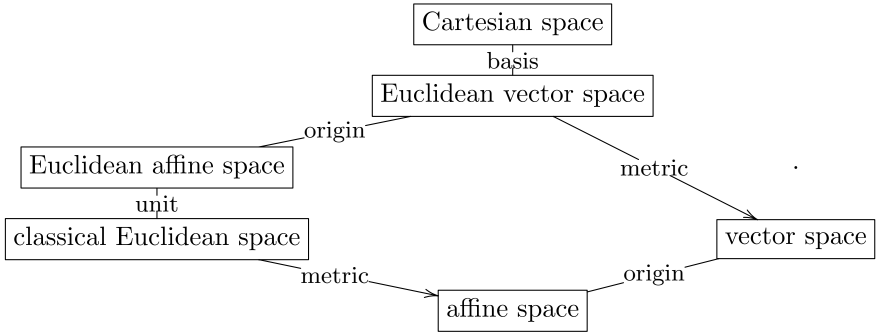

We may apply notions and results of Euclidean planimetry/stereometry in every 2-dimensional/3-dimensional subspace of an n-dimensional Euclidean affine space.

Topological notions are well-defined on every finite-dimensional vector or affine space.

All norms are equivalent on an arbitrary finite-dimensional vector space.

2b6 Proposition (Linearity of derivative)**.** Let S be a finite-dimensional affine space, V a finite-dimensional vector space, $f, g : S \to V$, $a, b \in \mathbb{R}$, and $x_0 \in S$. If f, g are differentiable at x_0 then also $af + bg$ is, and

$$\big(D(af + bg)\big)_{x_0} = a(Df)_{x_0} + b(Dg)_{x_0} \,.$$

2b8 Proposition (Product rule)**.** Let S be a finite-dimensional affine space, $f, g : S \to \mathbb{R}$, and $x_0 \in S$. If f, g are differentiable at x_0 then also fg is, and

$$\big(D(fg)\big)_{x_0} = f(x_0)(Dg)_{x_0} + g(x_0)(Df)_{x_0} \,.$$

2b11 Proposition (Chain rule)**.** Let S_1, S_2, S_3 be finite-dimensional affine spaces, $f : S_1 \to S_2$, $g : S_2 \to S_3$, and $x_0 \in S_1$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then $g \circ f$ is differentiable at x_0, and

$$\big(D(g \circ f)\big)_{x_0} = (Dg)_{f(x_0)} \circ (Df)_{x_0} \,.$$

(2d5)
$$\frac{\|\gamma(t_1) - \gamma(t_0)\|}{t_1 - t_0} \le \sup_{t \in (t_0, t_1)} \|\gamma'(t)\|$$

2f2 Lemma. Let a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at x_0, and $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ be the coordinate functions of f (that is, $f(x) = \big(f_1(x), \dots, f_m(x)\big)$. Then the following two conditions are equivalent:

(a) vectors $\nabla f_1(x_0), \dots, \nabla f_m(x_0)$ are linearly independent;

(b) the linear operator $(Df)_{x_0}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

2g1 Proposition. If $f \in C^k(\mathbb{R}^n \to \mathbb{R}^m)$ and $g \in C^k(\mathbb{R}^m \to \mathbb{R}^\ell)$ then $g \circ f \in C^k(\mathbb{R}^n \to \mathbb{R}^\ell)$.

$$f(x_0 + h) = f(x_0) + D_h f(x_0) + \frac{1}{2!} D_h D_h f(x_0) + \dots + \frac{1}{k!} D_h^k f(x_0) + o(|h|^k) \,.$$

3c2 Theorem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \mathbb{R}^n$. If f is continuously differentiable near x and the linear operator $(Df)_x$ is a homeomorphism, then f is a homeomorphism near x.

3f1 Theorem (Lagrange multipliers)**.** Assume that $x_0 \in \mathbb{R}^n$, $1 \le m \le n-1$, functions $f, g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable near x_0, $g_1(x_0) = \dots = g_m(x_0) = 0$, and vectors $\nabla g_1(x_0), \dots, \nabla g_m(x_0)$ are linearly independent. If x_0 is a local constrained extremum of f subject to $g_1(\cdot) = \dots = g_m(\cdot) = 0$ then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \dots + \lambda_m \nabla g_m(x_0) \,.$$

The system of $m + n$ equations proposed in Sect. 3f is only one way of finding local constrained extrema. Not necessarily the simplest way.

No need to find ∇f when $f(\cdot) = \varphi(g(\cdot))$; just find ∇g and note that ∇f is collinear to ∇g.

If Lagrange method does not solve a problem to the end, it may still give a useful information. Combine it with other methods as needed.

3i1 Proposition (Singular value decomposition)**.** Every linear operator from one finite-dimensional Euclidean vector space to another sends some orthonormal basis of the first space into an orthogonal system in the second space.

$$\frac{\partial}{\partial c_k}\Big|_{c=0} f(x(c)) = \lambda_k(0) \,.$$

It means that $\lambda_k = \lambda_k(0)$ is the sensitivity of the critical value to the level c_k of the constraint $g_k(x) = c_k$.

4c2 Theorem (Inverse function)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \mathbb{R}^n$. If f is continuously differentiable near x and the linear operator $(Df)_x$ is a diffeomorphism, then f is a diffeomorphism near x.

$$(Dg)_y = \big((Df)_x\big)^{-1} \quad \text{for } g = (f|_U)^{-1} \,, \ y = f(x) \,.$$

4c7 Proposition. Assume that $U, V \subset \mathbb{R}^n$ are open, $f : U \to V$ is a homeomorphism, continuously differentiable, and the operator $(Df)_x$ is invertible for all $x \in U$. Then the inverse mapping $f^{-1} : V \to U$ is continuously differentiable.

4c11 Exercise. (a) Let $f : U \to V$ be as in Prop. 4c7 and in addition $f \in C^2(U)$. Then $f^{-1} \in C^2(V)$

(b) The same holds for $C^k(\dots)$ where $k = 3, 4, \dots$

4d1 Proposition. Assume that $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable near x_0, Df is continuous at x_0, and the operator $T = (Df)_{x_0}$ is invertible. Then for every y near $y_0 = f(x_0)$ the iterative process

$$x_{n+1} = x_n + T^{-1}\big(y - f(x_n)\big) \quad \text{for } n = 0, 1, 2, \dots$$

is well-defined and converges to a solution x of the equation $f(x) = y$. In addition, $|x - x_0| = O(|y - y_0|)$.

5c1 Theorem (Implicit function)**.** Assume that $r, c \in \{1, 2, 3, \dots\}$, $n = r + c$, $x_0 \in \mathbb{R}^r$, $y_0 \in \mathbb{R}^c$, $g : \mathbb{R}^n \to \mathbb{R}^c$ is continuously differentiable near (x_0, y_0), $g(x_0, y_0) = 0$, and the operator $B = \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)}$ is invertible. Then there exist open neighborhoods U of x_0 and V of y_0 such that

(a) for every $x \in U$ there exists one and only one $y \in V$ satisfying $g(x, y) = 0$;

(b) a function $\varphi : U \to V$ defined by $g\big(x, \varphi(x)\big) = 0$ is continuously differentiable, and $(D\varphi)_{x_0} = -B^{-1}A$ where $A = \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)}$.

(6b16) $$\overline{\int_B} f = \sum_{C \in P} \overline{\int_C} f \,,$$

which means that the upper integral is an *additive box function.*

(6d9) $$\overline{\int} (f + g) \le \overline{\int} f + \overline{\int} g \,;$$

(6d10) $$\underline{\int} (f + g) \ge \underline{\int} f + \underline{\int} g \,;$$

(6d11) if f, g are integrable then $f + g$ is, and $$\int (f + g) = \int f + \int g \,.$$

$$\rho\big([f], [g]\big) = \|\,[f] - [g]\,\| = \overline{\int_B} |f - g| \,;$$

this is the *integral metric,* and the corresponding convergence is the *integral convergence.*

6e3 Exercise. (a) A function equivalent to an integrable function is integrable;

(b) equivalence classes of integrable functions are a closed set in the normed space of equivalence classes, and the functional $[f] \mapsto \int_B f$ on this set is continuous.

6f5 Proposition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support, and $\varepsilon > 0$. Then there exist continuous $g, h : \mathbb{R}^n \to \mathbb{R}$ with bounded support such that

$$g(\cdot) \le f(\cdot) \le h(\cdot) \,, \quad \int_{\mathbb{R}^n} (h - g) \le \varepsilon + \overline{\int_{\mathbb{R}^n}} f - \underline{\int_{\mathbb{R}^n}} f \,.$$

And, of course,

(6f6) $$\int_{\mathbb{R}^n} g \ge -\varepsilon + \underline{\int_{\mathbb{R}^n}} f \,, \quad \int_{\mathbb{R}^n} h \le \varepsilon + \overline{\int_{\mathbb{R}^n}} f \,.$$

6f7 Corollary. Continuous functions are dense among integrable functions (in the integral metric).

If f and g are integrable then $\min(f, g)$, $\max(f, g)$ and fg are integrable.

6g1 Definition. Let $E \subset \mathbb{R}^n$ be a bounded set.

$$v_*(E) = \underline{\int_{\mathbb{R}^n}} \mathbb{1}_E \,, \quad v^*(E) = \overline{\int_{\mathbb{R}^n}} \mathbb{1}_E \,.$$

If they are equal (that is, if $\mathbb{1}_E$ is integrable) then E is *Jordan measurable,* and its Jordan measure is

$$v(E) = \int_{\mathbb{R}^n} \mathbb{1}_E \,.$$

(6g5) $$v^*(E_1 \cup E_2) \le v^*(E_1) + v^*(E_2) \,,$$

(6g6) $$v_*(E_1 \uplus E_2) \ge v_*(E_1) + v_*(E_2) \,;$$

(6g7) if E_1, E_2 are Jordan measurable then $E_1 \uplus E_2$ is, and $$v(E_1 \uplus E_2) = v(E_1) + v(E_2) \,.$$

We may ignore values of integrands (as far as they are bounded) on sets of volume zero. We may ignore sets of volume zero when dealing with Jordan measure.

(6g16) $$\int_E f = \int_{\mathbb{R}^n} f \cdot \mathbb{1}_E \,.$$

(6g17) $$\int_{E_1 \uplus E_2} f = \int_{E_1} f + \int_{E_2} f \,.$$

(6g18) $$\int_E f = \int_E a \quad \text{where} \quad a = \frac{1}{v(E)} \int_E f \quad \text{is the } \textit{mean} \text{ (value) of } f \text{ on } E.$$

6h1 Proposition. Let $f : B \to [0, \infty)$ be an integrable function on a box $B \subset \mathbb{R}^n$, and

$$E = \{(x, t) : x \in B,\ 0 \le t \le f(x)\} \subset \mathbb{R}^{n+1} \,.$$

Then E is Jordan measurable (in $\mathbb{R}^{n+1}$), and

$$v(E) = \int_B f \,.$$

(7b1) $$|f(x) - f(y)| \le L|x - y| \quad \text{for all } x, y \,. \quad \text{(Lipschitz condition)}$$

7b4 Proposition. Let two boxes $B_1 \subset \mathbb{R}^m$, $B_2 \subset \mathbb{R}^n$ be given, and a Lipschitz function f on a box $B = B_1 \times B_2 \subset \mathbb{R}^{m+n}$. Then

(a) for every $x \in B_1$ the function f_x is Lipschitz continuous on B_2;

(b) the function $x \mapsto \int_{B_2} f_x$ is Lipschitz continuous on B_1;

(c) $$\int_B f = \int_{B_1} \Big(x \mapsto \int_{B_2} f_x \Big) \,.$$

7b6 Exercise. Prove that

$$\int_{B_1 \times B_2} f(x_1, \dots, x_m) g(y_1, \dots, y_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_m \, \mathrm{d}y_1 \dots \mathrm{d}y_n =$$
$$= \Big(\int_{B_1} f(x_1, \dots, x_m) \, \mathrm{d}x_1 \dots \mathrm{d}x_m \Big) \Big(\int_{B_2} g(y_1, \dots, y_n) \, \mathrm{d}y_1 \dots \mathrm{d}y_n \Big)$$

for Lipschitz functions $f : B_1 \to \mathbb{R}$, $g : B_2 \to \mathbb{R}$.

Existence of an iterated integral does not ensure existence of the two-dimensional integral.

7d1 Theorem. Let two boxes $B_1 \subset \mathbb{R}^m$, $B_2 \subset \mathbb{R}^n$ be given, and an integrable function f on the box $B = B_1 \times B_2 \subset \mathbb{R}^{m+n}$. Then the iterated integrals

$$\int_{B_1} \mathrm{d}x \, {}_*\!\!\int_{B_2} \mathrm{d}y \, f(x,y) \,, \qquad \int_{B_1} \mathrm{d}x \, {}^*\!\!\int_{B_2} \mathrm{d}y \, f(x,y) \,,$$

$$\int_{B_2} \mathrm{d}y \, {}_*\!\!\int_{B_1} \mathrm{d}x \, f(x,y) \,, \qquad \int_{B_2} \mathrm{d}y \, {}^*\!\!\int_{B_1} \mathrm{d}x \, f(x,y)$$

are well-defined and equal to

$$\iint_B f(x,y) \, \mathrm{d}x\mathrm{d}y \,.$$

7d3 Exercise. Generalize 7b6 to integrable functions
(a) assuming integrability of the function $(x,y) \mapsto f(x)g(y)$,
(b) deducing integrability of the function $(x,y) \mapsto f(x)g(y)$ from integrability of f and g (via sandwich).

7d6 Exercise. If $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^n$ are Jordan measurable sets then the set $E = E_1 \times E_2 \subset \mathbb{R}^{m+n}$ is Jordan measurable.

7d8 Corollary. Let $f : \mathbb{R}^{m+n} \to \mathbb{R}$ be integrable on every box, and $E \subset \mathbb{R}^{m+n}$ a Jordan measurable set; then

$$\int_E f = \int_{\mathbb{R}^m} \Big(x \mapsto \int_{E_x} f_x \Big)$$

where $E_x = \{ y : (x,y) \in E \} \subset \mathbb{R}^n$ for $x \in \mathbb{R}^m$.

(7d9) $$v_{m+n}(E) = \int_{\mathbb{R}^m} v_n(E_x) \, \mathrm{d}x \,.$$

7d10 Corollary (Cavalieri)**.** If Jordan measurable sets $E, F \subset \mathbb{R}^3$ satisfy $v_2(E_x) = v_2(F_x)$ for all x then $v_3(E) = v_3(F)$.

7d28 Exercise. Every $f \in C^0(\mathbb{R}^n)$ with bounded support is the limit of some uniformly convergent sequence of functions of $C^1(\mathbb{R}^n)$.

7e1 Theorem. Let $B \subset \mathbb{R}^n$ be a box, and $f, g : B \times [0,1] \to \mathbb{R}$ Lipschitz functions such that $f'_x(t) = g_x(t)$ for all $x \in B$, $t \in (0,1)$. Then $F'(t) = G(t)$ for all $t \in (0,1)$, where $F(t) = \int_B f(x,t) \, \mathrm{d}x$ and $G(t) = \int_B g(x,t) \, \mathrm{d}x$.

7e3 Exercise. (b) every $f \in C^0(\mathbb{R}^n)$ with bounded support is the limit of some uniformly convergent sequence of functions of $C^2(\mathbb{R}^n)$;
(c) the same as (b), but replace $C^2(\mathbb{R}^n)$ with $C^k(\mathbb{R}^n)$, $k = 1, 2, 3, \dots$

8b11 Proposition. ${}^*\!\!\int (f+g) = {}^*\!\!\int f + {}^*\!\!\int g$ for all upper semicontinuous bounded functions $f, g : \mathbb{R}^n \to \mathbb{R}$ with bounded support.

8c7 Lemma. If a superadditive box function F satisfies ${}_*F'(x) \ge 0$ for all $x \in \overline{B}_0$ (B_0 being a given box), then $F(\overline{B}_0) \ge 0$.

(8c10) $$F(B) = \int_B F' \quad \text{whenever } F' \text{ exists and is integrable on } B.$$

8c11 Exercise.
$$ {}_*\!\!\int_B {}_*F' \le F(B) \le {}^*\!\!\int_B {}^*F'$$
for every box B and additive box function F such that ${}_*F'$ and ${}^*F'$ are bounded on B.

8d2 Proposition. ${}^*\!\!\int f - {}_*\!\!\int f = {}^*\!\!\int \operatorname{Osc}_f$ for all bounded $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support.

8d3 Corollary. A bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable if and only if Osc_f is negligible.

8d4 Corollary. For every bounded $A \subset \mathbb{R}^n$,
(a) $v^*(A) - v_*(A) = v^*(\partial A)$;
(b) A is Jordan measurable if and only if ∂A is of volume zero.

(8d5) $$(f \text{ is integrable on a Jordan set } E) \iff (\operatorname{Osc}_f \text{ is negligible on } E^\circ) \,.$$

$$\text{Extended integral:} \quad {}^{\mathrm{e}}\!\!\int (f-g) = {}^*\!\!\int f - {}^*\!\!\int g \quad \text{for upper semicontinuous } f, g \,.$$
$${}^{\mathrm{e}}v(K) = v^*(K) \quad \text{for compact } K \subset \mathbb{R}^n \,,$$
$${}^{\mathrm{e}}v(G) = v_*(G) \quad \text{for open bounded } G \subset \mathbb{R}^n \,.$$

8e5 Definition. For a bounded set $A \subset \mathbb{R}^n$,
$$m_*(A) = \sup_{K \subset A} v^*(K) \,, \quad m^*(A) = \inf_{G \supset A} v_*(G)$$
(here K runs over compact sets, and G over open bounded sets); if these are equal, then A is *Lebesgue measurable,* and its *Lebesgue measure* is
$$m(A) = m_*(A) = m^*(A) \,.$$

8e6 Lemma. Every open bounded set is Lebesgue measurable. That is,
$$v_*(G) = \sup_{K \subset G} v^*(K) \quad \text{for every open bounded } G \subset \mathbb{R}^n \,,$$
the supremum being taken over all compact subsets of G.

8e7 Exercise. Every compact set is Lebesgue measurable. That is,
$$v^*(K) = \inf_{G \supset K} v_*(G) \quad \text{for every compact } K \subset \mathbb{R}^n \,,$$
the infimum being taken over all open bounded $G \supset K$.

8e9 Proposition. *(Monotone convergence for open sets)* For all open bounded sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,
$$G_i \uparrow G \implies v_*(G_i) \uparrow v_*(G) \,.$$

8e10 Corollary. $v_*(G_1 \cup G_2 \cup \dots) \le v_*(G_1) + v_*(G_2) + \dots$ for all open $G_1, G_2, \dots \subset \mathbb{R}^n$ whose union is bounded.

8e11 Exercise. *(Monotone convergence for compact sets)* For all compact sets $K, K_1, K_2, \dots \subset \mathbb{R}^n$,
$$K_i \downarrow K \implies v^*(K_i) \downarrow v^*(K) \,.$$

8f1 Theorem (Lebesgue's criterion)**.** A bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable if and only if it is continuous almost everywhere.