall Sect. 6d). We can use the function $f \cdot 1\!\!1_G$ equal f on G and 0 on $\mathbb{R}^n \setminus G$, but must be careful: $1\!\!1_G$ and $f \cdot 1\!\!1_G$ need not be continuous almost everywhere.

We define

$$\text{(9b1)} \qquad \int_G f = \sup \biggl\{ \int_{\mathbb{R}^n} g \,\bigg|\, g : \mathbb{R}^n \to \mathbb{R} \text{ integrable}, \\ 0 \le g \le f \text{ on } G,\ g = 0 \text{ on } \mathbb{R}^n \setminus G \biggr\} \in [0, \infty] \,.$$

The condition on g may be reformulated as $0 \le g \le f \cdot 1\!\!1_G$. If $f \cdot 1\!\!1_G$ is integrable (on $\mathbb{R}^n$), then clearly $\int_G f = \int_{\mathbb{R}^n} f \cdot 1\!\!1_G$, which generalizes 4d5. This happens if and only if $f \cdot 1\!\!1_G$ is bounded, with bounded support, and

$$f(x) \to f(x_0) = 0 \quad \text{as } G \ni x \to x_0$$

for almost all $x_0 \in \partial G$ (think, why). (Void if ∂G has measure 0.)

9b2 Exercise. (a) Without changing the supremum in (9b1) we may restrict ourselves to continuous g with bounded support; or, alternatively, to step functions g; and moreover, in both cases, WLOG, g has a compact support *inside G*;

[1] Additional literature (for especially interested):
M. Pascu (2006) "On the definition of multidimensional generalized Riemann integral", Bul. Univ. Petrol **LVIII**:2, 9–16.
(*Research level*) D. Maharam (1988) "Jordan fields and improper integrals", J. Math. Anal. Appl. **133**, 163–194.

[2] This condition will be used in 9b9.

[3] A bounded open set need not be admissible, even if it is diffeomorphic to a disk.

(b) if f is bounded (not necessarily a.e. continuous) and G is bounded, then $\int_G f = {}_*\!\int_{\mathbb{R}^n} f \cdot 1\!\!1_G$, and in particular, $\int_G 1 = v_*(G)$;[1]

(c) if f is bounded and G is admissible, then the integral defined by (9b1) is equal to the integral defined by 4d5.
Prove it.

There are many ways to treat the improper integral as the limit of (proper) Riemann integrals; here are some ways.

9b3 Exercise. Consider the case $G = \mathbb{R}^n$, and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ of the form[2] $\|x\| = (|x_1|^p + \dots + |x_n|^p)^{1/p}$ for $x = (x_1, \dots, x_n)$; here $p \in [1, \infty]$ is a parameter (and $\|x\| = \max(|x_1|, \dots, |x_n|)$ if $p = \infty$).

(a) Prove that

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\|x\|<k} \min\bigl(f(x), k\bigr)\,\mathrm{d}x\,.$$

(b) For a locally bounded[3] f prove that

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\|x\|<k} f(x)\,\mathrm{d}x\,.$$

(c) Can it happen that f is locally bounded, not bounded, and $\int_{\mathbb{R}^n} f < \infty$?

9b4 Example (Poisson)**.** Consider

$$I = \int_{\mathbb{R}^2} \mathrm{e}^{-|x|^2}\,\mathrm{d}x\,.$$

On one hand, by 9b3 for the Euclidean norm ($p = 2$),

$$I = \lim_{k\to\infty} \iint\limits_{x^2+y^2<k^2} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y = \lim_{k\to\infty} \int_0^k r\,\mathrm{d}r\,\mathrm{e}^{-r^2} \int_0^{2\pi} \mathrm{d}\theta = \lim_{k\to\infty} \pi \int_0^{k^2} \mathrm{e}^{-u}\,\mathrm{d}u = \pi\,.$$

On the other hand, by 9b3 for $\|(x,y)\| = \max(|x|,|y|)$ ($p = \infty$),

$$I = \lim_{k\to\infty} \iint\limits_{|x|<k,|y|<k} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y = \lim_{k\to\infty} \biggl(\int_{-k}^{k} \mathrm{e}^{-x^2}\,\mathrm{d}x\biggr)\biggl(\int_{-k}^{k} \mathrm{e}^{-y^2}\,\mathrm{d}y\biggr) = \biggl(\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\biggr)^2,$$

and we obtain the celebrated Poisson formula:

$$\boxed{\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,.}$$

[1] In fact, $v_*(G)$ is Lebesgue's measure of G.
[2] But in fact, the same holds for arbitrary norm.
[3] That is, bounded on every bounded subset of $\mathbb{R}^n$.

9b5 Exercise. Consider

$$I = \iint\limits_{x>0,y>0} x^a y^b \mathrm{e}^{-(x^2+y^2)} \, \mathrm{d}x\mathrm{d}y \in [0,\infty]$$

for given $a, b \in \mathbb{R}$. Prove that, on one hand,

$$I = \bigg(\int_0^\infty r^{a+b+1} \mathrm{e}^{-r^2} \, \mathrm{d}r \bigg) \bigg(\int_0^{\pi/2} \cos^a \theta \sin^b \theta \, \mathrm{d}\theta \bigg) ,$$

and on the other hand,

$$I = \bigg(\int_0^\infty x^a \mathrm{e}^{-x^2} \, \mathrm{d}x \bigg) \bigg(\int_0^\infty x^b \mathrm{e}^{-x^2} \, \mathrm{d}x \bigg) .$$

9b6 Exercise. Consider $f : \mathbb{R}^2 \to [0,\infty)$ of the form $f(x) = g(|x|)$ for a given $g : [0,\infty) \to [0,\infty)$.

(a) If g is integrable, then f is integrable and $\int_{\mathbb{R}^2} f = 2\pi \int_0^\infty g(r) \, r \, \mathrm{d}r$.

(b) If g is continuous on $(0,\infty)$, then $\int_{\mathbb{R}^2} f = 2\pi \int_0^\infty g(r) \, r \, \mathrm{d}r \in [0,\infty]$.

Prove it.[1]

9b7 Exercise. Let $\|\cdot\|$ be as in 9b3.[2] Consider $f : \mathbb{R}^n \to [0,\infty)$ of the form $f(x) = g\big(\|x\|\big)$ for a given $g : [0,\infty) \to [0,\infty)$.

(a) If g is integrable, then f is integrable, and $\int_{\mathbb{R}^n} f = nV \int_0^\infty g(r) \, r^{n-1} \, \mathrm{d}r$ where V is the volume of $\{x : \|x\| < 1\}$.

(b) If g is continuous on $(0,\infty)$, then $\int_{\mathbb{R}^n} f = nV \int_0^\infty g(r) \, r^{n-1} \, \mathrm{d}r \in [0,\infty]$.

c) Let g be continuous on $(0,\infty)$ and satisfy

$$g(r) \sim r^a \quad \text{for } r \to 0+ , \qquad g(r) \sim r^b \quad \text{for } r \to +\infty .$$

Then $\int f < \infty$ if and only if $b < -n < a$.
Prove it.[3]

9b8 Example. $\int_{\mathbb{R}^n} \mathrm{e}^{-\|x\|^2} \, \mathrm{d}x = nV \int_0^\infty r^{n-1} \mathrm{e}^{-r^2} \, \mathrm{d}r$; in particular, $\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2} \, \mathrm{d}x = nV_n \int_0^\infty r^{n-1} \mathrm{e}^{-r^2} \, \mathrm{d}r$ where V_n is the volume of the (usual) n-dimensional unit ball. On the other hand, $\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2} \, \mathrm{d}x = \big(\int_{\mathbb{R}} \mathrm{e}^{-x^2} \, \mathrm{d}x \big)^n = \pi^{n/2}$. Therefore

$$V_n = \frac{\pi^{n/2}}{n \int_0^\infty r^{n-1} \mathrm{e}^{-r^2} \, \mathrm{d}r} .$$

Not unexpectedly, $V_2 = \frac{\pi}{2 \int_0^\infty r \mathrm{e}^{-r^2} \, \mathrm{d}r} = \pi$.

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[1] Hint: (a) polar coordinates; (b) use (a).

[2] But in fact, the same holds for arbitrary norm.

[3] Hint: (a) first, $g = 1\!\!1_{[0,a]}$, second, a step function g, and third, sandwich; also, (a)$\Longrightarrow$(b)$\Longrightarrow$(c).

Clearly, $\int_G cf = c\int_G f$ for $c \in (0,\infty)$.

9b9 Proposition. $\int_G (f_1+f_2) = \int_G f_1 + \int_G f_2 \in [0,\infty]$ for all $f_1, f_2 \ge 0$ on G, continuous almost everywhere.

Proof. The easy part: $\int_G (f_1+f_2) \ge \int_G f_1 + \int_G f_2$.[1] Given integrable g_1, g_2 such that $0 \le g_1 \le f_1 \cdot 1\!\!1_G$ and $0 \le g_2 \le f_2 \cdot 1\!\!1_G$, we have $\int g_1 + \int g_2 = \int (g_1+g_2) \le \int_G (f_1+f_2)$, since g_1+g_2 is integrable and $0 \le g_1+g_2 \le (f_1+f_2)\cdot 1\!\!1_G$. The supremum in g_1, g_2 gives the claim.

The hard part: $\int_G (f_1+f_2) \le \int_G f_1 + \int_G f_2$, that is, $\int g \le \int_G f_1 + \int_G f_2$ for every integrable g such that $0 \le g \le (f_1+f_2)\cdot 1\!\!1_G$. We introduce $g_1 = \min(f_1, g)$, $g_2 = \min(f_2, g)$ (pointwise minimum on G; and 0 on $\mathbb{R}^n \setminus G$) and prove that they are continuous almost everywhere (on $\mathbb{R}^n$, not just on G). For almost every $x \in G$, both f_1 and g are continuous at x and therefore g_1 is continuous at x. For almost every $x \in \partial G$, g is continuous at x, which ensures continuity of g_1 at x (irrespective of continuity of f_1), since $g(x) = 0$ ($x \notin G$). Thus, g_1 is continuous almost everywhere; the same holds for g_2.

By Lebesgue's criterion 6d2, the functions g_1, g_2 are integrable. We have $g_1+g_2 \ge \min(f_1+f_2, g) = g$, since generally, $\min(a,c)+\min(b,c) \ge \min(a+b,c)$ for all $a,b,c \in [0,\infty)$ (think, why). Thus, $\int g \le \int (g_1+g_2) = \int g_1 + \int g_2 \le \int_G f_1 + \int_G f_2$, since $0 \le g_1 \le f_1 \cdot 1\!\!1_G$, $0 \le g_2 \le f_2 \cdot 1\!\!1_G$. □

9b10 Proposition (exhaustion)**.** For open sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,

$$G_k \uparrow G \quad \Longrightarrow \quad \int_{G_k} f \uparrow \int_G f \quad \in [0,\infty]$$

for all $f : G \to [0,\infty)$ continuous almost everywhere.

Proof. First of all, $\int_{G_k} f \le \int_{G_{k+1}} f$ (since $0 \le g \le f \cdot 1\!\!1_{G_k}$ implies $0 \le g \le f \cdot 1\!\!1_{G_{k+1}}$), and similarly, $\int_{G_k} f \le \int_G f$, thus $\int_{G_k} f \uparrow$ and $\lim_k \int_{G_k} f \le \int_G f$. We have to prove that $\int_G f \le \lim_k \int_{G_k} f$.

We take an integrable g, compactly supported inside G (recall 9b2(a)), such that $g \le f$ on G. By compactness, there exists k_0 such that $g \le f \cdot 1\!\!1_{G_{k_0}}$. Then $\int g \le \int_{G_{k_0}} f \le \lim_k \int_{G_k} f$. The supremum in g proves the claim. □

9b11 Corollary (monotone convergence for volume)**.** For open sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,[2]

$$G_k \uparrow G \quad \Longrightarrow \quad v_*(G_k) \uparrow v_*(G) \, .$$

9b12 Remark. Let $G_1, G_2, \dots \subset \mathbb{R}^n$ be (pairwise) disjoint open balls. Then

$$v_*(G_1 \uplus G_2 \uplus \dots) = v(G_1) + v_(G_2) + \dots$$

even if the union is dense in $\mathbb{R}^n$ (which can happen; think, why).

[1] Compare it with 4c7: $_*\!\int (f+g) \ge {}_*\!\int f + {}_*\!\int g$.

[2] Really, this is easy to prove without 9b10 (try it).

9c Special functions gamma and beta

The Euler *gamma function* Γ is defined by[1]

$$\Gamma(t) = \int_0^\infty x^{t-1} \mathrm{e}^{-x} \, \mathrm{d}x \quad \text{for } t \in (0,\infty) \,. \tag{9c1}$$

This integral is not proper for two reasons. First, the integrand is bounded near 0 for $t \in [1,\infty)$ but unbounded for $t \in (0,1)$. Second, the integrand has no bounded support. In every case, using 9b10,

$$\Gamma(t) = \lim_{k\to\infty} \int_{1/k}^k x^{t-1} \mathrm{e}^{-x} \, \mathrm{d}x < \infty \,,$$

since the integrand (for a given t) is continuous on $(0,\infty)$, is $O(x^{t-1})$ as $x \to 0$, and (say) $O(\mathrm{e}^{-x/2})$ as $x \to \infty$. Thus, $\Gamma : (0,\infty) \to (0,\infty)$.

Clearly, $\Gamma(1) = 1$. Integration by parts gives

$$\int_{1/k}^k x^t \mathrm{e}^{-x} \, \mathrm{d}x = -x^t \mathrm{e}^{-x} \Big|_{x=1/k}^k + t \int_{1/k}^k x^{t-1} \mathrm{e}^{-x} \, \mathrm{d}x \,;$$

$$\Gamma(t+1) = t\Gamma(t) \quad \text{for } t \in (0,\infty) \,. \tag{9c2}$$

In particular,

$$\Gamma(n+1) = n! \quad \text{for } n = 0,1,2,\dots \tag{9c3}$$

We note that

$$\int_0^\infty x^a \mathrm{e}^{-x^2} \, \mathrm{d}x = \frac{1}{2}\Gamma\Big(\frac{a+1}{2}\Big) \quad \text{for } a \in (-1,\infty) \,, \tag{9c4}$$

since $\int_0^\infty x^a \mathrm{e}^{-x^2} \, \mathrm{d}x = \int_0^\infty u^{a/2} \mathrm{e}^{-u} \frac{\mathrm{d}u}{2\sqrt{u}}$. For $a = 0$ the Poisson formula (recall 9b4) gives

$$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi} \,. \tag{9c5}$$

Thus,

$$\Gamma\Big(\frac{2n+1}{2}\Big) = \frac{1}{2} \cdot \frac{3}{2} \cdot \dots \cdot \frac{2n-1}{2} \sqrt{\pi} \,. \tag{9c6}$$

The volume V_n of the n-dimensional unit ball (recall 9b8) is thus calculated:

$$V_n = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma(\frac{n}{2})} \,. \tag{9c7}$$

[1] This is rather $\Gamma|_{(0,\infty)}$.

Not unexpectedly, $V_3 = \frac{\pi^{3/2}}{\frac{3}{2}\Gamma(\frac{3}{2})} = \frac{\pi^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{4}{3}\pi$.

By 9b5, $\frac{1}{2}\Gamma\big(\frac{a+b+2}{2}\big)\int_0^{\pi/2}\cos^a\theta\sin^b\theta\,\mathrm{d}\theta = \frac{1}{2}\Gamma\big(\frac{a+1}{2}\big)\cdot\frac{1}{2}\Gamma\big(\frac{b+1}{2}\big)$ for $a, b \in (-1, \infty)$; that is,

$$\int_0^{\pi/2}\cos^{\alpha-1}\theta\sin^{\beta-1}\theta\,\mathrm{d}\theta = \frac{1}{2}\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta}{2})} \quad \text{for } \alpha, \beta \in (0, \infty)\,. \tag{9c8}$$

In particular,

$$\int_0^{\pi/2}\sin^{\alpha-1}\theta\,\mathrm{d}\theta = \int_0^{\pi/2}\cos^{\alpha-1}\theta\,\mathrm{d}\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\big(\frac{\alpha}{2}\big)}{\Gamma\big(\frac{\alpha+1}{2}\big)}\,. \tag{9c9}$$

The trigonometric functions can be eliminated: $\int_0^{\pi/2}\cos^{\alpha-1}\theta\sin^{\beta-1}\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\pi/2}\cos^{\alpha-2}\theta\sin^{\beta-2}\theta\cdot 2\sin\theta\cos\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^1(1-u)^{\frac{\alpha-2}{2}}u^{\frac{\beta-2}{2}}\,\mathrm{d}u$; thus,

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \mathrm{B}(\alpha,\beta) \quad \text{for } \alpha, \beta \in (0, \infty)\,, \tag{9c10}$$

where

$$\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for } \alpha, \beta \in (0, \infty) \tag{9c11}$$

is another special function, the *beta function.*

9c12 Exercise. Check that $\mathrm{B}(x,x) = 2^{1-2x}B(x,\frac{1}{2})$.[1]

9c13 Exercise. Check the *duplication formula*:[2]

$$\boxed{\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\,\Gamma\Big(x+\frac{1}{2}\Big)}.$$

9c14 Exercise. Calculate $\int_0^1 x^4\sqrt{1-x^2}\,\mathrm{d}x$.
Answer: $\frac{\pi}{32}$.

9c15 Exercise. Calculate $\int_0^\infty x^m \mathrm{e}^{-x^n}\,\mathrm{d}x$.
Answer: $\frac{1}{n}\Gamma\big(\frac{m+1}{n}\big)$.

9c16 Exercise. Calculate $\int_0^1 x^m(\ln x)^n\,\mathrm{d}x$.
Answer: $\frac{(-1)^n n!}{(m+1)^{n+1}}$.

[1] Hint: $\int_0^{\pi/2}\big(\frac{2\sin\theta\cos\theta}{2}\big)^{2x-1}\,\mathrm{d}\theta$.
[2] Hint: use 9c12.

9c17 Exercise. Calculate $\int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{\cos x}}$.
Answer: $\frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}$.

9c18 Exercise. Check that $\Gamma(t)\Gamma(1-t) = \int_0^\infty \frac{x^{t-1}}{1+x}\,\mathrm{d}x$ for $0 < t < 1$.[1]

We mention without proof another useful formula

$$\int_0^\infty \frac{x^{t-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin \pi t} \qquad \text{for } 0 < t < 1\,.$$

There is a simple proof that uses the residues theorem from the complex analysis course. This formula yields that $\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin \pi t}$ for $0 < t < 1$.

Is the function Γ continuous?

For every compact interval $[t_0, t_1] \subset (0, \infty)$ the given function of two variables $(t, x) \mapsto x^{t-1}\mathrm{e}^{-x}$ is continuous on $[t_0, t_1] \times [\frac{1}{k}, k]$, therefore its integral in x is continuous in t on $[t_0, t_1]$ (recall 4e6(a)). Also,

$$\int_{1/k}^k x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x \to \Gamma(t) \quad \text{uniformly on } [t_0, t_1]\,,$$

since $\int_0^{1/k} x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x \le \int_0^{1/k} x^{t_0-1}\,\mathrm{d}x \to 0$ as $k \to \infty$ and $\int_k^\infty x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x \le \int_k^\infty x^{t_1-1}\mathrm{e}^{-x}\,\mathrm{d}x \to 0$ as $k \to \infty$. It follows that Γ is continuous on arbitrary $[t_0, t_1]$, therefore, on the whole $(0, \infty)$.

In particular, $t\Gamma(t) = \Gamma(t+1) \to \Gamma(1) = 1$ as $t \to 0+$; that is,

$$\Gamma(t) = \frac{1}{t} + o\Big(\frac{1}{t}\Big) \quad \text{as } t \to 0+\,.$$

9d Change of variables

9d1 Theorem (change of variables)**.** Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to [0, \infty)$. Then

(a) (f is continuous almost everywhere on V) $\iff$
($f \circ \varphi$ is continuous almost everywhere on U) $\iff$
($(f \circ \varphi)|\det D\varphi|$ is continuous almost everywhere on U);

(b) if they are continuous almost everywhere, then

$$\int_V f = \int_U (f \circ \varphi)|\det D\varphi| \quad \in [0, \infty]\,.$$

Item (a) follows easily from 8c1 (similarly to the proof of 8a1(a) in Sect. 8c but simpler: 8c4 is not needed now).

[1] Hint: change x to y via $(1+x)(1-y) = 1$.

9d2 Lemma. Let U, V, φ, f be as in Th. 9d1, and in addition, f be compactly supported within V. Then 9d1(b) holds.

Proof. This is basically Prop. 8d1; there U, V are admissible, since otherwise the integrals over U and V are not defined by 4d5. Now they are defined (see the paragraph after (9b1)): $\int_V f = \int_{\mathbb{R}^n} f \cdot 1\!\!1_V$ (and similarly for U), and the proof of 8d1 given in Sect. 8d applies (check it). $\square$

***Proof** of Th. 9d1(b).* First, we prove that

$$\int_V f \le \int_U (f \circ \varphi) |\det D\varphi| \,. \tag{9d3}$$

Assume the contrary. By 9b2(a) there exists integrable g, compactly supported within V, such that $g \le f$ on V and $\int_V g > \int_U (f \circ \varphi) |\det D\varphi|$. By 9d2, $\int_V g = \int_U (g \circ \varphi) |\det D\varphi| \le \int_U (f \circ \varphi) |\det D\varphi|$; this contradiction proves (9d3).

Second, we apply (9d3) to $\varphi_1 = \varphi^{-1} : V \to U$ and $f_1 = (f \circ \varphi) |\det D\varphi| : U \to [0, \infty)$:

$$\int_U f_1 \le \int_V (f_1 \circ \varphi_1) |\det D\varphi_1| \,.$$

By the chain rule, $\varphi \circ \varphi_1 = \mathrm{id}_V$ implies $\big((D\varphi) \circ \varphi_1\big)(D\varphi_1) = \mathrm{id}$, thus $\big((\det D\varphi) \circ \varphi_1\big)(\det D\varphi_1) = 1$. We get

$$f_1 \circ \varphi_1 = (f \circ \varphi \circ \varphi_1) |(\det D\varphi) \circ \varphi_1| = \frac{f}{|\det D\varphi_1|} \,;$$

$(f_1 \circ \varphi_1) |\det D\varphi_1| = f$; $\int_U (f \circ \varphi) |\det D\varphi| = \int_U f_1 \le \int_V f$. $\square$

9e Iterated integral

We consider an open set $G \subset \mathbb{R}^{m+n}$ and functions $f : G \to [0, \infty)$ continuous almost everywhere. Similarly to Sect. 5d, the section $f(x, \cdot)$ of f need not be continuous almost everywhere on the section $G_x = \{y : (x, y) \in G\}$ of G; thus, $\int_{G_x} f(x, \cdot)$ is generally ill-defined. Similarly to Th. 5d1 we need the lower integral (but no upper integral this time).

We define the lower integral by (9b1) again, but this time $f : G \to [0, \infty)$ is arbitrary (rather than continuous almost everywhere). That is, for open $G \subset \mathbb{R}^n$ (rather than $\mathbb{R}^{m+n}$, for now)

$$ {}_*\!\!\int_G f = \sup \left\{ \int_{\mathbb{R}^n} g \,\middle|\, g : \mathbb{R}^n \to \mathbb{R} \text{ integrable}, \; 0 \le g \le f \text{ on } G, \; g = 0 \text{ on } \mathbb{R}^n \setminus G \right\} \in [0, \infty] \,. \tag{9e1}$$

In particular, if f is continuous almost everywhere on G, then ${}_*\int_G f = \int_G f$.

As before, the condition on g may be reformulated as $0 \le g \le f \cdot 1\!\!1_G$. Still, 9b2(a) applies (check it). And 9b2(b) becomes: if f is bounded and G is bounded, then ${}_*\int_G f = {}_*\int_{\mathbb{R}^n} f \cdot 1\!\!1_G$, the latter integral being proper, that is, defined in Sect. 4c.

Similarly to 9b10, for open sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,

(9e2) $$G_k \uparrow G \quad \Longrightarrow \quad {}_*\int_{G_k} f \;\Big\uparrow\; {}_*\int_G f \quad \in [0,\infty]$$

for arbitrary $f : G \to [0,\infty)$. Similarly to 9b3(a),

$$G_k \uparrow G \quad \Longrightarrow \quad {}_*\int_{G_k} \min(f,k) \;\Big\uparrow\; {}_*\int_G f \quad \in [0,\infty] .$$

If, in addition, G_k are bounded, then we may rewrite it as

(9e3) $${}_*\int_{\mathbb{R}^n} \min(f,k) 1\!\!1_{G_k} \;\Big\uparrow\; {}_*\int_G f ,$$

the left-hand side integral being proper.

An increasing sequence of integrable functions can converge[1] to a function that is not almost everywhere continuous (and moreover, is discontinuous everywhere). Nevertheless, a limiting procedure is possible, as follows.

9e4 Proposition. If $g_1, g_2, \dots : \mathbb{R}^n \to [0,\infty)$ are integrable and $g_k \uparrow f : \mathbb{R}^n \to [0,\infty)$, then $\int_{\mathbb{R}^n} g_k \uparrow {}_*\int_{\mathbb{R}^n} f$.[2]

This claim follows easily from an important theorem (to be proved in Appendix).

9e5 Theorem (monotone convergence for Riemann integral)**.** If $g, g_1, g_2, \dots : \mathbb{R}^n \to \mathbb{R}$ are integrable and $g_k \uparrow g$, then $\int_{\mathbb{R}^n} g_k \uparrow \int_{\mathbb{R}^n} g$.

Proof *that Th. 9e5 implies Prop. 9e4.* Clearly, $g_k \uparrow f$ implies $\lim_k \int g_k \le {}_*\int f$; we have to prove that $\lim_k \int g_k \ge {}_*\int f$. Given an integrable $g \le f$, we have $\min(g_k, g) \uparrow \min(f, g) = g$ and, by 9e5, $\int \min(g_k, g) \uparrow \int g$. Thus, $\int g \le \lim_k \int g_k$; supremum in g gives ${}_*\int f \le \lim_k \int g_k$. □

We return to an open set $G \subset \mathbb{R}^{m+n}$ and its sections $G_x \subset \mathbb{R}^n$ for $x \in \mathbb{R}^m$.

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[1] Pointwise, not uniformly.

[2] Do you think that ${}_*\int g_k \uparrow {}_*\int f$ for arbitrary (not integrable) g_k? No, this is wrong. Recall f_k of 4e7 and consider $1 - f_k$.

9e6 Theorem (iterated improper integral)**.** If a function $f : G \to [0, \infty)$ is continuous almost everywhere, then

$$ {}_*\!\int_{\mathbb{R}^m} \mathrm{d}x \; {}_*\!\int_{G_x} \mathrm{d}y \, f(x,y) = \iint\limits_G f(x,y) \, \mathrm{d}x\mathrm{d}y \in [0, \infty] \, . $$

Unlike Th. 5d1, both integrals in the left-hand side are lower integrals. The function $x \mapsto {}_*\!\int_{G_x} \mathrm{d}y \, f(x,y)$ need not be almost everywhere continuous, even if $G = \mathbb{R}^2$ and f is continuous. Moreover, it can happen that $x \mapsto {}_*\!\int_{\mathbb{R}} f(x, \cdot)$ is unbounded on every interval, even if $f : \mathbb{R}^2 \to [0, \infty)$ is bounded, continuously differentiable, $\int_{\mathbb{R}^2} f < \infty$, and $f(x,y) \to 0$, $\nabla f(x,y) \to 0$ as $x^2 + y^2 \to \infty$. (Can you find a counterexample? Hint: construct separately $f|_{\mathbb{R} \times [2^k, 2^{k+1}]}$ for each k.)

It is easy to see (try it!) that $\int_G f$ does not exceed the iterated integral; but the equality needs more effort.

Proof. We take admissible open sets $G_k \subset \mathbb{R}^{m+n}$ such that $G_k \uparrow G$,[1] and introduce $f_k = \min(f, k) 1\!\!1_{G_k}$, that is,

$$ f_k(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in G_k \text{ and } f(x,y) \le k, \\ k, & \text{if } (x,y) \in G_k \text{ and } f(x,y) \ge k, \\ 0, & \text{if } (x,y) \notin G_k. \end{cases} $$

By Lebesgue's criterion 6d2, each f_k is integrable. By (9e3), $\int_{\mathbb{R}^{m+n}} f_k \uparrow \int_G f$, the left-hand side integral being proper.

Given $x \in \mathbb{R}^m$, we apply the same argument to the sections $f_k(x, \cdot)$, $f(x, \cdot)$, $(G_k)_x$, G_x, taking into account that $f_k(x, \cdot)$ need not be integrable, and we get

$$ {}_*\!\int_{\mathbb{R}^n} f_k(x, \cdot) = {}_*\!\int_{\mathbb{R}^n} \min\bigl(f(x, \cdot), k\bigr) 1\!\!1_{G_k}(x, \cdot) \; \Big\uparrow \; {}_*\!\int_{G_x} f(x, \cdot) \, . $$

By Th. 5d1 (applied to f_k), the function $x \mapsto {}_*\!\int_{\mathbb{R}^n} f_k(x, \cdot)$ is integrable, and its integral is equal to $\int_{\mathbb{R}^{m+n}} f_k$. Applying Prop. 9e4 to these functions we get

$$ \int_{\mathbb{R}^{m+n}} f_k = \int_{\mathbb{R}^m} \Bigl(x \mapsto {}_*\!\int_{\mathbb{R}^n} f_k(x, \cdot) \Bigr) \; \Big\uparrow \; {}_*\!\int_{\mathbb{R}^m} \Bigl(x \mapsto {}_*\!\int_{G_x} f(x, \cdot) \Bigr) ; $$

but on the other hand, $\int_{\mathbb{R}^{m+n}} f_k \uparrow \int_G f$. □

9e7 Corollary. The volume[2] of an open set $G \subset \mathbb{R}^{m+n}$ is equal to the lower integral of the volume of G_x (even if G is not admissible).

[1] For example, we may use the interior of the union of all N-pixels contained in $G \cap [-N, N]^n$.

[2] That is, $v_*(G)$ if G is bounded; and $\int_G 1$ (in fact, the Lebesgue measure of G) in general.

9f Multidimensional beta integrals of Dirichlet

9f1 Proposition.

$$\int\!\!\cdots\!\!\int_{\substack{x_1,\dots x_n>0,\\ x_1+\dots+x_n<1}} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n+1)}$$

for all $p_1,\dots p_n > 0$.

For the proof, we denote

$$I(p_1,\dots,p_n) = \int\!\!\cdots\!\!\int_{\substack{x_1,\dots x_n>0,\\ x_1+\dots+x_n<1}} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n\,.$$

This integral is improper, unless $p_1,\dots,p_n \ge 1$.

9f2 Lemma. $I(p_1,\dots,p_n) = \mathrm{B}(p_n, p_1+\dots+p_{n-1}+1)I(p_1,\dots,p_{n-1})$.

Proof. The change of variables $\xi = ax$ (that is, $\xi_1 = ax_1,\dots,\xi_n = ax_n$) gives (by Theorem 9d1[1])

$$\int\!\!\cdots\!\!\int_{\substack{\xi_1,\dots\xi_n>0,\\ \xi_1+\dots+\xi_n<a}} \xi_1^{p_1-1}\dots\xi_n^{p_n-1}\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_n = a^{p_1+\dots+p_n}I(p_1,\dots,p_n) \quad \text{for } a>0\,.$$

Thus, using 9e6 and (9c10),

$$I(p_1,\dots,p_n) = \int_0^1 \mathrm{d}x_n\, x_n^{p_n-1} \int\!\!\cdots\!\!\int_{\substack{x_1,\dots x_{n-1}>0,\\ x_1+\dots+x_{n-1}<1-x_n}} x_1^{p_1-1}\dots x_{n-1}^{p_{n-1}-1}\,\mathrm{d}x_1\dots\mathrm{d}x_{n-1} =$$
$$= \int_0^1 x_n^{p_n-1}(1-x_n)^{p_1+\dots+p_{n-1}} I(p_1,\dots,p_{n-1})\,\mathrm{d}x_n =$$
$$= I(p_1,\dots,p_{n-1})\mathrm{B}(p_n, p_1+\dots+p_{n-1}+1)\,.$$

□

Proof *of Prop. 9f1.*
Induction in the dimension n. For $n=1$ the formula is obvious:

$$\int_0^1 x_1^{p_1-1}\,\mathrm{d}x_1 = \frac{1}{p_1} = \frac{\Gamma(p_1)}{\Gamma(p_1+1)}\,.$$

[1]But a *linear* change of variables does not really need 9d1; it is a simple generalization of 7c1 or even (4h5).

From $n-1$ to n: using 9f2 (and (9c11)),

$$I(p_1,\dots,p_n) = \frac{\Gamma(p_n)\Gamma(p_1+\dots+p_{n-1}+1)}{\Gamma(p_1+\dots+p_n+1)} \cdot \frac{\Gamma(p_1)\dots\Gamma(p_{n-1})}{\Gamma(p_1+\dots+p_{n-1}+1)} = \\ = \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n+1)} \,.$$

$\square$

A seemingly more general formula,

$$\int\!\!\cdots\!\!\int_{\substack{x_1,\dots,x_n>0,\\ x_1^{\gamma_1}+\dots+x_n^{\gamma_n}<1}} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{1}{\gamma_1\dots\gamma_n}\cdot\frac{\Gamma\big(\frac{p_1}{\gamma_1}\big)\dots\Gamma\big(\frac{p_n}{\gamma_n}\big)}{\Gamma\big(\frac{p_1}{\gamma_1}+\dots+\frac{p_n}{\gamma_n}+1\big)}\,,$$

results from 9f1 by the (nonlinear!) change of variables $y_j = x_j^{\gamma_j}$.

A special case: $p_1=\dots=p_n=1$, $\gamma_1=\dots=\gamma_n=p$;

$$\int\!\!\cdots\!\!\int_{\substack{x_1,\dots,x_n>0\\ x_1^p+\dots+x_n^p<1}} \mathrm{d}x_1\dots\mathrm{d}x_n = \frac{\Gamma^n\big(\frac1p\big)}{p^n\Gamma\big(\frac np+1\big)}\,.$$

We've found the volume of the unit ball in the metric l_p:

$$\boxed{v\big(B_p(1)\big) = \frac{2^n\Gamma^n\big(\frac1p\big)}{p^n\Gamma\big(\frac np+1\big)}\,.}$$

If $p=2$, the formula gives us (again; see (9c7)) the volume of the standard unit ball:

$$\boxed{V_n = v\big(B_2(1)\big) = \frac{2\pi^{n/2}}{n\Gamma\big(\frac n2\big)}\,.}$$

We also see that the volume of the unit ball in the l_1-metric equals $\frac{2^n}{n!}$.

Question: what does the formula give in the $p\to\infty$ limit?

9f3 Exercise. Show that

$$\int\!\!\cdots\!\!\int_{\substack{x_1+\dots+x_n<1\\ x_1,\dots,x_n>0}} \varphi(x_1+\dots+x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{1}{(n-1)!}\int_0^1 \varphi(s)s^{n-1}\,\mathrm{d}s$$

for every "good" function $\varphi : [0,1] \to \mathbb{R}$ and, more generally,

$$\underset{\substack{x_1+\dots+x_n<1\\ x_1,\dots,x_n>0}}{\int\!\!\cdots\!\!\int} \varphi(x_1+\dots+x_n) x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n =$$

$$= \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n)} \int_0^1 \varphi(u) u^{p_1+\dots p_n-1}\,\mathrm{d}u\,.$$

Hint: consider

$$\int_0^1 \mathrm{d}s\,\varphi'(s) \underset{\substack{x_1+\dots+x_n<s\\ x_1,\dots,x_n>0}}{\int\!\!\cdots\!\!\int} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n\,.$$

9g Non-positive (signed) integrands

We define

$$\int_G (g-h) = \int_G g - \int_G h$$

whenever $g,h : G \to [0,\infty)$ are continuous almost everywhere and $\int_G g < \infty$, $\int_G h < \infty$; this definition is correct, that is,

$$\int_G g_1 - \int_G h_1 = \int_G g_2 - \int_G h_2 \quad \text{whenever } g_1 - h_1 = g_2 - h_2\,,$$

due to 9b9:

$$g_1 - h_1 = g_2 - h_2 \implies g_1 + h_2 = g_2 + h_1 \implies \int_G (g_1+h_2) = \int_G (g_2+h_1) \implies$$
$$\implies \int_G g_1 + \int_G h_2 = \int_G g_2 + \int_G h_1 \implies \int_G g_1 - \int_G h_1 = \int_G g_2 - \int_G h_2\,.$$

9g1 Lemma. The following two conditions on a function $f : G \to \mathbb{R}$ continuous almost everywhere are equivalent:

(a) there exist $g,h : G \to [0,\infty)$, continuous almost everywhere, such that $\int_G g < \infty$, $\int_G h < \infty$ and $f = g - h$;

(b) $\int_G |f| < \infty$.

Proof. (a)$\Longrightarrow$(b): $\int_G |g-h| \le \int_G \big(|g|+|h|\big) = \int_G |g| + \int_G |h| < \infty$.

(b)$\Longrightarrow$(a): we introduce the positive part f^+ and the negative part f^- of f,

$$\begin{gathered} f^+(x) = \max\big(0, f(x)\big)\,, \quad f^-(x) = \max\big(0, -f(x)\big)\,; \\ f^- = (-f)^+\,; \quad f = f^+ - f^-\,; \quad |f| = f^+ + f^-\,; \end{gathered} \tag{9g2}$$

they are continuous almost everywhere (think, why); $\int_G f^+ \le \int_G |f| < \infty$, $\int_G f^- \le \int_G |f| < \infty$; and $f^+ - f^- = f$. $\square$

We summarize:

$$\int_G f = \int_G f^+ - \int_G f^- \tag{9g3}$$

whenever $f : G \to \mathbb{R}$ is continuous almost everywhere and such that $\int_G |f| < \infty$. Such functions will be called *improperly integrable*[1] (on G).

9g4 Exercise. Prove linearity: $\int_G cf = c\int_G f$ for $c \in \mathbb{R}$, and $\int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2$.

Similarly to Sect. 4e, a function $f : G \to \mathbb{R}$ continuous almost everywhere will be called *negligible* if $\int_G |f| = 0$. Functions f, g continuous almost everywhere and such that $f - g$ is negligible will be called equivalent. The equivalence class of f will be denoted $[f]$.

Improperly integrable functions $f : G \to \mathbb{R}$ are a vector space. On this space, the functional $f \mapsto \int_G |f|$ is a seminorm. The corresponding equivalence classes are a normed space (therefore also a metric space). The integral is a continuous linear functional on this space.

If G is admissible, then the space of improperly integrable functions on G is embedded into the space of improperly integrable functions on $\mathbb{R}^n$ by $f \mapsto f \cdot 1\!\!1_G$.

9g5 Proposition (exhaustion)**.** For open sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,

$$G_k \uparrow G \quad \Longrightarrow \quad \int_{G_k} f \to \int_G f \quad \in \mathbb{R}$$

for all improperly integrable $f : G \to \mathbb{R}$.

9g6 Theorem (change of variables)**.** Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$. Then

(a) (f is continuous almost everywhere on V) $\Longleftrightarrow$
($f \circ \varphi$ is continuous almost everywhere on U) $\Longleftrightarrow$
($(f \circ \varphi)|\det D\varphi|$ is continuous almost everywhere on U);

(b) if they are continuous almost everywhere, then

$$\int_V |f| = \int_U |(f \circ \varphi) \det D\varphi| \quad \in [0, \infty] \, ;$$

(c) and if the integrals in (b) are finite, then

$$\int_V f = \int_U (f \circ \varphi)|\det D\varphi| \quad \in \mathbb{R} \, .$$

[1]In one dimension they are usually called absolutely (improperly) integrable.

9g7 Exercise. Prove 9g5 and 9g6.

9g8 Exercise. If $0 < t_0 < t_1 < \infty$, then the function $(x,t) \mapsto x^{t-1}\mathrm{e}^{-x}\ln x$ is improperly integrable on $(0,\infty)\times(t_0,t_1)$, and

$$\int_{t_0}^{t_1} \mathrm{d}t \int_0^\infty \mathrm{d}x\, x^{t-1}\mathrm{e}^{-x}\ln x = \Gamma(t_1) - \Gamma(t_0)\,.$$

Prove it.[1]

9g9 Exercise. (a) The function $t \mapsto \int_0^\infty x^{t-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x$ is continuous on $(0,\infty)$;

(b) the gamma function is continuously differentiable on $(0,\infty)$, and

$$\Gamma'(t) = \int_0^\infty x^{t-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x \quad \text{for } 0 < t < \infty\,;$$

(c) the gamma function is convex on $(0,\infty)$.

Prove it.

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[1] Hint: apply 9e6 twice, to f^+ and f^-.