1 Euclidean space $\mathbb{R}^n$

The n-dimensional space $\mathbb{R}^n$ may be treated as a Euclidean space, or just a vector space, etc. Its topology is uniquely determined by algebraic structure.

1a Prerequisites

LINEAR ALGEBRA

You should know the notion of: *Forgot? Then see:*

Vector space (=linear space) [Sh:p.26 "Vector space axioms"]
Isomorphism of vector spaces: a linear bijection.
Basis of a vector space [Sh:Def.2.1.2 on p.28]
Dimension of a finite-dimensional vector space: the number of vectors in every basis.
Two finite-dimensional vector spaces are isomorphic if and only if their dimensions are equal.
Subspace of a vector space.
Linear operator (=mapping=function) between vector spaces [Sh:3.1]
Inner product on a vector space: $\langle x, y \rangle$ [Sh:p.31 "Inner product properties"]
A basis of a subspace, being a linearly independent system, can be extended to a basis of the whole finite-dimensional vector space.

TOPOLOGY

You should know the notion of: *Forgot? Then see:*

A sequence of points of $\mathbb{R}^n$ [Sh:p.36][1]
Its convergence, limit [Sh:p.42–43]

[1] *Quote:* The only obstacle ... is notation ... n already denotes the dimension of the Euclidean space where we are working; and furthermore, the vectors can't be denoted with subscripts since a subscript denotes a component of an individual vector. ... As our work with vectors becomes more intrinsic, vector entries will demand less of our attention, and

Mapping $\mathbb{R}^n \to \mathbb{R}^m$; continuity (at a point; on a set)	[Sh:p.41–48]
Subsequence; Bolzano-Weierstrass theorem	[Sh:p.52–53]
Subset of $\mathbb{R}^n$, its limit points; closed set; bounded set	[Sh:p.51]
Compact set	[Sh:p.54]
Open set	[Sh:p.191][1]
Closure, boundary, interior	[Sh:p.311,314]
Open cover; Heine-Borel theorem	[Sh:p.312]
Open ball, closed ball, sphere	[Sh:p.50,191–192]
Open box, closed box	[Sh:p.246]

[Sh:Exer. 2.3.8–2.3.11, 2.4.1–2.4.8]

1a1 Exercise. For a function $f : (0,\infty) \times (0,\infty) \to \mathbb{R}$ defined by $f(x,y) = y \sin(1/x)$ prove that the limits

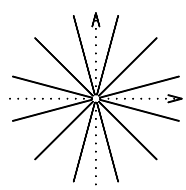
$$\lim_{(x,y)\to(0,0),x>0,y>0} f(x,y) \quad \text{and} \quad \lim_{x\to 0+} \lim_{y\to 0+} f(x,y)$$

exist and equal 0, but the second iterated limit

$$\lim_{y\to 0+} \lim_{x\to 0+} f(x,y)$$

does not exist.

1a2 Exercise.



Consider functions $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ constant on all rays from the origin; that is, $f(r\cos\varphi, r\sin\varphi) = h(\varphi)$ for some $h : \mathbb{R} \to \mathbb{R}$, $h(\varphi + 2\pi) = h(\varphi)$. Assume that h is continuous.

(a) Prove that the iterated limits

$$\lim_{x\to 0+} \lim_{y\to 0+} f(x,y) \quad \text{and} \quad \lim_{y\to 0+} \lim_{x\to 0+} f(x,y)$$

exist and are equal to $h(0)$ and $h(\pi/2)$ respectively.

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we will be able to denote vectors by subscripts.

More quote (p. 64–65): The author does not know any graceful way to avoid this notation collision, the systematic use of boldface or arrows to adorn vector names being heavyhanded, and the systematic use of the Greek letter ξ rather than its Roman counterpart x to denote scalars being alien. Since mathematics involves finitely many symbols and infinitely many ideas, the reader will in any case eventually need the skill of discerning meaning from context, a skill that may as well start receiving practice now.

[1] *Quote:* A set, however, is not a door: it can be neither open or closed, and it can be both open and closed. (Examples?)

(b) prove that the "full" limit

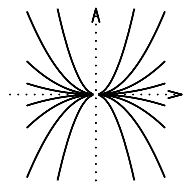
$$\lim_{(x,y)\to(0,0),x>0,y>0} f(x,y)$$

exists if and only if h is constant on $[0,\pi/2]$.

(c) It can happen that the two iterated limits exist and are equal, but the "full" limit does not exist. Give an example.

(d) The same as (c) and in addition, f is a rational function (that is, the ratio of two polynomials).[1]

(e) Generalize all that to arbitrary (not just positive) x, y.

1a3 Exercise.

Consider functions $g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ of the form $g(x,y) = f(x^2,y)$ where f is as in 1a2.

(a) Prove that the limit

$$\lim_{t\to0+} g(ta,tb)$$

exists for every $(a,b) \neq (0,0)$; calculate the limit in terms of the function h of 1a2.

(b) It can happen that the "full" limit

$$\lim_{(x,y)\to(0,0)} g(x,y)$$

does not exist. Give an example.

1a4 Exercise. *"Componentwise nature of continuity"* Prove or disprove: a mapping $f : \mathbb{R} \to \mathbb{R}^n$ is continuous if and only if each coordinate function $f_k : \mathbb{R} \to \mathbb{R}$ is continuous; here $f(x) = \big(f_1(x), \dots, f_n(x)\big)$. [Sh:Th.2.3.9]

1a5 Exercise. Prove or disprove: a mapping $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous if and only if it is continuous in each coordinate separately; that is, $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for every x, and $f(\cdot,y) : \mathbb{R} \to \mathbb{R}$ is continuous for every y.

1a6 Exercise. Prove the Bolzano-Weierstrass theorem and the Heine-Borel theorem.

1a7 Exercise. (a) Prove that finite union of closed sets is closed, but union of countably many closed sets need not be closed; moreover, every open set in $\mathbb{R}^n$ is such union. However, intersection of closed sets is always closed.

(b) Formulate and prove the dual statement (take the complement).

[1]Hint: try $x^2 + y^2$ in the denominator.

1a8 Exercise. Prove that a set $K \subset \mathbb{R}^n$ is compact if and only if *every* continuous function $f : K \to \mathbb{R}$ is bounded.

1a9 Exercise. Prove that a continuous image of a compact set is compact, but a continuous image of a bounded set need not be bounded, and a continuous image of a closed set need not be closed; moreover, every open set in $\mathbb{R}^n$ is a continuous image of a closed set.[1]

1a10 Exercise. Prove that every decreasing sequence of nonempty compact sets has a nonempty intersection. Does it hold for closed sets? for open sets?

1a11 Exercise. Let $K \subset \mathbb{R}^n$ be compact, and $f : K \to \mathbb{R}^m$ continuous. Prove that f is uniformly continuous, that is,
$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x, y \in K \, \big(\, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \, \big)$.

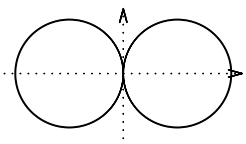
1a12 Exercise. Let $X \subset \mathbb{R}^n$ be a closed set, $f : X \to \mathbb{R}^m$ a continuous mapping. Prove that its *graph* $\Gamma_f = \{(x, f(x)) : x \in X\}$ is a closed subset of $\mathbb{R}^{n+m}$. Is the converse true?

1a13 Exercise. Prove existence of a bijection f from the open unit ball $\{x : |x| < 1\} \subset \mathbb{R}^n$ onto the whole $\mathbb{R}^n$ such that f and f^{-1} are continuous. (Such mappings are called *homeomorphisms*). What about the closed ball?

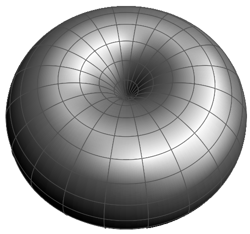
1a14 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bijection. Prove that $f^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous.

1a15 Exercise. Give an example of a continuous bijection $f : [0, 1) \to S^1 = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ such that $f^{-1} : S^1 \to [0, 1)$ fails to be continuous. The same for $f : [0, \infty) \to S^1$.

1a16 Exercise. Give an example of a continuous bijection $f : \mathbb{R} \to A = \{(x, y) : (|x| - 1)^2 + y^2 = 1\} \subset \mathbb{R}^2$ such that $f^{-1} : A \to \mathbb{R}$ fails to be continuous.



1a17 Exercise. Give an example of a continuous bijection $f : \mathbb{R}^2 \to B = \{(x, y, z) : (\sqrt{x^2 + y^2} - 1)^2 + z^2 = 1\} \subset \mathbb{R}^3$ such that $f^{-1} : B \to \mathbb{R}^2$ fails to be continuous.[2]



[1]Hint: the closed set need not be connected.

[2]What about a continuous bijection $f : \mathbb{R}^n \to \mathbb{R}^n$? In fact, f^{-1} is continuous, which can be proved using powerful means of topology (the Brouwer invariance of domain theorem); we'll return to this point later.

1b Structures on $\mathbb{R}^n$ and their isomorphisms

To a mathematician, the word space *doesn't connote volume but instead refers to a set endowed with some structure.* [Sh:p.24][1] [2]

Almost everything in contemporary mathematics is an example of a structured set. . .[3]

WHAT IS THE PROBLEM

The phrase "without loss of generality" (WLOG) makes proofs simpler. Here is a spectacular example.

A well-known result from Euclidean geometry: the three medians of a triangle intersect [Sh:p.27]. A short proof (sketch): WLOG, the triangle is equilateral. Now the medians evidently intersect in its center.

A faulty analogy: also, the three altitudes of a triangle intersect [Sh:p.37–38]. In this case the "WLOG" does not work. Why? In order to understand we need the notion of isomorphism between structures.

What is meant by "Euclidean space"? Here are three typical answers:

∗ A space in which Euclid's axioms apply.

∗ A 3-dimensional vector space endowed with a Euclidean metric.

∗ $\mathbb{R}^3$.

Is it all the same, or not? This is a matter of structures and isomorphism.

A SPACE IN WHICH EUCLID'S AXIOMS APPLY

For more than two thousand years, the adjective "Euclidean" was unnecessary because no other sort of geometry had been conceived.[4]

According to Euclid, geometry deals with straight lines, circles, and planes. The right angle is the unit for measuring angles. Distances are measured in relation to a line segment chosen arbitrarily as the unit of length. A rigorous version of Euclidean geometry was proposed in 1899 by Hilbert. It stipulates

[1] If you wonder why, see "Space (mathematics)" in Wikipedia.

[2] Additional sources:
"Affine and Euclidean geometry", chapter II of a course in Madrid Politech. Univ.;
"Basics of Euclidean geometry", chapter 6 of the book: J. Gallier, "Geometric Methods and Applications" (pdf or djvu);
"Vector spaces, affine spaces, and metric spaces", chapter 2 of the book: Bærentzen, J. Gravesen, F. Anton, H. Aanæs, "Guide to Computational Geometry Processing";
chapters 1, 2 of the book: M. Audin, "Geometry".

[3] nLab:Structured set#examples.

[4] Wikipedia:Euclidean geometry.

* 3 kinds of objects: points, (straight) lines, planes;
* 6 relations:
  – betweenness (for three points);
  – containment (3 relations: for a point and a line; a point and a plane; a line and a plane);
  – congruence (2 relations: for two line segments; for two angles); of course, segments and angles are defined in terms of points and lines;
* and 20 axioms: "incidence" (8 axioms), "order" (4), "congruence" (6), "parallels" (1), "continuity" (2).

In the ancient time of Euclid a line could not be treated as a set of points,[1] but now it can. Thus we may reformulate Hilbert's version in terms of a single kind of objects (called "points"). Denoting the set of all points by X, we consider the set $\mathcal{P}(X)$ of all subsets of X, and two sets of sets $L, P \subset \mathcal{P}(X)$ (that is, $L, P \in \mathcal{P}(\mathcal{P}(X))$); elements of L are called "lines", of P — "planes". The betweenness relation B is a subset of the (Cartesian) product $X \times X \times X$ (that is, $B \in \mathcal{P}(X \times X \times X)$); in other words, a set of triples of points. The containment relations need not be stipulated as special sets of pairs, since these relations are provided by the underlying set theory: $x \in l$, $x \in p$, and $l \subset p$ for $x \in X$, $l \in L$, $p \in P$. The congruence relations C_1, C_2 must be stipulated, but I omit the details.

Nowadays this approach (synthetic geometry) is rather out of fashion beyond the school. However, we all have some idea of it from the secondary school geometry. What should we do with it? Can we use notions and results of school geometry (say, the volume of a cylinder) in $\mathbb{R}^3$?[2] How does this notion of space relate to others? We'll return to this question soon.

Let us call such structured sets (X, L, P, B, C_1, C_2) "classical Euclidean spaces".

CARTESIAN SPACE

The invention of Cartesian coordinates in the 17-th century by René Descartes (Latinized name: Cartesius) revolutionized mathematics by providing the first systematic link between Euclidean geometry and algebra.[3]

[1] Before the second half of the 19th century infinity was only potential, never actual; uncountable set was unthinkable. See Wikipedia:Actual infinity.

[2] See also MathStackExchange:The status of high school geometry.

[3] Wikipedia:Cartesian coordinate system.

> *A Euclidean plane with a chosen Cartesian system is called a Cartesian plane.*[1]

That is, a Cartesian space is a structured set (X, f_1, f_2, f_3) where X is a set (whose elements are called "points"), and $f_1, f_2, f_3 : X \to \mathbb{R}$ are functions (called "coordinates") satisfying a single axiom:

$$\text{(1b1)} \qquad \text{the mapping} \quad X \ni x \mapsto \big(f_1(x), f_2(x), f_3(x)\big) \in \mathbb{R}^3 \quad \text{is bijective.}$$

Geometric notions are defined via coordinates (analytic geometry). For instance, a plane is $\{x \in X : a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) = b\}$ for given $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\}$, $b \in \mathbb{R}$. More formally, we may consider a combined structure (f_1, f_2, f_3, P) on X, where $f_1, f_2, f_3 : X \to \mathbb{R}$ and $P \in \mathcal{P}(\mathcal{P}(X))$, with two axioms: (1b1) and

$$\text{(1b2)} \quad \forall p \in \mathcal{P}(X) \; \Big(p \in P \quad \Longleftrightarrow \quad \exists (a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\} \; \exists b \in \mathbb{R} \\ p = \{x \in X : a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) = b\} \Big) \,.$$

Continuing this way (I omit the details) we get a combined structure $(f_1, f_2, f_3, L, P, B, C_1, C_2)$ on X whose axioms are (1b1), (1b2), and several more complicated excerpts from analytic geometry. Significantly, these axioms ensure that (L, P, B, C_1, C_2) is a classical Euclidean structure on X; that is, the 20 axioms can be deduced from the analytic geometry! (I omit the proof, of course; otherwise this month would be devoted to geometry rather than analysis.) This is our first example of the so-called "deduction procedure" for mathematical structures. Quite a few more examples will appear soon. Thus, we need some general ideas, notation and terminology.

Given a set X, we denote by $T_1(X)$ the set of all Cartesian structures on X, and by $T_2(X)$ the set of all classical Euclidean structures on X. That is, $T_1(X)$ is the set of all (f_1, f_2, f_3) satisfying (1b1); just the set of all bijections $X \to \mathbb{R}^3$. (Of course, it is empty unless X is of cardinality continuum.) Similarly, $T_2(X)$ is the set of all (L, P, B, C_1, C_2) satisfying the 20 axioms of classical Euclidean space. Further, we define $T(X) \subset T_1(X) \times T_2(X)$ as the set of all pairs (σ_1, σ_2) where $\sigma_1 = (f_1, f_2, f_3) \in T_1(X)$, $\sigma_2 = (L, P, B, C_1, C_2) \in T_2(X)$ satisfying the list of axioms starting with (1b2). Clearly, $T(X)$ is a binary relation between structures σ_1 and σ_2. Significantly, this relation is a mapping $T_1(X) \to T_2(X)$. That is,

$$\forall \sigma_1 \in T_1(X) \; \exists!\, \sigma_2 \in T_2(X) \;\; (\sigma_1, \sigma_2) \in T(X) \,.$$

[1] Wikipedia:Cartesian coordinate system#Higher dimensions.

In other words,[1]

$$\forall X \;\; T(X) : T_1(X) \to T_2(X) \,.$$

Moreover, the mapping is onto (=surjective). That is,

$$\forall \sigma_2 \in T_2(X) \; \exists \sigma_1 \in T_1(X) \;\; (\sigma_1, \sigma_2) \in T(X) \,,$$

since every classical Euclidean space admits (at least one) Cartesian coordinate system. On the other hand, this mapping is not one-to-one (=not injective);[2] different Cartesian coordinates exist on a classical Euclidean space.

On a classical Euclidean space, coordinate system is an additional structure, but its existence is not an additional property.

A classical Euclidean space can be *upgraded* to a Cartesian space by choosing Cartesian coordinates.

A Cartesian space can be *downgraded* to a classical Euclidean space by forgetting the coordinates while retaining the corresponding Euclidean geometry.

An n-dimensional Cartesian space is defined similarly. (Of course, Euclid did not treat dimensions other than 2 and 3.)

1b3 Remark. We'll use various kinds of structured sets (X, σ), where σ is a structure on a set X. We'll use various changes of structure, from (X, σ_1) to (X, σ_2), mostly upgrades and downgrades (but not only). I emphasize that (X, σ_1) and (X, σ_2) contain the same X. Elements of the set X never lose their identity.[3] If a point $x_1 \in X$ was considered in context of (X, σ_1), we may still consider this point in context of (X, σ_2). It may happen that x_1 was, say, a vector of length 1 before the change, and becomes a vector of length 7, or 0, or maybe of no length at all (or even not a vector at all) after the change. But in every case, it is still $x_1 \in X$. Likewise, if a function $f : X \to \mathbb{R}$ was considered before the change, it may still be considered after. Maybe it was continuous before and becomes discontinuous after. But if it was bounded, it remains bounded. Why? Since boundedness of f (unlike continuity) is well-defined on a set X with no structure. (This is "null structure", like the number 0, the empty set etc.)

[1] It holds for all X, but is vacuous unless X is of cardinality continuum.

[2] For X of cardinality continuum, of course; otherwise $T_1(X), T_2(X), T(X)$ are empty.

[3] There are attempts to build new foundations of mathematics, based on category theory rather than set theory. There, elements have no identity unless a structure gives them identity. That may be exciting; but for now, following the mainstream, we work within the set theory.

Digression: terminological conventions ("abuse of language")

> *As far as possible we have drawn attention in the text to abuse of language, without which any mathematical text runs the risk of pedantry not to say unreadability.* (Bourbaki)
> *The student of mathematics has to develop a tolerance for ambiguity. Pedantry can be the enemy of insight.* (Gila Hanna)[1]

"A set in a classical Euclidean space is called a cylinder, if..." — but wait; what happens? A classical Euclidean space is a tuple (X, L, P, B, C_1, C_2). A set in the tuple? What's it? No, do not take it literally; a subset of X is meant. Dealing with a structured set $(X, \dots)$ it is convenient and habitual to think of it as, first of all, a set. Yes, endowed with something, and still a set.

More formally, X is the so-called *principal base set* of the structured set $(X, \dots)$. Nevertheless we say routinely "the space X", denoting by the same letter (X) both the tuple and its first element.

"A vector space over $\mathbb{R}$ is either uncountable, or a single point." It is meant, of course, that its principal base set is. By the way, $\mathbb{R}$ is here the so-called *auxiliary base set.* But only the principal base set is meant by default.

In some cases, two (and more) principal base sets are used. "Let A be a set in a graph." Oops, this is problematic. Still, this phrase occurs in texts based on the "edges without own identity" approach, and means that A is a subset of the set of all vertices of the graph. In texts based on the other approach, "edges with own identity", one uses instead such phrases as "subset of a graph's vertices", "subset of a graph's edges", "vertex subset of a graph", "edge set in a graph".

Here is a completely different abuse of language. Given three sets X, Y, Z, (Cartesian) products $X \times (Y \times Z)$ and $(X \times Y) \times Z$ are routinely treated as equal. In fact, they are not, since the pairs $(x, (y, z))$ and $((x, y), z)$ differ; and both differ from the triple (x, y, z). A trouble![2] We routinely ignore such troubles. It means, canonical bijections, such as $X \times (Y \times Z) \longleftrightarrow X \times Y \times Z \longleftrightarrow (X \times Y) \times Z$ are inserted as needed by default without being mentioned.

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[1] Both quotes borrowed from: MathStackExchange:Why is 'abuse of notation' tolerated?

[2] You could say: let us treat a pair as a sequence of length 2, and concatenate sequences... This is problematic, since a sequence of length 2 is, by definition, a function on $\{1, 2\}$, and a function is, by definition, a set of pairs!

VECTOR SPACE WITH EUCLIDEAN METRIC

By a Euclidean metric on a vector space we mean either an inner product $x, y \mapsto \langle x, y\rangle$ on this space, or the corresponding norm $x \mapsto |x| = \sqrt{\langle x, x\rangle}$ (called "Euclidean norm"), or the corresponding metric $x, y \mapsto |x-y|$. These correspond to each other bijectively. The inner product may be reconstructed from the norm:

$$\langle x, y\rangle = \frac{1}{4}\big(|x+y|^2 - |x-y|^2\big)\,; \tag{1b4}$$

other transitions are evident.

An n-dimensional *Euclidean vector space* is, by definition, an n-dimensional vector space (over $\mathbb{R}$) endowed with a Euclidean metric. More formally, this is a structured set[1] (X, "$+$", "$\cdot$", "$|\cdot|$") where X is a set (whose elements are called "vectors" or "points"), "$+$" $: X \times X \to X$ a binary operation (called "addition of vectors"), "$\cdot$" $: \mathbb{R} \times X \to X$ another operation (called "multiplication by scalars"), and "$|\cdot|$" $: X \to [0, \infty)$ a function (called "norm"). A number of axioms must be satisfied:[2]

∗ the axioms of vector space;

∗ the vector space is n-dimensional;

∗ existence[3] of an inner product corresponding to the norm;

∗ axioms of inner product.

Given an n-dimensional Cartesian space, we define on it linear operations and inner product:

$$\begin{gathered} f_k(x+y) = f_k(x) + f_k(y)\,, \quad f_k(\lambda x) = \lambda f_k(x)\,, \\ \langle x, y\rangle = \sum_{k=1}^{n} f_k(x) f_k(y) \end{gathered} \tag{1b5}$$

where $f_1, \dots, f_n$ are the coordinate functions. This is a procedure of deduction of a Euclidean vector space structure from a Cartesian structure. More formally, we denote (again) by $T_1(X)$ the set of all Cartesian structures on X, and by $T_2(X)$ the set of all n-dimensional Euclidean vector space structures on X. Further, we define $T(X) \subset T_1(X) \times T_2(X)$ as the set of all pairs (σ_1, σ_2) where $\sigma_1 \in T_1(X)$, $\sigma_2 \in T_2(X)$ satisfy (1b5). Again,

$$\forall \sigma_1 \in T_1(X)\ \exists!\, \sigma_2 \in T_2(X)\ \ (\sigma_1, \sigma_2) \in T(X)\,.$$

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[1] One out of several equivalent formulations. One may add a constant $0 \in X$ ("the origin"), or/and replace the norm with the inner product, etc.

[2] Again, one out of several equivalent formulations. The "parallelogram equality" $|a-b|^2 + |a+b|^2 = 2|a|^2 + 2|b|^2$ may be used instead of the inner product.

[3] Add "and uniqueness" if you like. Or do not; it holds anyway by (1b4).

and the mapping is onto. Indeed, every n-dimensional Euclidean vector space has orthonormal bases; every such basis $(e_1, \dots, e_n)$ leads to coordinate functions $f_k : x \mapsto \langle x, e_k \rangle$, $x = \sum_k f_k(x) e_k$. Cartesian structures on a given n-dimensional Euclidean vector space correspond bijectively to orthonormal bases.

> On an n-dimensional Euclidean vector space, orthonormal basis is an additional structure, but its existence is not an additional property.

> An n-dimensional Euclidean vector space can be upgraded to a Cartesian space by choosing an orthonormal basis.

> A Cartesian space can be downgraded to an n-dimensional Euclidean vector space by forgetting the basis.

Now we turn to a seemingly unrelated matter: two well-known inequalities

$$|x_1 y_1 + \dots + x_n y_n| \le \sqrt{x_1^2 + \dots + x_n^2} \sqrt{y_1^2 + \dots + y_n^2} \, , \tag{1b6}$$

$$\sqrt{(x_1 + y_1)^2 + \dots + (x_n + y_n)^2} \le \sqrt{x_1^2 + \dots + x_n^2} + \sqrt{y_1^2 + \dots + y_n^2} \tag{1b7}$$

for all $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$. They can be treated algebraically, but geometry can help a lot. We rewrite them as

$$-|x||y| \le \langle x, y \rangle \le |x||y| \, , \tag{1b8}$$

$$|x + y| \le |x| + |y| \tag{1b9}$$

for $x, y \in \mathbb{R}^n$. Two vectors x, y span an (at most) two-dimensional subspace of $\mathbb{R}^2$. Thus, we have two vectors in a two-dimensional Euclidean *vector* space. On the other hand, on a *classical* Euclidean plane both inequalities are evident; (1b9) is the triangle inequality, and (1b8) holds, since $\langle x, y \rangle = |x||y| \cos \theta$ where θ is the angle between x and y. In both cases the inequality is strict unless the vectors are collinear.

Can we turn a 2-dimensional Euclidean vector space into a classical Euclidean plane? Yes, easily. We first upgrade the given space to a Cartesian space, and then downgrade the latter to a classical Euclidean space.

> Euclidean planimetry or stereometry applies in every 2-dimensional or 3-dimensional subspace of an n-dimensional Euclidean vector space.

1b10 Exercise. Instead of using Euclidean planimetry, give a simple algebraic proof after choosing a 2-dimensional coordinate system adjusted to the given vectors.[1]

[1]Hint. To (1b8): $x_2 = 0$; that is, $x = (x_1, 0)$ and $y = (y_1, y_2)$. To (1b9): $x_2 + y_2 = 0$.

When we turn a 2-dimensional (or 3-dimensional) Euclidean vector space into a classical Euclidean space, is it upgrade, downgrade, or neither? We did it via a basis (a Cartesian space); and for now we do not know, whether the result depends on the basis, or not. But we can get rid of the basis. We can define a line as $\{\lambda a + b : \lambda \in \mathbb{R}\}$ for $a \in E \setminus \{0\}$, $b \in E$; a plane as $\{\lambda_1 a_1 + \lambda_2 a_2 + b : \lambda_1, \lambda_2 \in \mathbb{R}\}$ for linearly independent $a_1, a_2 \in E$ and arbitrary $b \in E$; $B = \{(a, (1-\lambda)a + \lambda b, b) : 0 < \lambda < 1, a, b \in E, a \neq b\}$; and C_1, C_2 are easily constructed from the metric.[1] Thus, once again, our relation $T(X) \subset T_1(X) \times T_2(X)$ is a mapping,

$$T(X) : T_1(X) \to T_2(X)\,;$$

this time $T_1(X)$ consists of Euclidean vector spaces (structures), and $T_2(X)$ of classical Euclidean spaces. Again, the mapping is onto (think, why). But is it a bijection (equivalence) or not (downgrade)? In other words, can we restore the Euclidean vector space from the classical Euclidean space?

In geometry, vectors are usually defined either as equivalence classes of pairs of points (or "directed line segments"), or as shifts (of the whole space). They are a vector space. In order to make it a Euclidean vector space, a unit of length must be chosen. And still, vectors are not points![2] The set E of points is a classical Euclidean space; the set $\vec{E}$ of vectors is a Euclidean vector space. A special element 0 of $\vec{E}$ is singled out by its unique property $0 + 0 = 0$. In contrast, elements of E (points) are mutually congruent and cannot be distinguished by their properties.

Choosing a point $O \in E$ ("the origin") we get a bijection between E and $\vec{E}$:

$$A \longleftrightarrow \overrightarrow{OA}\,.$$

A classical Euclidean space can be upgraded to a Euclidean vector space by choosing the origin and the unit of length.

AFFINE SPACE

An affine space is nothing more than a vector space whose origin we try to forget about.[3 4]

[1] $[a, b]$ and $[c, d]$ are congruent when $|a - b| = |c - d|$; for angles, use $\langle a, b\rangle = |a||b|\cos\theta$.

[2] "Points can't be added; vectors can" (Hubbard, p. 37); see also MathStackExchange:What is the difference between a point and a vector.

[3] Marcel Berger, "Geometry I", p. 32.

[4] Quite a few equivalent definitions of an affine space are used. See Wikipedia:Affine space, nLab:Affine space.

We like the algebraic approach of Descartes (more than the synthetic approach of Euclid), but we do not like the fixed origin. Indeed, if $\mathbb{R}^n$ is the space of vectors, then, where are the corresponding points? How to get rid of this undesirable byproduct of vector algebra? Can we modify the vector space structure as to make it invariant under translations?

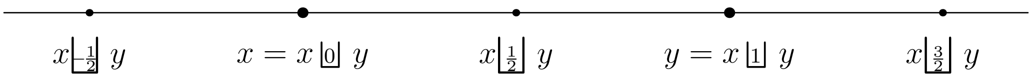
The vector space structure is not invariant under translations, since the relation $x + y = z$ is not equivalent to $(x + a) + (y + a) = z + a$ (unless $a = 0$), and $\lambda x = y$ is not equivalent to $\lambda(x + a) = y + a$. More generally, $\lambda_1 x_1 + \dots + \lambda_k x_k = y$ is not equivalent to $\lambda_1(x_1 + a) + \dots + \lambda_k(x_k + a) = y + a$, unless $a = 0$ or $\lambda_1 + \dots + \lambda_k = 1$. Aha! The latter is the clue!

An *affine combination* is, by definition, a linear combination $\lambda_1 x_1 + \dots + \lambda_k x_k$ such that $\lambda_1 + \dots + \lambda_k = 1$.

The idea is, to restrict ourselves to affine combinations. Thus, $x + y$ and λx are bad; but

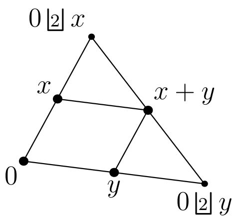
$$x \lfloor\lambda\rfloor y = x + \lambda(y - x)$$

is good. Let us denote it as shown above,[1] and look closely.



We note that

$$
\begin{aligned}
0 \lfloor\lambda\rfloor x &= \lambda x\,, \\
x \lfloor\tfrac{1}{2}\rfloor y &= \tfrac{1}{2}(x + y)\,, \qquad (1b11)\\
\Big(0 \lfloor 2 \rfloor x\Big) \lfloor\tfrac{1}{2}\rfloor \Big(0 \lfloor 2 \rfloor y\Big) &= x + y\,.
\end{aligned}
$$



1b12 Definition. An *affine space* is a structured set $(X, \lfloor\,\rfloor)$ where X is a set and "$\lfloor\,\rfloor$" is a mapping $X \times \mathbb{R} \times X \to X$, denoted $(x, \lambda, y) \mapsto x \lfloor\lambda\rfloor y$, satisfying the following axiom:

there exists a vector space structure ("+" , "·") on X such that

$$\forall x, y \in X\ \forall \lambda \in \mathbb{R}\ \ x \lfloor\lambda\rfloor y = x + \lambda(y - x)\,.$$

By (1b11), the vector space structure ("+" , "·") is uniquely determined by its origin 0. Thus, the axiom in 1b12 is equivalent to this one:

the operations $x, y \mapsto \big(0 \lfloor 2 \rfloor x\big) \lfloor\frac{1}{2}\rfloor \big(0 \lfloor 2 \rfloor y\big)$ and $\lambda, x \mapsto 0 \lfloor\lambda\rfloor x$ are a vector space structure on X.

[1] Not a standard notation.

1b13 Exercise. For arbitrary $\mathcal{O} \in X$, operations $x, y \mapsto \big(\mathcal{O}\,\underline{\lvert 2\rvert}\,x\big)\underline{\left\lvert \tfrac{1}{2}\right\rvert}\big(\mathcal{O}\,\underline{\lvert 2\rvert}\,y\big)$ and $\lambda, x \mapsto \mathcal{O}\,\underline{\lvert \lambda\rvert}\,x$ are a vector space structure on X.

Prove it.[1]

> An affine space can be upgraded to a vector space by choosing the origin.

1b14 Exercise. In an affine space, an affine combination does not depend on the choice of the origin.

Prove it.[2]

> Affine combinations are well-defined in every affine space.

Accordingly, a relation $\lambda_1 x_1 + \dots + \lambda_k x_k = 0$ between $x_1, \dots, x_k$ makes sense in an affine space whenever $\lambda_1 + \dots + \lambda_k = 0$.[3]

However, what about vectors as equivalence classes of pairs of points (or "directed line segments") over an affine space S "as is", not upgraded, with no origin chosen? Now we are in position to do it. For $x, y \in S$, the pair (x, y) may be thought of as a bound vector $\overrightarrow{xy}$, and we define equivalence relation: $\overrightarrow{x_1y_1} \sim \overrightarrow{x_2y_2}$ when $y_1 - x_1 = y_2 - x_2$, that is, $y_1 - x_1 - y_2 + x_2 = 0$; this relation is well-defined, since the sum of coefficients is 0. Equivalence classes of bound vectors are called free vectors. Similarly, we define the relation $\overrightarrow{x_1y_1} + \overrightarrow{x_2y_2} \sim \overrightarrow{x_3y_3}$ as $(y_1 - x_1) + (y_2 - x_2) = y_3 - x_3$, and $\lambda \cdot \overrightarrow{x_1y_1} \sim \overrightarrow{x_2y_2}$ as $\lambda(y_1 - x_1) = y_2 - x_2$. But for now these are relations, not operations, and we do not know, whether free vectors are a vector space, or not.

In order to prove that they are, we do choose the origin $0 \in S$, upgrade S to a vector space S_0, and treat $\overrightarrow{xy}$ as the vector $y - x \in S_0$. We have a bijection between free vectors and elements of S_0 (think, why) and note that the relation $\overrightarrow{x_1y_1} + \overrightarrow{x_2y_2} \sim \overrightarrow{x_3y_3}$ is equivalent to the relation $(y_1 - x_1) + (y_2 - x_2) = y_3 - x_3$ between the corresponding elements of S_0; the same holds for the other relation, $\lambda \cdot \overrightarrow{x_1y_1} \sim \overrightarrow{x_2y_2}$. Knowing that axioms of vector space hold in S_0 we conclude that they hold in the set of free vectors as well.

The vector space of free vectors over an affine space S is denoted $\vec{S}$ and called the *difference space*. We have

> $x \in S\,,\ a \in \vec{S} \quad \Longrightarrow \quad x + a \in S;$
> $x, y \in S \quad \Longrightarrow \quad x - y \in \vec{S};$

[1] Hint: $\big(\mathcal{O}\,\underline{\lvert 2\rvert}\,x\big)\underline{\left\lvert \tfrac{1}{2}\right\rvert}\big(\mathcal{O}\,\underline{\lvert 2\rvert}\,y\big) = z \iff (x - \mathcal{O}) + (y - \mathcal{O}) = z - \mathcal{O}$, and $\mathcal{O}\,\underline{\lvert \lambda\rvert}\,x = y \iff \lambda(x - \mathcal{O}) = y - \mathcal{O}$.

[2] Hint: use the hint to 1b13; calculating a linear combination relative to $\mathcal{O}$ we get $\mathcal{O} + \lambda_1(x_1 - \mathcal{O}) + \dots + \lambda_k(x_k - \mathcal{O})$.

[3] Rewrite it as $x_1 = (\lambda_1 + 1)x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k$.

and usual algebraic rules apply; in particular, $(x + a) + b = x + (a + b)$ for $x \in S$ and $a, b \in \vec{S}$.

A mapping $S \to S$ of the form $x \mapsto x + a$ (where $a \in \vec{S}$) is called a *translation.* The difference space may be thought of as the space of translations.

1b15 Example. Given a continuous function $g : \mathbb{R} \to \mathbb{R}$, consider the set $S_1 = \{f : f' = g\}$ of all its antiderivatives (that is, indefinite integrals). This S_1 is an affine space with the difference space $P_0 = \{f : f' = 0\}$ of all constant functions.

More generally, $S_n = \{f : f^{(n)} = g\}$ is an affine space with the difference space $P_{n-1} = \{f : f^{(n)} = 0\}$ of all polynomials of degree (at most) $n - 1$.

1b16 Exercise. Fill in the details in 1b15. What about a more general linear differential equation?

An affine space is called n-dimensional if its difference space is n-dimensional.

1b17 Exercise. Recall S_n of 1b15. Is it finite-dimensional? What is its dimension? Check that it is a hyperplane[1] in the vector space $\{f : \exists c \in \mathbb{R} \;\; f^{(n)} = cg\}$ (unless $g = 0$).

ISOMORPHISM

In elementary geometry, two triangles (or other figures) may be congruent (or not), but they are situated in the same Euclidean space, "the space". In contrast, we deal with "a space". What about congruence between two spaces? This is a matter of isomorphism.

Isomorphism of vector spaces is, by definition, a linear bijection. That is, a bijection $\varphi : V_1 \to V_2$ that preserves linear operations:[2]

$$\begin{aligned} x + y = z \quad &\Longleftrightarrow \quad \varphi(x) + \varphi(y) = \varphi(z)\,; \\ \lambda x = y \quad &\Longleftrightarrow \quad \lambda\varphi(x) = \varphi(y)\,. \end{aligned} \tag{1b18}$$

Isomorphism of Euclidean vector spaces is, by definition, an isometric linear bijection: (1b18) and in addition,

$$|x| = |\varphi(x)|\,. \tag{1b19}$$

[1] That is, a set of the form $\{x : \ell(x) = c\}$ where ℓ is a (real-valued) linear function (not identically zero) on the vector space, and $c \in \mathbb{R}$.

[2] Algebraists prefer an equivalent formulation: $\varphi(x+y) = \varphi(x)+\varphi(y)$, $\varphi(\lambda x) = \lambda\varphi(x)$, since in algebra, a bijective homomorphism is always an isomorphism. But this may fail for non-algebraic structures.

Isomorphism of n-dimensional Cartesian spaces (X_1, f_1) and (X_2, f_2) (where $f_1 : X_1 \to \mathbb{R}^n$ and $f_2 : X_2 \to \mathbb{R}^n$) is, by definition, a bijection $\varphi : X_1 \to X_2$ that preserves the coordinates:

$$\varphi(x_1) = x_2 \quad \Longrightarrow \quad f_1(x_1) = f_2(x_2) \,. \tag{1b20}$$

Generally, isomorphism between two structured sets is a bijection that preserves the given structure.

Thus, isomorphism between two classical Euclidean spaces $(X_1, L_1, P_1, B_1, C_{1,1}, C_{1,2})$ and $(X_2, L_2, P_2, B_2, C_{2,1}, C_{2,2})$ is a bijection $\varphi : X_1 \to X_2$ such that

* φ preserves lines; that is, a subset of X_1 belongs to L_1 if and only if the corresponding subset of X_2 (the image) belongs to L_2;
* φ preserves planes (similarly);
* φ preserves betweenness; that is, $(x, y, z) \in B_1 \quad \Longleftrightarrow \quad \big(\varphi(x), \varphi(y), \varphi(z)\big) \in B_2$;
* φ preserves congruence (I omit the details).

A well-known theorem of elementary geometry (abbreviated CPCTC) states that corresponding parts of congruent triangles are congruent. This is a tiny special case of a very general “isomorphism argument”:

Isomorphism preserves the given structure, therefore it preserves everything that is derived from this structure.

For example, isomorphism between vector spaces preserves dimension.

Why “isomorphism argument” rather than “isomorphism theorem”? Well, it is a theorem of the general theory of mathematical structures. If you are interested, see Appendix A. However, it is easier to prove each special case separately, when needed. If you did it few times, you can do it always.

1b21 Exercise. An isomorphism between vector spaces preserves: linear independence; basis; dimension; subspace; inner product.

Formulate it accurately,[1] and prove.

Likewise, an isomorphism between classical Euclidean spaces preserves: triangle; right-angled triangle; acute-angled triangle; obtuse-angled triangle; isosceles triangle; equilateral triangle; circle; ellipse; sphere; cube; cylinder; cone; etc.

[1]It does not mean that the isomorphism sends a *given* basis of one space to a *given* basis of the other space, or a *given* inner product on one space to a *given* inner product on the other space. (Indeed, isomorphism of vector spaces need not be isomorphism of Cartesian spaces, nor Euclidean vector spaces.) Rather, it means that every basis of one space is sent to *some* basis of the other space, and every inner product on one space is sent to *some* inner product on the other space.

You know that all n-dimensional vector spaces are mutually isomorphic. How do you prove it? Via bases? Here is another way. (Or is it another form of the same way?)

First, all n-dimensional Cartesian spaces are mutually isomorphic (for a trivial reason; do you see it?).

Second, every n-dimensional vector space can be upgraded to an n-dimensional Cartesian space (think, why).

Third, every isomorphism between Cartesian spaces is also an isomorphism between the corresponding vector spaces. Why? Since the vector space structure can be deduced from the Cartesian space structure (recall (1b5), the first line).

This is a useful general argument: having two isomorphic structures and downgrading both we get again two isomorphic structures.

This way we get, with no additional effort, more facts.

∗ All classical Euclidean spaces are mutually isomorphic. (I mean 3-dimensional, but the same holds for 2-dimensional.)

∗ All n-dimensional Euclidean vector spaces are mutually isomorphic.

∗ All n-dimensional affine spaces are mutually isomorphic.

Isomorphisms to itself are called *automorphisms*.[1]

For an n-dimensional Euclidean vector space, automorphisms include (and are generated by) rotations and reflections (but not translations, nor homotheties); in a given basis they are described by orthogonal matrices.[2]

For a classical Euclidean space, automorphisms include (and are generated by) translations, rotations, reflections and homotheties (called also similarities).

For a Cartesian space, the group of automorphisms of a Cartesian space is trivial (only id).

1b22 Exercise. (a) Consider an isomorphism between affine spaces. Is it uniquely determined by the corresponding isomorphism between their difference spaces?

(b) Define an affine subspace of an affine space. Is it uniquely determined by the corresponding subspace of the difference space? What about an affine subspace of a vector space?

1b23 Exercise. (a) Let V_1, V_2 be vector planes (that is, 2-dimensional vector spaces), $u_1, v_1 \in V_1$ linearly independent, and $u_2, v_2 \in V_2$ linearly indepen-

[1] They are a subgroup of the group of all bijections $X \to X$.

[2] For the Euclidean vector space $\mathbb{R}^n$, the group of automorphisms is the $\frac{(n-1)n}{2}$-dimensional Lie group known as the orthogonal group $\mathrm{O}(n) = \mathrm{O}(n, \mathbb{R})$.

dent. Then one and only one isomorphism $V_1 \to V_2$ sends u_1 to u_2 and v_1 to v_2.

(b) Let S_1, S_2 be affine planes (that is, 2-dimensional affine spaces), $a_1, b_1, c_1 \in S_1$ not on a line,[1] and $a_2, b_2, c_2 \in S_2$ not on a line. Then one and only one isomorphism $S_1 \to S_2$ sends a_1 to a_2, b_1 to b_2 and c_1 to c_2.

Prove it.[2]

Thus, up to isomorphism there is only one affine plane ("*the* affine plane") and only one triangle on it![3]

1b24 Exercise. Let S be an affine space.

(a) Given two different points in S, define the corresponding line segment (a subset of S) and its midpoint (a point).

(b) Given three points in S not on a line, define the corresponding triangle (a subset of S) and its medians (line segments). Prove that the three medians intersect.

1b25 Exercise. Let S be an affine space, and $a, b, c \in S$ three points not on a line. Consider the automorphism $\varphi : S \to S$ such that $\varphi(a) = b$, $\varphi(b) = c$, $\varphi(c) = a$. Prove existence and uniqueness of $x \in S$ such that $\varphi(x) = x$.

Thus, "the affine triangle" has only one center. In contrast, in Euclidean geometry, more than 5 000 centers of a triangle are introduced.[4]

AFFINE SPACE WITH EUCLIDEAN METRIC

By a Euclidean metric on an affine space we mean a metric of the form

$$a, b \mapsto |a - b|$$

where $|\cdot|$ is a Euclidean norm on the difference space (here a, b are points and $a - b$ is a vector).

An n-dimensional Euclidean affine space is, by definition, an n-dimensional affine space endowed with a Euclidean metric.

Automorphisms of an n-dimensional Euclidean affine space include (and are generated by) translations, rotations and reflections (but not homotheties).[5]

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[1] A line is a 1-dimensional affine subspace.

[2] Hint to (b): apply (a) to the difference spaces.

[3] In contrast, in Euclidean geometry some triangles are right-angled, acute-angled, obtuse-angled, isosceles, equilateral etc. Nothing like this can happen on the affine plane.

[4] Wikipedia:Encyclopedia of Triangle Centers.

[5] For the Euclidean affine space $\mathbb{R}^n$, the group of automorphisms is the $\frac{n(n+1)}{2}$-dimensional Lie group known as the Euclidean group $\mathrm{E}(n) = \mathrm{E}(n, \mathbb{R})$ as well as the inhomogeneous orthogonal group $\mathrm{IO}(n)$.

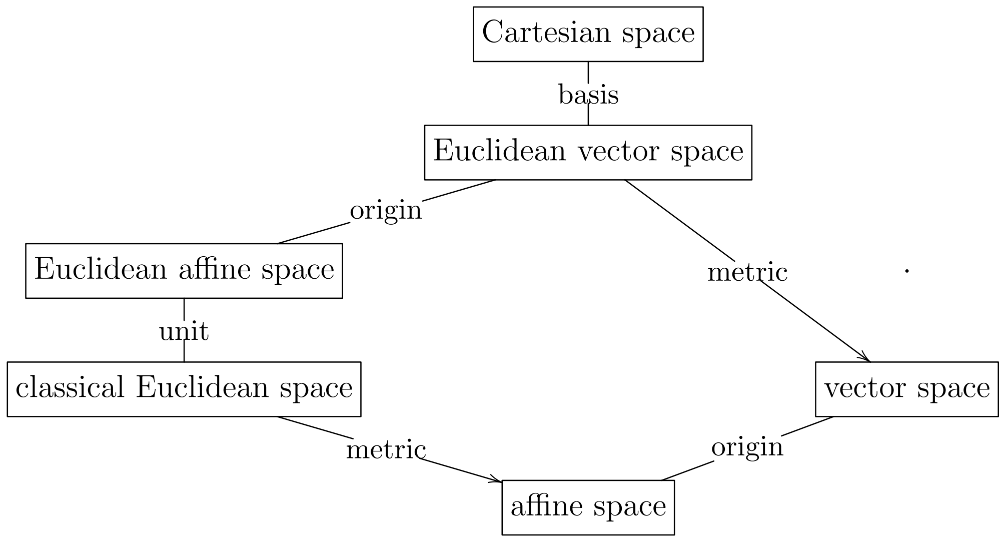
Choosing an origin we upgrade a Euclidean affine space to a Euclidean vector space; forgetting the origin we downgrade a Euclidean vector space to a Euclidean affine space.

All n-dimensional Euclidean affine spaces are mutually isomorphic. The argument is as before: we upgrade both to Euclidean vector spaces; these are isomorphic.

Every n-dimensional affine space may be upgraded to a Euclidean affine space (since "the" n-dimensional vector space may be upgraded to a Cartesian space, the more so to a Euclidean vector space). Try to do it for the affine space S_n of 1b15. Have you any idea of a *canonical* metric for this space?

A 3-dimensional Euclidean affine space may be downgraded to a classical Euclidean space. This procedure forgets the unit of length. Choosing a unit of length in a classical Euclidean space we upgrade it to a Euclidean affine space.

Thus, we may apply notions and results of Euclidean planimetry/stereometry in every 2-dimensional/3-dimensional subspace of an n-dimensional Euclidean affine space.



1c Metric and topology

A *metric* on a set X is, by definition, a function $\rho : X \times X \to [0, \infty)$ such that

$$\begin{array}{lll} \forall x, y \in X & \rho(x,y) = \rho(y,x)\,, & \text{symmetry} \\ \forall x, y, z \in X & \rho(x,z) \le \rho(x,y) + \rho(y,z)\,, & \text{triangle inequality} \\ \forall x, y \in X & \bigl(\rho(x,y) = 0 \iff x = y\bigr)\,. & \end{array}$$

A set endowed with a metric is called a *metric space.*[1] [2] Isomorphism between metric spaces, so-called isometry, is a bijection that preserves distances: $\rho_1(x,y) = \rho_2(\varphi(x), \varphi(y))$.

Every subset of a metric space is itself a metric space:[3] $A \subset X$, $\rho_A(x,y) = \rho(x,y)$ for $x, y \in A$.

An n-dimensional Euclidean affine space is a metric space (being endowed with the Euclidean metric). All its subsets are metric spaces as well.

Let (X, ρ) be a metric space, $A \subset X$ and $x \in A$. If

$$\exists \varepsilon > 0 \;\; \forall y \in X \;\; \bigl(\rho(x,y) < \varepsilon \implies y \in A\bigr)\,,$$

then x is called an *interior point* of A, and A is called a *neighborhood* of x.

The intersection of two (or finitely many) neighborhoods of x is a neighborhood of x; for infinitely many neighborhoods this is not the case.

An *open* set in a metric space is, by definition, a set that is a neighborhood of every point of this set. In other words: every point of this set is its interior point.

A *topology* on a set X is, by definition, a set $\tau \subset \mathcal{P}(X)$ of subsets of X (called open sets) such that

$$\begin{gathered} \emptyset \in \tau\,, \quad X \in \tau\,; \\ \forall U, V \in \tau \quad U \cap V \in \tau\,; \\ \forall \mathcal{A} \subset \tau \quad \Bigl(\bigcup_{U \in \mathcal{A}} U\Bigr) \in \tau\,. \end{gathered}$$

(That is, a finite intersection of open sets must be open, and an arbitrary union of open sets must be open.) A set endowed with a topology is called a *topological space.*

---

[1] Note one principal base set X and one auxiliary base set $[0, \infty)$.

[2] Many authors require X to be nonempty.

[3] In striking contrast to algebraic structures; an arbitrary subset of a vector space is not at all a vector space.

An isomorphism between topological spaces (X_1, τ_1) and (X_2, τ_2), so-called *homeomorphism,* is a bijection $\varphi : X_1 \to X_2$ such that a subset of X_1 belongs to τ_1 if and only if the corresponding subset of X_2 (the image) belongs to τ_2.

1c1 Exercise. The set of all open sets in a metric space is a topology.
Prove it.

A metric space can be downgraded to a topological space by forgetting the metric while retaining the corresponding topology.

Two metrics (on the same set) are called equivalent if they correspond to the same topology. For example, the metric $2\rho : x, y \mapsto 2\rho(x, y)$ is equivalent to ρ. More examples: $\sqrt{\rho}$; $\min\big(1, \rho\big)$.

1c2 Exercise. (a) Metrics ρ_1, ρ_2 on X are equivalent if and only if

$$\rho_1(x_n, x) \to 0 \quad \Longleftrightarrow \quad \rho_2(x_n, x) \to 0$$

for arbitrary $x, x_1, x_2, \dots \in X$. Prove it.

(b) Equivalence of metrics does not mean that

$$\rho_1(x_n, y_n) \to 0 \quad \Longleftrightarrow \quad \rho_2(x_n, y_n) \to 0$$

for arbitrary $x_1, x_2, \dots \in X$ and $y_1, y_2, \dots \in X$. Find a counterexample.

On a topological space, metric is an additional structure, and its existence is an additional property.

A topology (as well as a topological space) is called *metrizable*, if it corresponds to some metric. For example, on a two-point set there exist four topologies, one metrizable and three non-metrizable (think, why). We restrict ourselves to metrizable topological spaces. Every subset of a metrizable space is itself a metrizable space. An n-dimensional Euclidean affine space is a metrizable topological space (and its subsets are).

Treating a set $X \subset \mathbb{R}^n$ as a metrizable space we have open sets in X, and they need not be open in $\mathbb{R}^n$ (unless X is open in $\mathbb{R}^n$); in order to avoid confusion, they are called *relatively open* (in X). The same holds for relatively closed sets. Note that X always is both relatively open and relatively closed in X.

1c3 Exercise. For arbitrary $X \subset \mathbb{R}^n$ prove that

(a) for every open $G \subset \mathbb{R}^n$ the set $G \cap X$ is relatively open in X;

(b) all relatively open subsets of X are such $G \cap X$.

Formulate and prove a similar fact for closed sets.

1c4 Exercise. Define all topological notions mentioned in Sect. 1a (convergence and limit of a sequence; continuity; limit point of a set; closed set; closure, boundary, interior of a set) in an arbitrary metrizable space.

Thus, all these notions are invariant under homeomorphisms (by the isomorphism argument). For example, if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism then

* for arbitrary $A \subset \mathbb{R}^n$, A is closed if and only if $\varphi(A)$ is closed;
* for arbitrary $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, x is a limit point of A if and only if $\varphi(x)$ is a limit point of $\varphi(A)$;
* for arbitrary $x_1, x_2, \dots \in \mathbb{R}^n$, the sequence $(x_k)_k$ converges if and only if the sequence $\big(\varphi(x_k)\big)_k$ converges.

The same applies to a homeomorphism $\varphi : X \to Y$ between sets $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$. For example, for every $A \subset X$, A is relatively closed in X if and only if $\varphi(A)$ is relatively closed in Y.

WARNING. It does not mean that X is closed (in $\mathbb{R}^n$) if and only if Y is closed (in $\mathbb{R}^m$). Find a counterexample (and compare it with 1a9). *"Closed" is a property of a set in a space, not a property of a space.* The same holds for "bounded".

In contrast, the combined property "closed and bounded" (in $\mathbb{R}^n$) is equivalent to a property of the space (rather than set); recall Bolzano-Weierstrass theorem or Heine-Borel theorem. For an arbitrary metrizable space compactness may be defined according to Bolzano-Weierstrass, or Heine-Borel.[1]

1c5 Exercise. (a) Is $\mathbb{Z}$ homeomorphic to $A = \{\arctan n : n \in \mathbb{Z}\}$? Describe all relatively open sets in A.

(b) Is A homeomorphic to $B = A \cap [0, \infty)$?

(c) Is the closure $\overline{A}$ homeomorphic to $\overline{B}$? Describe all relatively open sets in $\overline{A}$.

As was seen in 1a15–1a17, a continuous bijection need not be a homeomorphism.[2]

Two functions on a metrizable space X are said to be equal *near* a given point $x \in X$, if they are equal on some neighborhood of x. Equality near x is an equivalence relation. Its equivalence classes are called *germs* (of functions) at x. The germ of f at x is denoted by $[f]_x$. The same applies to mappings from X to any Y, as well as from a neighborhood of x to Y.

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[1]In fact, these two definitions of compactness are equivalent for all metrizable spaces, but not all topological spaces. By the way, in a normed space of infinite dimension, the closed unit ball is not compact.

[2]In striking contrast to algebraic structures; there, every bijective homomorphism is an isomorphism.

Many properties of functions apply readily to germs, according to the pattern
$[f]_x$ is called $\boxed{}$ when f is $\boxed{}$ near x;
here $\boxed{}$ may be "linear", "bounded", "continuous", "one-to-one" etc.

1c6 Exercise. "Continuous at x" is not the same as "continuous near x". Find a counterexample.

If $[f_1]_x = [f_2]_x$ then $\lim_{y\to x} f_1(y) = \lim_{y\to x} f_2(y)$ in the following sense: either both limits exist and coincide, or neither limit exists. This way the notion of limit applies to germs; it is a local notion.

When locality is evident, I do not hesitate writing "let $f : \mathbb{R}^n \to \mathbb{R}^m$" rather than "let $f : U \to \mathbb{R}^m$ where $U \subset \mathbb{R}^n$ is a neighborhood of x".

A bit about connectedness.

Let X_1, X_2 be disjoint sets, and (X_1, ρ_1), (X_2, ρ_2) metric spaces. Assume for convenience that the distances never exceed 1. Here is a useful metric on $X = X_1 \cup X_2$ (assuming that $\rho_1(\cdot,\cdot) \le 1$ and $\rho_2(\cdot,\cdot) \le 1$):

$$\rho(x, y) = \begin{cases} \rho_1(x, y) & \text{if } x, y \in X_1, \\ \rho_2(x, y) & \text{if } x, y \in X_2, \\ 1 & \text{otherwise.} \end{cases}$$

In the metric space (X, ρ) the sets X_1, X_2 are *clopen,* that is, both closed and open.

1c7 Exercise. Let X be a metric space, and $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$. Then the following two conditions are equivalent:

(a) the metric

$$\tilde\rho(x, y) = \begin{cases} \rho(x, y) & \text{if } x, y \in X_1 \text{ or } x, y \in X_2, \\ 1 & \text{otherwise} \end{cases}$$

is equivalent to ρ;

(b) the sets X_1, X_2 are clopen.

Prove it.

1c8 Definition. A topological space is *connected* if the only clopen sets are the empty set and the whole space.

1c9 Exercise. A subset of $\mathbb{R}$ is connected if and only if it is an interval.[1] Prove it.

[1] Be it open or not, closed or not, bounded or not, empty or not.

1c10 Exercise. Which of these subsets of $\mathbb{R}^2$ are connected? Prove.

$$A = \{(x,y) : x^2y^2 = 1\} ;$$
$$B = \{(x, \sin\tfrac{1}{x}) : x \ne 0\} ;$$
$$C = B \cup \{(0,0)\} .$$

1c11 Exercise. Let $G \subset \mathbb{R}^n$ be an open set, and $x \in G$. Then the set of all points $y \in G$ that can be connected with x by a polygonal line inside G is a relatively clopen subset of G.

Prove it.

We see that every connected open set G is polygonally connected, and therefore we may introduce a metric d_G on G (sometimes called shortest path metric or geodesic metric) by

$$d_G(x,y) = \inf\{\text{length}(P) : P \text{ is polygonal line from } x \text{ to } y\} .$$

Clearly, d_G is a metric on G, and $d_G \ge \rho_G$ where ρ_G is the Euclidean metric, $\rho_G(x,y) = |x-y|$.

1c12 Exercise. (a) Prove that $d_G \sim \rho_G$, that is, $d_G(x_k,x) \to 0 \iff |x_k - x| \to 0$ whenever $x, x_1, x_2, \dots \in G$.

(b) The relation $|x_k - y_k| \to 0$ does not imply $d_G(x_k, y_k) \to 0$; find a counterexample.[1]

A metric space (X,ρ) is called bounded, if $\sup_{x,y\in X} \rho(x,y) < \infty$. A set A in (X,ρ) is called bounded, if $\sup_{x,y\in A} \rho(x,y) < \infty$.

1c13 Exercise. (a) Boundedness of (G, ρ_G) does not imply boundedness of (G, d_G). Find a counterexample.[2]

(b) If $K \subset G$ is compact (in $\mathbb{R}^n$) then K is a bounded set in (G, d_G). Prove it.[3]

(c) If $K \subset G$ is compact (in $\mathbb{R}^n$) then $\displaystyle\sup_{x,y\in K, x\ne y} \frac{d_G(x,y)}{|x-y|} < \infty$.

Prove it.[4]

[1]Hint: try a slit domain.

[2]Hint: recall the set B of 1c10.

[3]Hint: otherwise $d_G(x_k,y_k) \to \infty$, $|x_k - x| \to 0$, $|y_k - y| \to 0$.

[4]Hint: otherwise $\frac{d_G(x_k,y_k)}{|x_k-y_k|} \to \infty$, $|x_k - x| \to 0$, $|y_k - y| \to 0$.

1d Linearity and continuity

The general form of a linear function $f : \mathbb{R}^n \to \mathbb{R}$ [Sh:p.65]:

$$f(x) = \langle a, x \rangle \quad \text{for some } a \in \mathbb{R}^n \,.$$

Such f is continuous [Sh:p.65] (being a linear combination of coordinate functions).

The general form of a linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$ [Sh:p.68]:

$$T(x) = Ax \quad \text{for some } m \times n \text{ matrix } A \,.$$

Such T is continuous [Sh:Th.3.1.5] (since each coordinate of Tx is a linear function of x).

Affine function $f : \mathbb{R}^n \to \mathbb{R}$:

$$f(x) = \langle a, x \rangle + t \quad \text{for some } a \in \mathbb{R}^n,\, t \in \mathbb{R} \,.$$

Affine operator $T : \mathbb{R}^n \to \mathbb{R}^m$:

$$T(x) = Ax + b \quad \text{for some } m \times n \text{ matrix } A \text{ and some } b \in \mathbb{R}^m \,.$$

Such f and T are continuous.

Thus, every linear (as well as affine) bijection $T : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism.

Given an n-dimensional Cartesian space X, we introduce a topology on X by transporting to X the topology of $\mathbb{R}^n$ via the given bijection between X and $\mathbb{R}^n$. This bijection becomes a homeomorphism.

Given an n-dimensional affine space S, we introduce a topology on S by upgrading S to a Cartesian space and using the topology introduced above. The result does not depend on the choice of coordinates. Every isomorphism between n-dimensional affine spaces becomes also a homeomorphism.[1]

Topological notions are well-defined on every finite-dimensional vector or affine space.

1d1 Exercise. Recall the n-dimensional vector space P_{n-1} of polynomials discussed in 1b15. Prove that the following conditions on $f, f_1, f_2, \dots \in P_{n-1}$ are equivalent:

(a) $f_k \to f$ in P_{n-1};

(b) $f_k(0) \to f(0)$, $f'_k(0) \to f'(0)$, $\dots$, $f_k^{(n-1)}(0) \to f^{(n-1)}(0)$;

[1] In infinite dimension the situation is utterly different.

(c) $f_k(0) \to f(0)$, $f_k(1) \to f(1)$, ..., $f_k(n-1) \to f(n-1)$;[1]
(d) $f_k(\cdot) \to f(\cdot)$ pointwise; that is, $f_k(x) \to f(x)$ for every $x \in \mathbb{R}$;
(e) $f_k(\cdot) \to f(\cdot)$ locally uniformly; that is, $\max_{|x|\le M} |f_k(x) - f(x)| \to 0$ for every M.

1d2 Exercise. The same as 1d1 for the n-dimensional affine space $S_n = \{f : f^{(n)} = g\}$ discussed in 1b15.[2]

1d3 Exercise. Let V be an n-dimensional vector space, and $V_1 \subset V$ its subspace.
(a) Upgrade V to $\mathbb{R}^n$ (by choosing a basis) getting $V_1 = \{(x_1, \dots, x_n) : x_{m+1} = \dots = x_n = 0\}$; here $m = \dim V_1$.
(b) Conclude that every subspace of an n-dimensional vector or affine space is closed (topologically).[3]

1e Norms of vectors and operators

1e1 Definition. The norm $\|T\|$ of a linear operator $T : E_1 \to E_2$ between finite-dimensional Euclidean vector spaces E_1, E_2 is

$$\|T\| = \sup_{x \in E_1, x \ne 0} \frac{|T(x)|}{|x|} \, .$$

Also,

$$\|T\| = \max_{|x|\le 1} |T(x)|$$

(think, why); this is the maximum of a continuous function on a compact set [Sh:p.73].

The *operator norm* $\|A\|$ of an $m \times n$ matrix A is, by definition, the norm of the corresponding operator $\mathbb{R}^n \to \mathbb{R}^m$.

1e2 Exercise. If a matrix $A = (a_{i,j})_{i,j}$ is diagonal then

$$\|A\| = \max_{i=1,\dots,\min(m,n)} |a_{i,i}|.$$

Prove it.

The set $M_{m,n}(\mathbb{R})$ of all $m \times n$ matrices (with real elements) evidently is an mn-dimensional vector space. Does the operator norm turn it to a Euclidean space? No, it does not. Even if we restrict ourselves to $M_{2,2}(\mathbb{R})$, and even

[1] Hint: (c) consider the linear operator $P_{n-1} \ni g \mapsto \big(g(0), g(1), \dots, g(n-1)\big) \in \mathbb{R}^n$.
[2] Hint: use 1d1.
[3] In infinite dimension the situation is strikingly different.

to its 2-dimensional subspace of diagonal matrices, we get (by 1e2, up to isomorphism) $\mathbb{R}^2$ with the norm

$$\|(s,t)\| = \max(|s|,|t|)\,,$$

its unit ball $\{x : \|x\| \le 1\}$ being the square $[-1,1]\times[-1,1]$. This is not the Euclidean plane! For two non-collinear vectors $a=(1,1)$ and $b=(1,-1)$ we have $\|a\|=1$, $\|b\|=1$ and $\|a+b\|=2$, which never happens on the Euclidean plane. Also, the "parallelogram equality" $|a-b|^2+|a+b|^2 = 2|a|^2+2|b|^2$ holds for arbitrary vectors a,b of a Euclidean space, but fails for the operator norm.

1e3 Definition. (a) A *norm* on a vector space V is a function $V \ni x \mapsto \|x\| \in [0,\infty)$ such that

$$\begin{gathered}\|tx\| = |t|\cdot\|x\| \quad \text{for all } x\in V,\ t\in\mathbb{R}\,;\\ \|x+y\| \le \|x\|+\|y\| \quad \text{for all } x,y\in V\,;\\ \|x\|>0 \quad \text{whenever } x\ne 0\,.\end{gathered}$$

(b) A *normed space* is a vector space endowed with a norm.

Every normed space is also a metric space (with the metric $x,y \mapsto \|x-y\|$), therefore, also a topological space.

Euclidean vector spaces are a special case of normed spaces.[1] Distances are well-defined in normed spaces, but angles — only in Euclidean spaces.

1e4 Exercise. Prove that

$$-\|x-y\| \le \|x\|-\|y\| \le \|x-y\|$$

for all $x,y\in V$.

1e5 Exercise. Prove that the operator norm is indeed a norm on $M_{m,n}(\mathbb{R})$.

1e6 Lemma. Every norm on $\mathbb{R}^n$ is continuous.

Proof. For arbitrary $t_1,\dots,t_n\in\mathbb{R}$,

$$\begin{aligned}\|(t_1,\dots,t_n)\| = \|t_1e_1+\dots+t_ne_n\| \le |t_1|\cdot\|e_1\|+\dots+|t_n|\cdot\|e_n\| \le\\ \le (\|e_1\|+\dots+\|e_n\|)\cdot\max(|t_1|,\dots,|t_n|) \le C\sqrt{t_1^2+\dots+t_n^2}\end{aligned}$$

where[2] $C=\|e_1\|+\dots+\|e_n\|$ (and $e_1,\dots,e_n$ are the standard basis). Thus, $\|x\|\le C|x|$ for all $x\in\mathbb{R}^n$. Now, if $|x_n-x|\to 0$ then $\|x_n-x\|\to 0$, and by 1e4, $\|x_n\|\to\|x\|$. $\square$

[1] In fact, a normed space is Euclidean iff the norm satisfies the parallelogram equality.

[2] Even better, $C=\sqrt{\|e_1\|^2+\dots+\|e_n\|^2}$ fits.

The sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ being compact, a norm $\|\cdot\|$ reaches its minimum c and maximum C on S^{n-1}:

$$c = \min_{|x|=1} \|x\| \,, \quad C = \max_{|x|=1} \|x\| \,;$$

$0 < c \le C < \infty$ (think, why $c > 0$). Thus,

$$\forall x \quad c|x| \le \|x\| \le C|x| \,;$$
$$\|x_n\| \to 0 \quad \text{if and only if} \quad |x_n| \to 0 \,;$$

one says that $\|\cdot\|$ and $|\cdot|$ are equivalent norms. Clearly, equivalent norms lead to equivalent metrics. Taking into account that every n-dimensional vector space is isomorphic to $\mathbb{R}^n$, we conclude.

1e7 Proposition. For every finite-dimensional vector space V,[1]

(a) for every norm $\|\cdot\|$ on V,

$$(x_n \to x) \quad \Longleftrightarrow \quad (\|x_n - x\| \to 0) \quad \text{for all } x, x_1, x_2, \dots \in V \,;$$

(b) for every pair of norms $\|\cdot\|_1$, $\|\cdot\|_2$ on V,

$$\exists c, C \in (0,\infty) \;\; \forall x \in V \quad c\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \,.$$

All norms are equivalent on an arbitrary finite-dimensional vector space.

Thus, in finite dimension all norms lead to the same topology; this is the topology introduced in Sect. 1d (think, why).

1e8 Exercise. Generalize 1e1 and 1e5 to the space $\mathcal{L}(X,Y)$ of all linear operators [Sh:p.71] between normed (not just Euclidean) finite-dimensional spaces X, Y.[2]

1e9 Exercise. If $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(Y,Z)$ then $TS \in \mathcal{L}(X,Z)$ and $\|TS\| \le \|T\| \cdot \|S\|$. Prove it.

1e10 Exercise. (a) Prove equivalence of two definitions of the *Hilbert-Schmidt norm* $\|A\|_{\text{HS}}$ of an $m \times n$ matrix $A = (a_{i,j})_{i,j}$:
$\|A\|_{\text{HS}} = \bigl(\sum_{j,k} a_{j,k}^2\bigr)^{1/2}$;
$\|A\|_{\text{HS}} = \sqrt{\operatorname{trace}(A^*A)}$.

(b) Is $(M_{m,n}(\mathbb{R}), \|\cdot\|_{\text{HS}})$ a normed space? a Euclidean space?

(c) Prove that $\|A\| \le \|A\|_{\text{HS}} \le \sqrt{n}\|A\|$.[3]

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[1] In infinite dimension the situation is utterly different.

[2] Linear operators between spaces of operators are also well-defined, and sometimes called superoperators (mostly by physicists); see also "Superoperator" in Wikipedia.

[3] Hint to $\|A\| \le \|A\|_{\text{HS}}$: using the Cauchy-Schwarz inequality, estimate first y_k^2 and then $\sum_{k=1}^m y_k^2$; here $y_k = \sum_j a_{k,j} x_j$.
Hint to $\|A\|_{\text{HS}} \le \sqrt{n}\|A\|$: $|Ae_j| \le \|A\|$ for each $j = 1, \dots, n$.

A bit about convexity.

1e11 Definition. (a) A set C in a vector or affine space is *convex* if for all $x, y \in C$ the segment $[x, y] = \{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\}$ is contained in C.

(b) A real-valued function f on a vector or affine space, or on a convex set therein, is called *convex* if

$$f\big((1-\lambda)x + \lambda y\big) \le (1-\lambda)f(x) + \lambda f(y)$$

for all $\lambda \in [0, 1]$ and all x, y in the domain of f.

1e12 Exercise. Prove that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is convex if and only if the set $\{(x, y, z) : z \ge f(x, y)\} \subset \mathbb{R}^3$ is convex.

1e13 Exercise. Prove that convexity of the sets $\{x : f(x) \le t\}$ for all $t \in \mathbb{R}$ is necessary but not sufficient for convexity of a function f.[1]

1e14 Exercise. Prove that the second condition of 1e3 ($\|x+y\| \le \|x\|+\|y\|$) is equivalent (given the other two conditions) to (a) convexity of the norm, and also to (b) convexity of the ball $\{x \in V : \|x\| \le 1\}$.[2]

1e15 Exercise. Let $p \in [1, \infty)$. Prove that the function

$$\mathbb{R}^n \ni (t_1, \dots, t_n) \mapsto \big(|t_1|^p + \dots + |t_n|^p\big)^{1/p} \in [0, \infty)$$

is a norm on $\mathbb{R}^n$.[3]

This norm is often denoted $\|\cdot\|_p$.

In the limit $p \to \infty$ we get

$$\|(t_1, \dots, t_n)\|_\infty = \max(|t_1|, \dots, |t_n|)\,.$$

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[1] Hint: for "but not sufficient" try dimension one.

[2] Hint: (b) $\frac{x+y}{\|x\|+\|y\|} = \theta \frac{x}{\|x\|} + (1-\theta)\frac{y}{\|y\|}$.

[3] Hint: the function $(t_1, \dots, t_n) \mapsto |t_1|^p + \dots + |t_n|^p$ is convex (being the sum of convex functions), therefore the set $\{(t_1, \dots, t_n) : |t_1|^p + \dots + |t_n|^p \le 1\}$ is convex.

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