14 Divergence, flux, Laplacian

Divergence and flux are widely used in order to relate volume integrals and surface integrals in a geometrically natural way.

14a What is the problem

The "integral of derivative" (13b3) deserves a generalization. The most straightforward generalization is

(14a1) $$\int_{\mathbb{R}^n} Df = 0 \quad \text{if } f \in C^1(\mathbb{R}^n \to \mathbb{R}^m) \text{ has a bounded support;}$$

but this is boring. Indeed, $(Df)_x$ may be thought of as a matrix whose rows are gradients of the coordinate functions $f_1, \dots, f_m \in C^1(\mathbb{R}^n)$ of f (recall Sect. 2e), and (14a1) is just (13b3) applied rowwise.

Restricting ourselves to the case $m = n$, we may think about $\det(Df)$; definitely an interesting function of Df. We cannot expect $\int \det(Df)$ to vanish, since the determinant is a nonlinear function of a matrix. But we know (recall 2e9) that

(14a2) $$\det(I + H) = 1 + \operatorname{tr}(H) + o(H)$$

for small H. The trace being a linear function of a matrix, we have

(14a3) $$\int_{\mathbb{R}^n} \operatorname{tr}(Df) = 0 \quad \text{if } f \in C^1(\mathbb{R}^n \to \mathbb{R}^n) \text{ has a bounded support.}$$

Now the question is, what is $\operatorname{tr}(Df)$ good for?

Consider a one-parameter family of diffeomorphisms $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ given for $t \in \mathbb{R}$; we assume that the mapping $(x, t) \mapsto \varphi_t(x)$ belongs to $C^2(\mathbb{R}^{n+1} \to \mathbb{R}^n)$, and $\varphi_0(x) = x$ for all $x \in \mathbb{R}^n$. Then $(D\varphi_0)_0 = I$ and $(D\varphi_t)_0 = I + tA + o(t)$ where $A = \frac{d}{dt}\big|_{t=0} (D\varphi_t)_0$; thus, $\det(D\varphi_t)_0 = 1 + t \operatorname{tr} A + o(t)$ for small t. If $\operatorname{tr} A > 0$, then $\det(D\varphi_t)_0 > 1$ for small $t > 0$, which means that

$v\bigl(\varphi_t(U)\bigr) > v(U)$ for a small enough neighborhood U of 0 in $\mathbb{R}^n$. Moreover, $v\bigl(\varphi_t(U)\bigr) \approx (1 + t \operatorname{tr} A) v(U)$.

In mechanics, a flowing matter may be described this way; every point x flows to another point $\varphi_t(x)$ during the time interval $(0, t)$. A small drop of the flowing matter inflates if $\operatorname{tr} A > 0$ and deflates if $\operatorname{tr} A < 0$. The rate of this inflation/deflation is $\operatorname{tr} A$.

The vector $F(x) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \varphi_t(x)$ is the velocity of the flow at a point x and the instant 0. This mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called the *velocity field* of the flow. We have

$$A = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (D\varphi_t)_0 = \Bigl(D\Bigl(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi_t \Bigr) \Bigr)_0 = (DF)_0 \,,$$

thus, the inflation/deflation rate at the origin is $\operatorname{tr} A = \operatorname{tr}(DF)_0$, and similarly, at a point x it is $\operatorname{tr}(DF)_x$.

The velocity field is a *vector field.* The word "field" in "vector field" is not related to the algebraic notion of a field. Rather, it is related to the physical notion of a force field (gravitational, for example), or the velocity field of a moving matter (usually liquid or gas). Mathematically, a vector field formally is just a mapping $\mathbb{R}^n \to \mathbb{R}^n$; less formally, a vector is attached to each point.

A vector field on an affine space is a mapping from this space to its difference space. Note that the determinant is well-defined in a (finite-dimensional) vector space; metric is irrelevant. The same holds for the trace.

14a4 Definition. The *divergence* of a mapping ("vector field") $F \in C^1(\mathbb{R}^n \to \mathbb{R}^n)$ is the function ("scalar field") $\operatorname{div} F \in C(\mathbb{R}^n)$,

$$\operatorname{div} F = \operatorname{tr}(DF) \,.$$

That is, for $F(x) = \bigl(F_1(x), \dots, F_n(x)\bigr)$ we have

$$\operatorname{div} F = D_1 F_1 + \dots + D_n F_n = (\nabla F_1)_1 + \dots + (\nabla F_n)_n \,;$$
$$\operatorname{div} F(x_1, \dots, x_n) = \frac{\partial}{\partial x_1} F_1(x_1, \dots, x_n) + \dots + \frac{\partial}{\partial x_n} F_n(x_1, \dots, x_n) \,.$$

Once again: if F is a velocity field, then $\operatorname{div} F$ is the inflation/deflation rate.

For a vector field $F \in C^1(V \to V)$ on an n-dimensional vector space V, still, $\operatorname{div} F = \operatorname{tr}(DF)$; here $(DF)_x : V \to V$.

For a vector field $F \in C^1(S \to \vec{S})$ on an n-dimensional affine space S, also, $\operatorname{div} F = \operatorname{tr}(DF)$; here $(DF)_x : \vec{S} \to \vec{S}$.

Clearly,

(14a5) $$\int_{\mathbb{R}^n} \operatorname{div} F = 0 \quad \text{if } F \text{ has a bounded support.}$$

Similarly to the singular gradient (treated in Sect. 13b), we want to introduce singular divergence; and then, similarly to Theorem 13b9, we want to generalize (14a5) to a vector field continuous up to a surface.

14b Integral of derivative (again)

Similarly to Sect. 13b we consider a hypersurface, that is, an n-dimensional manifold M in $\mathbb{R}^N$, $N = n + 1$. Similarly to 13b5, for a vector field $F : \mathbb{R}^N \setminus \overline{M} \to \mathbb{R}^N$ we define the notion "continuous up to M". Clearly, $F = (F_1, \dots, F_N)$ is continuous up to M if and only if $F_1, \dots, F_N$ are continuous up to M (as defined by 13b5). The one-sided limits F_-, F_+ are now vector-valued, and the jump $F_+(x_0) - F_-(x_0)$ is a vector; its sign depends on the side indicator. Recall the unit normal vector $\mathbf{n}_x \in \mathbb{R}^N$; its sign also depends on the side indicator. Here is a definition similar to 13b7. As before, we denote $F(x - 0\mathbf{n}_x) = F_-(x)$ and $F(x + 0\mathbf{n}_x) = F_+(x)$.

14b1 Definition. The *singular divergence*[1] $\operatorname{div}_{\text{sng}} F(x)$ at $x \in M$ of a mapping $F : \mathbb{R}^N \setminus \overline{M} \to \mathbb{R}^N$ continuous up to M is the number

$$\operatorname{div}_{\text{sng}} F(x) = \langle F(x + 0\mathbf{n}_x) - F(x - 0\mathbf{n}_x), \mathbf{n}_x \rangle \,.$$

As before, the singular divergence does not depend on the side indicator (and $\mathbf{n}_x$). It is a continuous function $\operatorname{div}_{\text{sng}} F : M \to \mathbb{R}$.

Less formally, the singular divergence is the jump of the normal component of the vector field.

Here is the singular counterpart of the formula

$$\operatorname{div} F = \sum_k (\nabla F_k)_k \,.$$

14b2 Lemma.

$$\operatorname{div}_{\text{sng}} F = \sum_{k=1}^N \bigl(\nabla_{\text{sng}} F_k\bigr)_k \,.$$

Proof.

$$\begin{aligned} \sum_k \bigl(\nabla_{\text{sng}} F_k(x)\bigr)_k &= \sum_k \Bigl(\bigl(F_k(x + 0\mathbf{n}_x) - F_k(x - 0\mathbf{n}_x)\bigr)\mathbf{n}_x\Bigr)_k = \\ &= \sum_k \bigl(F(x + 0\mathbf{n}_x) - F(x - 0\mathbf{n}_x)\bigr)_k (\mathbf{n}_x)_k = \\ &= \langle F(x + 0\mathbf{n}_x) - F(x - 0\mathbf{n}_x), \mathbf{n}_x \rangle = \operatorname{div}_{\text{sng}} F(x) \,. \end{aligned}$$

□

[1] Not a standard terminology.

A theorem, similar to 13b9, follows easily.

14b3 Theorem. Let $M \subset \mathbb{R}^{n+1}$ be an n-manifold, $K \subset M$ a compact subset, and $F : \mathbb{R}^{n+1} \setminus K \to \mathbb{R}^{n+1}$ a mapping such that

(a) F is continuously differentiable (on $\mathbb{R}^{n+1} \setminus K$);

(b) $F|_{\mathbb{R}^{n+1}\setminus\overline{M}}$ is continuous up to M;

(c) F has a bounded support, and DF is bounded (on $\mathbb{R}^{n+1} \setminus K$).

Then

$$\int_{\mathbb{R}^{n+1}\setminus K} \operatorname{div} F + \int_M \operatorname{div}_{\mathrm{sng}} f = 0\,.$$

Proof. We have $F(x) = \big(F_1(x), \dots, F_N(x)\big)$, and Theorem 13b9 applies to each F_k, giving

$$\int_{\mathbb{R}^{n+1}\setminus K} \nabla F_k + \int_M \nabla_{\mathrm{sng}} F_k = 0\,.$$

It remains to take the k-th coordinate, and sum up over k. □

14c Divergence and flux

We return to the case treated before, in the end of Sect. 13b: $G \subset \mathbb{R}^N$ is a bounded regular open set, and $\partial G \subset \mathbb{R}^N$ a (necessarily compact) hypersurface (that is, n-manifold for $n = N - 1$). Recall the outward unit normal vector $\mathbf{n}_x$ for $x \in \partial G$.

14c1 Definition. For a continuous $F : \partial G \to \mathbb{R}^n$, the (outward) *flux* of (the vector field) F through ∂G is

$$\int_{\partial G} \langle F, \mathbf{n} \rangle\,.$$

(The integral is interpreted according to (13a8).)

If a vector field F on $\mathbb{R}^3$ is the velocity field of a fluid, then the flux of F through a surface is the amount[1] of fluid flowing through the surface (per unit time).[2] If the fluid is flowing parallel to the surface then, evidently, the flux is zero.

We continue similarly to Sect. 13b. Let $F : \overline{G} \to \mathbb{R}^N$ be continuous, $F|_G \in C^1(G \to \mathbb{R}^N)$, with DF bounded (on G). Then the mapping $\tilde{F} : \mathbb{R}^N \setminus \partial G \to \mathbb{R}^N$ defined by

$$\tilde{F}(x) = \begin{cases} F(x) & \text{for } x \in G, \\ 0 & \text{for } x \notin \overline{G} \end{cases}$$

[1] The volume is meant, not the mass. However, these are proportional if the density (kg/m^3) of the matter is constant (which often holds for fluids).

[2] See also mathinsight.

is continuous up to ∂G, and

$$\tilde F(x - 0\mathbf{n}_x) = F(x)\,, \quad \tilde F(x + 0\mathbf{n}_x) = 0\,;$$
$$\operatorname{div}_{\text{sng}} \tilde F(x) = -\langle F(x), \mathbf{n}_x \rangle\,.$$

By Theorem 14b3 (applied to $\tilde F$ and $K = \partial G$),

$$\int_G \operatorname{div} F = \int_{\partial G} \langle F, \mathbf{n} \rangle\,, \tag{14c2}$$

just the flux. The divergence theorem, formulated below, is thus proved.[1]

14c3 Theorem (*Divergence theorem*)**.** Let $G \subset \mathbb{R}^{n+1}$ be a bounded regular open set, ∂G an n-manifold, $F : \overline{G} \to \mathbb{R}^{n+1}$ continuous, $F|_G \in C^1(G \to \mathbb{R}^{n+1})$, with DF bounded on G.

Then the integral of $\operatorname{div} F$ over G is equal to the (outward) flux of F through ∂G.

In particular, if $\operatorname{div} F = 0$, then $\int_{\partial G} \langle F, \mathbf{n} \rangle = 0$.

14c4 Exercise. (a) For every $f \in C^1(G)$, boundedness of ∇f on G ensures that f extends to $\overline{G}$ by continuity (and therefore is bounded).

(b) For every $F \in C^1(G \to \mathbb{R}^{n+1})$, boundedness of DF on G ensures that F extends to $\overline{G}$ by continuity (and therefore is bounded).
Prove it.[2]

In such cases we'll always mean this extension.

14c5 Exercise. $\operatorname{div}(fF) = f \operatorname{div} F + \langle \nabla f, F \rangle$ whenever $f \in C^1(G)$ and $F \in C^1(G \to \mathbb{R}^N)$

Prove it.

Thus, the divergence theorem, applied to fF when $f \in C^1(G)$ with bounded ∇f, and $F \in C^1(G \to \mathbb{R}^N)$ with bounded DF, gives a kind of integration by parts, similar to (13b13):

$$\int_G \langle \nabla f, F \rangle = \int_{\partial G} f \langle F, \mathbf{n} \rangle - \int_G f \operatorname{div} F\,. \tag{14c6}$$

In particular, if $\operatorname{div} F = 0$, then $\int_G \langle \nabla f, F \rangle = \int_{\partial G} f \langle F, \mathbf{n} \rangle$

[1] Divergence is often explained in terms of sources and sinks (of a moving matter). But be careful; the flux of a velocity field is the amount (per unit time) as long as “amount” means “volume”. If by “amount” you mean “mass”, then you need the vector field of momentum, not velocity; multiply the velocity by the density of the matter. However, the problem disappears if the density is constant (which often holds for fluids).

[2] Hint: recall the proof of 13b4.

14d Divergence of gradient: Laplacian

Some (but not all) vector fields are gradients of scalar fields.

14d1 Definition. (a) The *Laplacian* Δf of a function $f \in C^2(G)$ on an open set $G \subset \mathbb{R}^n$ is

$$\Delta f = \operatorname{div} \nabla f \,.$$

(b) f is *harmonic,* if $\Delta f = 0$.

We have $\nabla f = (D_1 f, \dots, D_n f)$, thus, $\operatorname{div} \nabla f = D_1(D_1 f) + \dots + D_n(D_n f)$; in this sense,

$$\Delta = D_1^2 + \dots + D_n^2 = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \,,$$

the so-called Laplace operator, or Laplacian.

Any n-dimensional Euclidean affine space may be used instead of $\mathbb{R}^n$. Indeed, the gradient is well-defined in such space, and the divergence is well-defined even without Euclidean metric.

The divergence theorem 14c3 gives the so-called *first Green formula*

$$\text{(14d2)} \qquad \int_G \Delta f = \int_{\partial G} \langle \nabla f, \mathbf{n} \rangle = \int_{\partial G} D_{\mathbf{n}} f \,,$$

where $\big(D_{\mathbf{n}} f\big)(x) = \big(D_{\mathbf{n}_x} f\big)_x$ is the directional derivative of f at x in the normal direction $\mathbf{n}_x$. Here $f \in C^2(G)$, with bounded second derivatives.

Here is another instance of integration by parts. Let $u \in C^1(G)$, with bounded gradient, and $v \in C^2(G)$, with bounded second derivatives. Applying (14c6) to $f = u$ and $F = \nabla v$ we get $\int_G \langle \nabla u, \nabla v \rangle = \int_{\partial G} u \langle \nabla v, \mathbf{n} \rangle - \int_G u \Delta v$, that is,

$$\text{(14d3)} \qquad \int_G (u \Delta v + \langle \nabla u, \nabla v \rangle) = \int_{\partial G} \langle u \nabla v, \mathbf{n} \rangle = \int_{\partial G} u D_{\mathbf{n}} v \,,$$

the *second Green formula.* It follows that

$$\text{(14d4)} \qquad \int_G (u \Delta v - v \Delta u) = \int_{\partial G} (u D_{\mathbf{n}} v - v D_{\mathbf{n}} u) \,,$$

the *third Green formula;* here $u, v \in C^2(G)$, with bounded second derivatives. In particular,

$$\int_{\partial G} u D_{\mathbf{n}} v = \int_{\partial G} v D_{\mathbf{n}} u \quad \text{for harmonic } u, v \,.$$

Rewriting (14d4) as

$$\int_G u\Delta v = \int_G v\Delta u - \int_{\partial G} vD_{\mathbf{n}}u + \int_{\partial G} (D_{\mathbf{n}}v)u \tag{14d5}$$

we may say that really $\int (u1\!\!1_G)\Delta v = \int v\Delta(u1\!\!1_G)$ where $\Delta(u1\!\!1_G)$ consists of the usual Laplacian $(\Delta u)1\!\!1_G$ sitting on G and the singular Laplacian sitting on ∂G, of two terms, so-called single layer $(-D_{\mathbf{n}}u)$ and double layer $uD_{\mathbf{n}}$. Why two layers? Because the Laplacian (unlike gradient and divergence) involves second derivatives.

14d6 Exercise. Consider homogeneous polynomials on $\mathbb{R}^2$:

$$f(x,y) = \sum_{k=0}^{m} c_k x^k y^{m-k}\,.$$

For $m = 1, 2$ and 3 find all harmonic functions among these polynomials.[1]

14d7 Exercise. On $\mathbb{R}^2$,

(a) a function of the form

$$f(x,y) = \sum_{k=1}^{m} c_k \mathrm{e}^{a_k x + b_k y} \quad (a_k, b_k, c_k \in \mathbb{R})$$

is harmonic only if it is constant;

(b) a function of the form

$$f(x,y) = \mathrm{e}^{ax}\cos by$$

is harmonic if and only if $|a| = |b|$.[2]
Prove it.

14d8 Exercise. Consider $f : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ of the form $f(x) = g(|x|)$ for a given $g \in C^2(0,\infty)$. Prove that[3]

(a) $f \in C^2(\mathbb{R}^N \setminus \{0\})$;

(b) $f(r+\varepsilon, \delta, 0, \dots, 0) = g(r) + g'(r)\varepsilon + \frac12\big(g''(r)\varepsilon^2 + \frac1r g'(r)\delta^2\big) + o(\varepsilon^2+\delta^2)$;

(c) $\Delta f(x) = g''(|x|) + \frac{N-1}{|x|}g'(|x|)$.

Thus, f is harmonic if and only if $g''(r) + \frac{N-1}{r}g'(r) = 0$ for all r; that is: $\big(\log g'(r)\big)' = -\frac{N-1}{r} = -(N-1)(\log r)'$; $\log g'(r) = -(N-1)\log r + \text{const}$; $g'(r) = \text{const}\cdot r^{-(N-1)}$; $g(r) = \text{const}_1 \cdot r^{-(N-2)} + \text{const}_2$;

$$f(x) = \begin{cases} \frac{c_1}{|x|^{N-2}} + c_2 & \text{if } N \ne 2; \\ c_1 \log|x| + c_2 & \text{if } N = 2. \end{cases} \tag{14d9}$$

[1] In fact, they are $\operatorname{Re}(x+\mathrm{i}y)^m$, $\operatorname{Im}(x+\mathrm{i}y)^m$ and their linear combinations.

[2] That is, $f(x,y) = \operatorname{Re}\big(\mathrm{e}^{x+\mathrm{i}y}\big)$.

[3] Hint: (a,b) $|x| = \sqrt{|x|^2}$; (c) rotation invariance.

14e Laplacian at a singular point

The function $g(x) = 1/|x|^{N-2}$ is harmonic on $\mathbb{R}^N \setminus \{0\}$, thus, for every $f \in C^2$ compactly supported within $\mathbb{R}^N \setminus \{0\}$,

$$\int g\Delta f = \int f \Delta g = 0 \,.$$

It appears that for $f \in C^2(\mathbb{R}^N)$ with a compact support,

$$\int g\Delta f = \text{const} \cdot f(0) \,;$$

in this sense g has a kind of singular Laplacian at the origin.

14e1 Lemma.

$$\int_{\mathbb{R}^N} \frac{\Delta f(x)}{|x|^{N-2}} \, \mathrm{d}x = -(N-2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f(0)$$

for every $N > 2$ and $f \in C^2(\mathbb{R}^N)$ with a compact support.

This improper integral converges, since $1/|x|^{N-2}$ is improperly integrable near 0 (recall 10b7(c)). The coefficient $\frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the $(N-1)$-dimensional volume of the unit sphere (recall (13c9)).

Proof. For arbitrary $\varepsilon > 0$ we consider the function $g_\varepsilon(x) = 1/\bigl(\max(|x|, \varepsilon)\bigr)^{N-2}$, and $g(x) = 1/|x|^{N-2}$. Clearly, $\int |g_\varepsilon - g| \to 0$ (as $\varepsilon \to 0$), and $\int |g_\varepsilon - g||\Delta f| \to 0$, thus, $\int g_\varepsilon \Delta f \to \int g \Delta f$. We take $R \in (0, \infty)$ such that $f(x) = 0$ for $|x| \ge R$, introduce regular open sets $G_1 = \{x : |x| < \varepsilon\}$, $G_2 = \{x : \varepsilon < |x| < R\}$, and apply (14d4), taking into account that $\Delta g_\varepsilon = 0$ on G_1 and G_2:

$$\int g_\varepsilon \Delta f = \biggl(\int_{G_1} + \int_{G_2} \biggr) g_\varepsilon \Delta f = \biggl(\int_{\partial G_1} + \int_{\partial G_2} \biggr) \bigl(g_\varepsilon D_{\mathbf{n}} f - f D_{\mathbf{n}} g_\varepsilon \bigr) \,;$$

however, these $D_{\mathbf{n}}$ must be interpreted differently under $\int_{\partial G_1}$ and $\int_{\partial G_2}$:

$$\int_{\partial G_1} g_\varepsilon D_{\mathbf{n}_1} f = \int_{|x|=\varepsilon} \frac{1}{\varepsilon^{N-2}} D_{\mathbf{n}} f \,,$$
$$\int_{\partial G_2} g_\varepsilon D_{\mathbf{n}_2} f = \int_{|x|=\varepsilon} \frac{1}{\varepsilon^{N-2}} D_{-\mathbf{n}} f$$

where $\mathbf{n}$ is the outward normal of G_1 and inward normal of G_2; these two summands cancel each other. Further, $\int_{\partial G_1} f D_{\mathbf{n}_1} g_\varepsilon = \int_{|x|=\varepsilon} f \cdot 0 = 0$ since g_ε is constant on G_1; and

$$\int_{\partial G_2} f D_{\mathbf{n}_2} g_\varepsilon = \int_{|x|=\varepsilon} f \cdot \frac{N-2}{\varepsilon^{N-1}} \,,$$

since $g_\varepsilon(x) = 1/|x|^{N-2}$ on G_2, and $f(x) = 0$ when $|x| = R$. Finally,

$$\int g_\varepsilon \Delta f = -(N-2)\frac{1}{\varepsilon^{N-1}} \int_{|x|=\varepsilon} f = -(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} f_\varepsilon \,,$$

where f_ε is the mean value of f on the ε-sphere. By continuity, $f_\varepsilon \to f(0)$ as $\varepsilon \to 0$; and, as we know, $\int g_\varepsilon \Delta f \to \int g \Delta f$. □

14e2 Remark. For $N = 2$ the situation is similar:

$$\int_{\mathbb{R}^2} \Delta f(x) \log |x| \, \mathrm{d}x = 2\pi f(0)$$

for every compactly supported $f \in C^2(\mathbb{R}^2)$.

When the boundary consists of a hypersurface and an isolated point, we get a combination of (14d5) and 14e1: a singular point and two layers.

14e3 Remark. Let $G \subset \mathbb{R}^N$ be a bounded regular open set, ∂G an n-manifold, $f \in C^2(G)$ with bounded second derivatives, and $0 \in G$. Then

$$\int_G \frac{\Delta f(x)}{|x|^{N-2}} \, \mathrm{d}x = -(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) - \\ - \int_{\partial G} \Big(x \mapsto f(x) D_{\mathbf{n}} \frac{1}{|x|^{N-2}} \Big) + \int_{\partial G} \Big(x \mapsto (D_{\mathbf{n}} f(x)) \frac{1}{|x|^{N-2}} \Big) \,.$$

The proof is very close to that of 14e1. The case $N = 2$ is similar to 14e2, of course.

The case $G = \{x : |x| < R\}$ is especially interesting. Here $\partial G = \{x : |x| = R\}$; on ∂G,

$$\frac{1}{|x|^{N-2}} = \frac{1}{R^{N-2}} \quad \text{and} \quad D_{\mathbf{n}_x} \frac{1}{|x|^{N-2}} = -\frac{N-2}{R^{N-1}} \,;$$

thus,

$$\int_{|x|<R} \frac{\Delta f(x)}{|x|^{N-2}} \, \mathrm{d}x = -(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) + \frac{N-2}{R^{N-1}} \int_{|\cdot|=R} f + \frac{1}{R^{N-2}} \int_{|\cdot|=R} D_{\mathbf{n}} f \,.$$

Taking into account that $\int_{|\cdot|=R} D_{\mathbf{n}} f = \int_{|\cdot|<R} \Delta f$ by (14d2) we get

$$(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) = -\int_{|x|<R} \Big(\frac{1}{|x|^{N-2}} - \frac{1}{R^{N-2}} \Big) \Delta f(x) \, \mathrm{d}x + \frac{N-2}{R^{N-1}} \int_{|\cdot|=R} f$$

for $N > 2$; and similarly,

$$2\pi f(0) = -\int_{|x|<R} \big(\log R - \log|x|\big)\Delta f(x)\,\mathrm{d}x + \frac{1}{R}\int_{|\cdot|=R} f$$

for $N = 2$. In particular, for a harmonic f,

$$f(0) = \frac{\Gamma(N/2)}{2\pi^{N/2}}\frac{1}{R^{N-1}}\int_{|\cdot|=R} f = \frac{\int_{|\cdot|=R} f}{\int_{|\cdot|=R} 1}$$

for $N \ge 2$; the following result is thus proved (and holds also for $N = 1$, trivially).

14e4 Proposition (*Mean value property*)**.** For every harmonic function on a ball, with bounded second derivatives, its value at the center of the ball is equal to its mean value on the boundary of the ball.[1]

14e5 Remark. Now it is easy to understand why harmonic functions occur in physics ("the stationary heat equation"). Consider a homogeneous material solid body (in three dimensions). Fix the temperature on its boundary, and let the heat flow until a stationary state is reached. Then the temperature in the interior is a harmonic function (with the given boundary conditions).

14e6 Remark. Can the mean value property be generalized to a non-spherical boundary? We leave this question to more special courses (PDE, potential theory). But here is the idea. In 14e3 we may replace $\int_G \frac{\Delta f(x)}{|x|^{N-2}}\,\mathrm{d}x$ with $\int_G \Big(\frac{1}{|x|^{N-2}} + g(x)\Big)\Delta f(x)\,\mathrm{d}x$ where g is a harmonic function satisfying $\frac{1}{|x|^{N-2}} + g(x) = 0$ for all $x \in \partial G$ (if we are lucky to have such g). Then the double layer $\int_{\partial G}(D_{\mathbf{n}}v)u$ in (14d5), and the corresponding term in 14e3, disappears, and we get

$$(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)}f(0) = \int_{\partial G}\Big(x \mapsto f(x)D_{\mathbf{n}}\Big(\frac{1}{|x|^{N-2}} + g(x)\Big)\Big)\,.$$

14e7 Exercise (*Maximum principle for harmonic functions*)**.**
Let u be a harmonic function on a connected open set $G \subset \mathbb{R}^N$. If $\sup_{x\in G} u(x) = u(x_0)$ for some $x_0 \in G$ then u is constant.

Prove it.[2]

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[1] In fact, the mean value property is also sufficient for harmonicity, even if differentiability is not assumed.

[2] Hint: the set $\{x_0 : u(x_0) = \sup_{x\in G} u(x)\}$ is both open and closed in G.

It appears that

$$\Delta f(x) = 2N \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\Big(\big(\text{mean of } f \text{ on } \{y : |y-x| = \varepsilon\}\big) - f(x)\Big) . \tag{14e8}$$

14e9 Exercise. (a) Prove that, for $N > 2$,

$$\frac{1}{R^2}\int_{|x|<R}\Big(\frac{1}{|x|^{N-2}} - \frac{1}{R^{N-2}}\Big)\,\mathrm{d}x \quad \text{does not depend on } R;$$

and for $N = 2$, $\frac{1}{R^2}\int_{|x|<R}\big(\log R - \log|x|\big)\,\mathrm{d}x$ does not depend on R. (No need to calculate these integrals.)[1]

(b) For f of class C^2 near the origin, prove that the mean value of f on $\{x : |x| = \varepsilon\}$ is $f(0) + c_N\varepsilon^2\Delta f(0) + o(\varepsilon^2)$ as $\varepsilon \to 0$, for some $c_2, c_3, \dots \in \mathbb{R}$ (not dependent on f).

(c) Applying (b) to $f(x) = |x|^2$, find $c_2, c_3, \dots$ and prove (14e8).

14e10 Exercise. (a) For every f integrable (properly) on $\{x : |x| < R\}$,

$$\frac{\int_{|\cdot|<R} f}{\int_{|\cdot|<R} 1} = \int_0^R \frac{\int_{|\cdot|=r} f}{\int_{|\cdot|=r} 1}\,\frac{\mathrm{d}r^N}{R^N} .$$

(b) For every bounded harmonic function on a ball, its value at the center of the ball is equal to its mean value on the ball.

Prove it.[2]

14e11 Proposition. *(Liouville's theorem for harmonic functions)*

Every harmonic function $\mathbb{R}^N \to [0,\infty)$ is constant.

Proof. *(Nelson's short proof)*

For arbitrary $x, y \in \mathbb{R}^N$ and $R > 0$ we have

$$f(x) = \frac{\int_{|z-x|<R} f(z)\,\mathrm{d}z}{\int_{|z-x|<R}\mathrm{d}z} \le \frac{\int_{|z-y|<R+|x-y|} f(z)\,\mathrm{d}z}{\int_{|z-x|<R}\mathrm{d}z} =$$

$$= \Big(\frac{R+|x-y|}{R}\Big)^N \frac{\int_{|z-y|<R+|x-y|} f(z)\,\mathrm{d}z}{\int_{|z-y|<R+|x-y|}\mathrm{d}z} = \Big(\frac{R+|x-y|}{R}\Big)^N f(y) ,$$

since the R-neighborhood of x is contained in the $(R+|x-y|)$-neighborhood of y. In the limit $R\to\infty$ we get $f(x) \le f(y)$; similarly, $f(y) \le f(x)$. □

[1] Hint: change of variable.

[2] Hint: (a) recall 13c8.

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