# 6 Lebesgue's criterion for Riemann integrability



## 6a Introduction

Consider a bounded function $f : (0,1) \to \mathbb{R}$. If $f$ is continuous then it is integrable (even if it is not uniformly continuous, like $\sin(1/x)$). A step function is (generally) discontinuous, and still, integrable; its set of discontinuity points is finite. Non-integrable functions mentioned in 4c3 are "very discontinuous", having intervals of discontinuity points. The function of 4c5 (or 4f12) has a dense set of discontinuity points, and still, is integrable. Can integrability be decided via the set of discontinuity points? An affirmative answer was given by Lebesgue, it involves the notion of Lebesgue measure zero (rather than volume zero).

"This aesthetically pleasing integrability criterion has little practical value" (Bichteler).[1] Well, if you use it when proving simple facts, such as integrability of $\sqrt[3]{f}$ or $fg$ (for integrable $f$ and $g$), you may find far more elementary proofs. But here is a harder case. The so-called improper integral (to be treated later) may be applied to unbounded functions $f$ on $(0,1)$ such that the function

$$\operatorname{mid}(-M, f, M) : x \mapsto \begin{cases} -M & \text{when } f(x) \le -M, \\ f(x) & \text{when } -M \le f(x) \le M, \\ M & \text{when } M \le f(x) \end{cases}$$

is integrable for all $M > 0$. The sum of two such functions is also such function. This fact follows easily from Lebesgue's criterion. You may discover another proof, but I doubt it will be simpler!

[1] From book "Integration — a functional approach" by Klaus Bichteler (1998); see Exercise 6.16 on p. 27.

A natural quantitative measure of non-integrability is the difference

$$A = {}^{*}\!\!\int_{(0,1)} f - {}_{*}\!\!\int_{(0,1)} f \ \in [0, \infty) \, .$$

What about a natural quantitative measure of discontinuity of $f$? At a given point $x_0 \in (0, 1)$ it is the oscillation,

$$\mathrm{Osc}_f(x_0) = \inf_{r>0} \mathrm{Osc}_f\big((x_0 - r, x_0 + r)\big) \, ,$$

where

$$\mathrm{Osc}_f(U) = \mathrm{diam} f(U) = \sup_{x\in U} f(x) - \inf_{x\in U} f(x) \, . \tag{6a1}$$

But it depends on $x_0$. In order to get a number we integrate the oscillation function:

$$B = {}^{*}\!\!\int_{(0,1)} \mathrm{Osc}_f \ .$$

We would be happy to know that $B = 0 \implies A = 0$, even happier to know that $B = 0 \iff A = 0$, but here is a surprise:

$$A = B \, .$$

Qualitatively,

$$(f \text{ is integrable}) \quad \Longleftrightarrow \quad (\mathrm{Osc}_f \text{ is negligible}) \, .$$

And of course, we need a multidimensional theory; $(0, 1)$ is only the simplest case.

It may seem that the equality $A = B$ is an easy matter, since $\mathrm{Osc}_f = f^* - f_*$ where

$$f_*(x_0) = \sup_{r>0} \inf_{|x-x_0|<r} f(x) \, , \quad f^*(x_0) = \inf_{r>0} \sup_{|x-x_0|<r} f(x) \, ,$$

and so, $B = \int \mathrm{Osc}_f = \int f^* - \int f_* = {}^*\!\!\int f - {}_*\!\!\int f = A$. However, $f^*$ and $f_*$ need not be integrable. In fact, ${}^*\!\!\int f^* = {}^*\!\!\int f$, ${}_*\!\!\int f_* = {}_*\!\!\int f$ (which is rather easy to see), and ${}^*\!\!\int (f^* - f_*) = {}^*\!\!\int f^* - {}_*\!\!\int f_*$, that is, ${}^*\!\!\int \big(f^* + (-f_*)\big) = {}^*\!\!\int f^* + {}^*\!\!\int (-f_*)\big)$, which is not easy, and a surprise, since the upper integral is not linear![1] The equality $A = B$ will be proved, but not this way.

**6a2 Exercise.** For the function $f$ of 4c5

(a) $\mathrm{Osc}_f(x) = 2^{-m}$ if $x = \frac{2k+1}{2^{2m+1}}$ for $m = 0, 1, \dots$ and $k = 0, \dots, 2^{2m} - 1$; otherwise $\mathrm{Osc}_f(x) = 0$;

(b) $\mathrm{Osc}_f$ is negligible.

Prove it (not using results of Sect. 6).[2]

[1] In fact, ${}^*\!\!\int (f + g) = {}^*\!\!\int f + {}^*\!\!\int g$ when $f$ and $g$ are (bounded, with bounded support, and) *upper semicontinuous,* that is, $f^* = f$ and $g^* = g$.

[2] Hint: (b) recall 4f12.

## 6b Integral of oscillation

We consider a bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support, and its oscillation function

$$\operatorname{Osc}_f(x_0) = \inf_{r>0} \operatorname{Osc}_f\big(\{x : |x - x_0| < r\}\big) \,, \tag{6b1}$$

where $\operatorname{Osc}_f(U)$ is still defined by (6a1).

**6b2 Theorem.**

$$\int^*_{\mathbb{R}^n} f - {}_*\!\!\int_{\mathbb{R}^n} f = \int^*_{\mathbb{R}^n} \operatorname{Osc}_f \,.$$

Here is the easy part.

**6b3 Proposition.**

$$\int^*_{\mathbb{R}^n} f - {}_*\!\!\int_{\mathbb{R}^n} f \ge \int^*_{\mathbb{R}^n} \operatorname{Osc}_f \,.$$

***Proof.*** Similarly to 4g9 (or combining 4g9 with 4g7), given $\varepsilon > 0$, there exist continuous $g, h$ with bounded support such that $g \le f \le h$ and

$$\int_{\mathbb{R}^n} h \le \frac{\varepsilon}{2} + \int^*_{\mathbb{R}^n} f \,, \quad \int_{\mathbb{R}^n} g \ge -\frac{\varepsilon}{2} + {}_*\!\!\int_{\mathbb{R}^n} f \,,$$

therefore

$$\int_{\mathbb{R}^n} (h - g) \le \varepsilon + \int^*_{\mathbb{R}^n} f - {}_*\!\!\int_{\mathbb{R}^n} f \,.$$

For arbitrary $U \subset \mathbb{R}^n$,

$$\operatorname{Osc}_f(U) = \sup_{x\in U} f(x) - \inf_{x \in U} f(x) \le \sup_{x\in U} h(x) - \inf_{x\in U} g(x) \,;$$

by (6b1) and continuity of $g$ and $h$,

$$\operatorname{Osc}_f(x_0) \le h(x_0) - g(x_0)$$

for all $x_0$. Thus,

$$\int^*_{\mathbb{R}^n} \operatorname{Osc}_f \le \int^*_{\mathbb{R}^n} (h - g) = \int_{\mathbb{R}^n} (h - g) \le \varepsilon + \int^*_{\mathbb{R}^n} f - {}_*\!\!\int_{\mathbb{R}^n} f$$

for all $\varepsilon > 0$. $\square$

Now, the hard part.

**6b4 Proposition.**

$$\int^*_{\mathbb{R}^n} f - \int_{*\mathbb{R}^n} f \le \int^*_{\mathbb{R}^n} \operatorname{Osc}_f \, .$$

**6b5 Lemma** (Lebesgue's covering number)**.** Let $K \subset \mathbb{R}^n$ be a compact set, $U_1, \dots, U_m \subset \mathbb{R}^n$ open sets, and $K \subset U_1 \cup \dots \cup U_m$. Then[1]

$$\exists \delta > 0 \; \forall x \in K \; \exists i \in \{1, \dots, m\} \; \forall y \; \big( |y - x| < \delta \implies y \in U_i \big) \, .$$

***Proof.*** Assume the contrary: for every $k$ there exists $x_k \in K$ whose $\frac{1}{k}$-neighborhood is not covered by a single $U_i$. By compactness, there exists an accumulation point $x_0 \in K$ of the sequence $(x_k)_k$. We take $i$ such that $x_0 \in U_i$, and then $\delta > 0$ such that $U_i$ contains the $2\delta$-neighborhood of $x_0$. For all $k$ such that $\frac{1}{k} < \delta$ we know that the $\delta$-neighborhood of $x_k$ is not contained in $U_i$, and therefore $|x_k - x_0| \ge \delta$; a contradiction. $\square$

Recall Sect. 4b (Darboux sums).

***Proof** of Prop. 6b4.* We take a natural $M$ such that $\{x : f(x) \ne 0\} \subset (-2^M, 2^M)^n$, and introduce the compact set $K = [-2^M, 2^M]^n$.

Given $\varepsilon > 0$, we take a continuous $h$ with bounded support such that $\operatorname{Osc}_f \le h$ and $\int_{\mathbb{R}^n} h \le \varepsilon + {}^*\!\!\int_{\mathbb{R}^n} \operatorname{Osc}_f$.

For every $x_0 \in K$ there exists $\delta > 0$ such that the neighborhood $U = \{x : |x - x_0| < \delta\}$ satisfies

$$\operatorname{Osc}_f(U) \le \frac{\varepsilon}{2} + \operatorname{Osc}_f(x_0) \, , \quad \operatorname{Osc}_h(U) \le \frac{\varepsilon}{2} \, ,$$

then

$$\operatorname{Osc}_f(U) \le \varepsilon + \inf_{x \in U} h(x) \tag{6b6}$$

(since $\operatorname{Osc}_f(x_0) \le h(x_0)$ and $h(x_0) - \inf_{x \in U} h(x) \le \operatorname{Osc}_h(U) \le \frac{\varepsilon}{2}$).

By compactness, $K \subset U_1 \cup \dots \cup U_m$ for some $U_i$ satisfying (6b6). Lemma 6b5 gives us Lebesgue's covering number $\delta$ for this covering; we take natural $N$ such that $\frac{1}{2}\sqrt{n} 2^{-N} \le \delta$; then each "pixel" $2^{-N}(Q + k)$ (where $Q = [0, 1]^n$ and $k \in \mathbb{Z}_n$) contained in $K$, being contained in the $\frac{1}{2}\sqrt{n} 2^{-N}$-neighborhood of its center, is contained in some $U_i$, and therefore, by (6b6),

$$\operatorname{Osc}_f\big(2^{-N}(Q + k)\big) \le \varepsilon + \inf_{x \in 2^{-N}(Q+k)} h(x) \, ,$$

[1] Note the quantifier complexity: $\exists \forall \exists \forall$ (and globally, $\forall \exists \forall \exists \forall$). Wow!

that is,

$$U_{N,k}(f) - L_{N,k}(f) \le 2^{-nN}\varepsilon + L_{N,k}(h)\,.$$

The sum over $k \in \big(\mathbb{Z} \cap [-2^{M+N}, 2^{M+n} - 1]\big)^n$ gives

$$U_N(f) - L_N(f) \le 2^{nM}\varepsilon + L_N(h)\,,$$

whence, taking $N \to \infty$,

$$\int^*_{\mathbb{R}^n} f - \int_{*\,\mathbb{R}^n} f \le 2^{nM}\varepsilon + \int_{\mathbb{R}^n} h \le (2^{nM}+1)\varepsilon + \int^*_{\mathbb{R}^n} \operatorname{Osc}_f$$

for all $\varepsilon > 0$. $\square$

**6b7 Corollary.** A bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable if and only if $\operatorname{Osc}_f$ is negligible.

**6b8 Exercise.** For a set $E \subset \mathbb{R}^n$,

(a) $\operatorname{Osc}_{\mathbb{1}_E} = \mathbb{1}_{\partial E}$;

(b) $E$ is admissible if and only if $\partial E$ has volume 0;

(c) $v^*(E) - v_*(E) = v^*(\partial E)$;

(d) if $E$ is admissible, then $E^\circ$ and $\overline{E}$ are admissible, and $v(E^\circ) = v(E) = v(\overline{E})$.

Prove it.

**6b9 Exercise.** For sets $E, F \subset \mathbb{R}^n$,

(a) prove that $\partial(E \cup F) \subset \partial E \cup \partial F$, $\partial(E \cap F) \subset \partial E \cup \partial F$, $\partial(E \setminus F) \subset \partial E \cup \partial F$,

(b) give another proof of 4f14.

**6b10 Exercise.** For $f, g : \mathbb{R}^n \to [-M, M]$,

(a) prove that $\operatorname{Osc}_{fg} \le M(\operatorname{Osc}_f + \operatorname{Osc}_g)$;

(b) give another proof of 4f9.

**6b11 Exercise.** Give another proof of 4f10 and 4f16, via oscillation.

If $E$ is admissible, then integrability of $f$ on $E$ is well-defined (recall 4d5), it is integrability on $\mathbb{R}^n$ of the function

$$f \cdot \mathbb{1}_E : x \mapsto \begin{cases} f(x) & \text{for } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

By 6b7, this integrability is equivalent to negligibility of $\operatorname{Osc}_{f\cdot\mathbb{1}_E}$. Note that

$$\operatorname{Osc}_{f\cdot\mathbb{1}_E} = \begin{cases} \operatorname{Osc}_f & \text{on } E^\circ, \\ \text{something bounded} & \text{on } \partial E, \\ 0 & \text{outside } \overline{E}. \end{cases}$$

Taking into account that $\partial E$ is of volume zero by 6b8(b) we see that $\operatorname{Osc}_{f\cdot 1\!\!1_E}$ is equivalent to $\operatorname{Osc}_f \cdot 1\!\!1_{E^\circ}$. Thus,

$$(f \text{ is integrable on } E) \iff (\operatorname{Osc}_f \text{ is negligible on } E^\circ)\,. \tag{6b12}$$

If the set $\{x : \operatorname{Osc}_f(x) \ne 0\}$ is of volume zero, then $\operatorname{Osc}_f$ is negligible by (4d8), thus $f$ is integrable. However, an integrable function can be discontinuous on a dense set; for example, see 4c5 (or 4f12).

**6b13 Remark.** It is tempting to invent an appropriate notion "negligible set" such that[1]

(a) $f$ is negligible if and only if $\{x : f(x) \ne 0\}$ is negligible,

and therefore

(b) $f$ is integrable if and only if $\{x : \operatorname{Osc}_f(x) \ne 0\}$ is negligible.

Is this possible? Yes and no...

Bad news: it can happen that $\{x : f(x) \ne 0\} = \{x : g(x) \ne 0\}$, $f$ is negligible, but $g$ is not.

Good news: it cannot happen that $\{x : \operatorname{Osc}_f(x) \ne 0\} = \{x : \operatorname{Osc}_g(x) \ne 0\}$, $f$ is integrable, but $g$ is not.

That is, (b) succeeds, but not due to (a). Rather, (b) succeeds in spite of the fact that (a) fails.[2]

## 6c Measure zero

**6c1 Definition.** A set $Z \subset \mathbb{R}^n$ has *measure* 0 if for every $\varepsilon > 0$ there exist boxes $B_1, B_2, \dots \subset \mathbb{R}^n$ such that $Z \subset \cup_{k=1}^\infty B_k$ and $\sum_{k=1}^\infty v(B_k) \le \varepsilon$.

**6c2 Proposition.** Countable union of sets of measure 0 has measure 0.

***Proof.*** Let $Z = Z_1 \cup Z_2 \cup \dots$ and each $Z_k$ has measure 0. Given $\varepsilon > 0$, we take $\varepsilon_1, \varepsilon_2, \dots > 0$ such that $\varepsilon_1 + \varepsilon_2 + \dots \le \varepsilon$ (for instance, $\varepsilon_k = 2^{-k}\varepsilon$), and for each $k$ we take boxes $B_{k,\ell}$ such that $Z_k \subset \cup_{\ell=1}^\infty B_{k,\ell}$ and $\sum_{\ell=1}^\infty v(B_{k,\ell}) \le \varepsilon_k$. We get $Z \subset \cup_{k,\ell} B_{k,\ell}$, and $\sum_{k,\ell} v(B_{k,\ell}) \le \varepsilon$. (And all pairs $(k,\ell)$ are a countable set, of course.) $\square$

Every set of volume 0 has measure 0 (think, why). Thus, countable union of sets of volume 0 has measure 0 (even if dense in the whole $\mathbb{R}^n$). In particular, every countable set has measure 0. Also, many sets of cardinality continuum have measure 0 (see 4i3).[3]

[1] Assuming that $f$ is bounded, with bounded support, of course.

[2] Puzzled? Here is an explanation: $\operatorname{Osc}_f$ is not just a function; it is an upper semicontinuous function. For upper semicontinuous $f, g$ it cannot happen that $\{x : f(x) \ne 0\} = \{x : g(x) \ne 0\}$, $f$ is negligible, but $g$ is not.

[3] In dimension 1 the Cantor set is such example.

**6c3 Proposition.** A compact set has measure 0 if and only if it has volume 0.

***Proof.*** "If": trivial. "Only if": let $K \subset \mathbb{R}^n$ be compact, of measure 0. Given $\varepsilon > 0$, we take boxes $B_k$ as in 6c1, and boxes $A_k$ such that $B_k \subset A_k^\circ$ and $v(A_k) \le 2v(B_k)$ (think, how). By compactness, $K \subset A_1^\circ \cup \dots \cup A_m^\circ$ for some $m$. Thus, $v^*(K) \le v(A_1) + \dots + v(A_m) \le 2v(B_1) + \dots + 2v(B_m) \le 2\varepsilon$. $\square$

**6c4 Exercise.** (a) If $Z$ has measure 0, then $Z^\circ = \emptyset$, and $v_*(Z) = 0$.
Prove it.[1]

However, $v^*(Z)$ need not be 0, of course.

## 6d Continuity almost everywhere

**6d1 Definition.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous *almost everywhere,* if its points of discontinuity are a set of measure 0.

For an example, recall 6a2.

More generally, a property of a point of $\mathbb{R}^n$ is said to hold almost everywhere if it holds except on a set of measure zero.

**6d2 Theorem** (Lebesgue's criterion)**.** A bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable if and only if it is continuous almost everywhere.

**6d3 Lemma.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. If $f$ is negligible then $f(\cdot) = 0$ almost everywhere.[2]

***Proof.*** We consider sets $A = \{x : f(x) \ne 0\}$ and $A_i = \{x : |f(x)| \ge \frac{1}{i}\}$; $A = \cup_i A_i$. For each $i$ we have $1\!\!1_{A_i} \le i|f|$, thus $v^*(A_i) \le i \,{}^*\!\!\int |f| = 0$, which implies that $A_i$ has measure 0 and, by 6c2, $A$ has measure 0. $\square$

**6d4 Lemma.** The set $\{x : \operatorname{Osc}_f(x) \ge \varepsilon\}$ is compact, for every $\varepsilon > 0$.

***Proof.*** Boundedness is evident. We'll prove that its complement, $\{x : \operatorname{Osc}_f(x) < \varepsilon\}$, is open. Given $\operatorname{Osc}_f(x_0) < \varepsilon$, we have $\operatorname{Osc}_f(U) < \varepsilon$ for some neighborhood $U$ of $x_0$. Thus, $\operatorname{Osc}_f(x) \le \operatorname{Osc}_f(U) < \varepsilon$ for all $x \in U$. $\square$

[1] Hint: for $Z^\circ = \emptyset$ use 6c3; for $v_*(Z) = 0$ consider Darboux sums, or use 6b8(d).

[2] The converse fails; try indicator of a dense countable set.

***Proof*** *of Theorem 6d2.* By 6b7 it is sufficient to prove that the function $\varphi = \mathrm{Osc}_f$ is negligible if and only if $f$ is continuous almost everywhere, that is, $\varphi = 0$ almost everywhere.

"Only if": just by 6d3 applied to $\varphi$.

"If": for every $\varepsilon > 0$ the set $\{x : \varphi(x) \ge \varepsilon\}$ has measure 0; by 6d4 and 6c3, this set has volume 0. By (4d8), $\varphi$ is equivalent (recall 4e) to the function $\min(\varphi, \varepsilon) : x \mapsto \min\bigl(\varphi(x), \varepsilon\bigr)$. We take $M \in (0, \infty)$ such that $\{x : f(x) > 0\} \subset [-M, M]^n$, then also $\{x : \varphi(x) > 0\} \subset [-M, M]^n$, and we get

$$\int^*_{\mathbb{R}^n} \varphi = \int^*_{\mathbb{R}^n} \min(\varphi, \varepsilon) \le \varepsilon (2M)^n$$

for all $\varepsilon > 0$. $\square$

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