8 Change of variables

Change of variables is the most powerful tool for calculating multidimensional integrals. Two kinds of differentiation are instrumental: of mappings (treated in Sections 2–5) and of set functions (treated here).

8a What is the problem

The area of a disk $\{(x,y) : x^2 + y^2 \le 1\} \subset \mathbb{R}^2$ may be calculated by iterated integral,

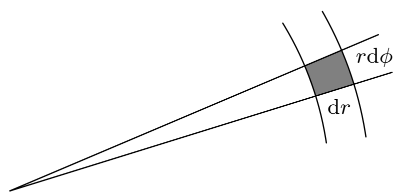
$$\int_{-1}^{1} \mathrm{d}x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y = \int_{-1}^{1} 2\sqrt{1-x^2}\,\mathrm{d}x = \dots$$

or alternatively, in polar coordinates,

$$\int_0^1 r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\varphi = \int_0^1 2\pi r\,\mathrm{d}r = \pi\,;$$

the latter way is much easier! Note "$r\mathrm{d}r$" rather than "$\mathrm{d}r$" (otherwise we would get 2π instead of π).

Why the factor r? In analogy to the one-dimensional theory we may expect something like $\frac{\mathrm{d}x\,\mathrm{d}y}{\mathrm{d}r\,\mathrm{d}\varphi}$; is it r? Well, basically, it is r because an infinitesimal rectangle $[r, r+\mathrm{d}r] \times [\varphi, \varphi+\mathrm{d}\varphi]$ of area $\mathrm{d}r \cdot \mathrm{d}\varphi$ on the (r,φ)-plane corresponds to an infinitesimal rectangle or area $\mathrm{d}r \cdot r\mathrm{d}\varphi$ on the (x,y)-plane.



The factor r is nothing but $|\det T|$ of Sect. 6n, where T is the linear approximation to the nonlinear mapping $(r, \varphi) \mapsto (x, y) = (r \cos \varphi, r \sin \varphi)$ near a point (r, φ).

Thus, we need a generalization of Theorem 6n1 (the linear transformation) to nonlinear transformations. Naturally, the nonlinear case needs more effort.

8a1 Definition. A diffeomorphism[1] between open sets $U, V \subset \mathbb{R}^n$ is an invertible mapping $\varphi : U \to V$ such that both φ and φ^{-1} are continuously differentiable.

By the inverse function theorem 4c5, a homeomorphism $\varphi : U \to V$ is a diffeomorphism if and only if φ is continuously differentiable and $(D\varphi)_x$ is an invertible operator for all $x \in U$ (equivalently, the Jacobian $\det(D\varphi)_x$ does not vanish on U).

And do not forget: in contrast to dimension one, the condition $\det(D\varphi)_x \neq 0$ does not guarantee that φ is one-to-one (as noted in 4b).

8a2 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $E \subset U$. Then the following two conditions are equivalent.

(a) E is Jordan measurable and contained in a compact subset of U;

(b) $\varphi(E)$ is Jordan measurable and contained in a compact subset of V.

8a3 Definition. A function $f : E \to \mathbb{R}$ on a Jordan measurable set $E \subset \mathbb{R}^n$ is *integrable* (on E) if the function $x \mapsto \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{otherwise} \end{cases}$ is integrable on $\mathbb{R}^n$. And in this case the integral of the latter function (over $\mathbb{R}^n$) is $\int_E f$.

8a4 Exercise. (a) Let $E_1 \subset E_2$ be Jordan measurable, and $f : E_2 \to \mathbb{R}$ integrable; then $f|_{E_1}$ is integrable.

(b) Let E_1, E_2 be Jordan measurable, and $f : E_1 \cup E_2 \to \mathbb{R}$; if $f|_{E_1}, f|_{E_2}$ are integrable then f is integrable.

Prove it.

8a5 Theorem. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, $E \subset U$ a Jordan measurable set contained in a compact subset of U, and $f : \varphi(E) \to \mathbb{R}$ an integrable function. Then $f \circ \varphi : E \to \mathbb{R}$ is integrable, and

$$\int_{\varphi(E)} f = \int_E (f \circ \varphi) |\det D\varphi| \, .$$

[1]Namely, C^1 diffeomorphism.

On the other hand, it can happen that an open set is not Jordan measurable (even if bounded); worse, it can happen that $U \subset \mathbb{R}^2$ is a disk but $V = \varphi(U)$ is open, bounded but not Jordan measurable.[1]

8a6 Corollary. If, in addition, U and V are Jordan measurable and $D\varphi$ is bounded on U then integrability of $f : V \to \mathbb{R}$ implies integrability of $(f \circ \varphi)|\det D\varphi| : U \to \mathbb{R}$, and

$$\int_V f = \int_U (f \circ \varphi)|\det D\varphi| \,.$$

The proofs, given in Sect. 8h, are based on a transition from set functions to (ordinary) functions, inverse to integration. (Basically, we'll prove that $|\det D\varphi|$ is the derivative of the set function $E \mapsto v(\varphi(E))$.) This form of differentiation, introduced and examined in 8c–8e, may be partially new to you even in dimension one.

8b Examples and exercises

In this section we take for granted Proposition 8a2, Theorem 8a5 and Corollary 8a6 (to be proved later).

8b1 Exercise. (spherical coordinates in $\mathbb{R}^3$)

Consider the mapping $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$, $\Psi(r, \varphi, \theta) = (r \cos\varphi \sin\theta, r \sin\varphi \sin\theta, r \cos\theta)$.

(a) Draw the images of the planes $r = \text{const}$, $\varphi = \text{const}$, $\theta = \text{const}$, and of the lines $(\varphi, \theta) = \text{const}$, $(r, \theta) = \text{const}$, $(r, \varphi) = \text{const}$.

(b) Show that Ψ is surjective but not injective.

(c) Show that $|\det D\Psi| = r^2 \sin\theta$. Find the points (r, φ, θ), where the operator $D\Psi$ is invertible.

(d) Let $V = (0, \infty) \times (-\pi, \pi) \times (0, \pi)$. Prove that $\Psi|_V$ is injective. Find $U = \Psi(V)$.

8b2 Exercise. Compute the integral $\iiint_{x^2+y^2+(z-2)^2 \le 1} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{x^2+y^2+z^2}$.

Answer: $\pi\left(2 - \frac{3}{2}\log 3\right)$.[2]

8b3 Exercise. Compute the integral $\iint \frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2}$ over one loop of the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$.[3]

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[1]The Riemann mapping theorem is instrumental. See Sect. 18.8 "Change of variables" in book: D.J.H. Garling, "A course in mathematical analysis", vol. 2 (2014).

[2]Hint: $1 < r < 3$; $\cos\theta > \frac{r^2+3}{4r}$.

[3]Hints: use polar coordinates; $-\frac{\pi}{4} < \varphi < \frac{\pi}{4}$; $0 < r < \sqrt{\cos 2\varphi}$; $1 + \cos 2\varphi = 2\cos^2\varphi$; $\int \frac{\mathrm{d}\varphi}{\cos^2\varphi} = \tan\varphi$.

8b4 Exercise. Compute the integral over the four-dimensional unit ball: $\iiiint_{x^2+y^2+u^2+v^2\le 1} e^{x^2+y^2-u^2-v^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v$.[1]

8b5 Exercise. Compute the integral $\iiint |xyz|\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ over the ellipsoid $\{x^2/a^2+y^2/b^2+z^2/c^2\le 1\}$.

Answer: $\frac{a^2b^2c^2}{6}$.[2]

8b6 Exercise. Find the volume cut off from the unit ball by the plane $lx+my+nz=p$.[3]

The *mean* (value) of an integrable function f on a Jordan measurable set $E\subset\mathbb{R}^n$ of non-zero volume is (by definition)

$$\frac{1}{v(E)}\int_E f\,.$$

The *centroid*[4] of E is the point $C_E\in\mathbb{R}^n$ such that for every linear (or affine) $f:\mathbb{R}^n\to\mathbb{R}$ the mean of f on E is equal to $f(C_E)$. That is,

$$C_E=\frac{1}{v(E)}\biggl(\int_E x_1\,\mathrm{d}x,\dots,\int_E x_n\,\mathrm{d}x\biggr)\,,$$

which is often abbreviated to $C_E=\frac{1}{v(E)}\int_E x\,\mathrm{d}x$.

8b7 Exercise. Find the centroids of the following bodies in $\mathbb{R}^3$:

(a) The cone built over the unit disk, the height of the cone is h.

(b) The tetrahedron bounded by the three coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

(c) The hemispherical shell $\{a^2\le x^2+y^2+z^2\le b^2,\ z\ge 0\}$.

(d) The octant of the ellipsoid $\{x^2/a^2+y^2/b^2+z^2/c^2\le 1,\ x,y,z\ge 0\}$.

The *solid torus* in $\mathbb{R}^3$ with minor radius r and major radius R (for $0<r<R<\infty$) is the set

$$\tilde\Omega=\{(x,y,z):\bigl(\sqrt{x^2+y^2}-R\bigr)^2+z^2\le r^2\}\subset\mathbb{R}^3$$

generated by rotating the disk

$$\Omega=\{(x,z):(x-R)^2+z^2\le r^2\}\subset\mathbb{R}^2$$

---

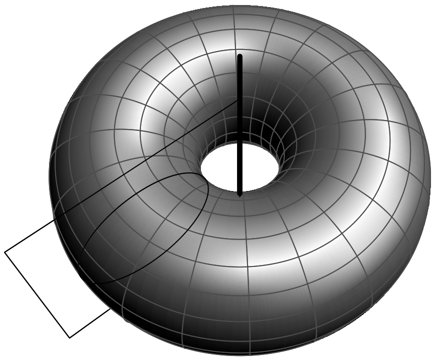
[1] Hint: The integral equals $\iint_{x^2+y^2\le 1} e^{x^2+y^2}\bigl(\iint_{u^2+v^2\le 1-(x^2+y^2)} e^{-(u^2+v^2)}\,\mathrm{d}u\mathrm{d}v\bigr)\,\mathrm{d}x\mathrm{d}y$. Now use the polar coordinates.

[2] Hint: 6e14 can help.

[3] Hint: 6m4 can help.

[4] In other words, the barycenter of (the uniform distribution on) E.

on the (x, z) plane (with the center $(R, 0)$ and radius r) about the z axis.



Interestingly, the volume $2\pi^2 R r^2$ of $\tilde{\Omega}$ is equal to the area πr^2 of Ω multiplied by the distance $2\pi R$ traveled by the center of Ω. (Thus, it is also equal to the volume of the cylinder $\{(x, y, z) : (x, z) \in \Omega, \, y \in [0, 2\pi R].\}$) Moreover, this is a special case of a general property of all solids of revolution.

8b8 Proposition. (The second Pappus's centroid theorem)[1] [2] Let $\Omega \subset (0, \infty) \times \mathbb{R} \subset \mathbb{R}^2$ be a Jordan measurable set and $\tilde{\Omega} = \{(x, y, z) : \big(\sqrt{x^2 + y^2}, z\big) \in \Omega\} \subset \mathbb{R}^3$. Then $\tilde{\Omega}$ is Jordan measurable, and

$$v_3(\tilde{\Omega}) = v_2(\Omega) \cdot 2\pi x_{C_E} \, ;$$

here $C_E = (x_{C_E}, y_{C_E}, z_{C_E})$ is the centroid of E.

8b9 Exercise. Prove Prop. 8b8.[3]

8c Differentiating set functions

As was noted in the end of Sect. 6a, in dimension one an (ordinary) function $\tilde{F} : \mathbb{R} \to \mathbb{R}$ leads to a set function $F : [s, t) \mapsto \tilde{F}(t) - \tilde{F}(s)$; clearly, F is additive: $F([r, s)) + F([s, t)) = F([r, t))$. Moreover, every additive set function F defined on one-dimensional boxes corresponds to some $\tilde{F}$ (unique up to adding a constant); namely, $\tilde{F}(t) = F([0, t))$.

If $\tilde{F}$ is differentiable, $\tilde{F}' = f$, then F and f are related by

$$\frac{F([t - \varepsilon, t))}{\varepsilon} \to f(t) \, , \quad \frac{F([t, t + \varepsilon))}{\varepsilon} \to f(t) \quad \text{as } \varepsilon \to 0+ \, .$$

[1] Pappus of Alexandria (≈ 0290–0350) was one of the last great Greek mathematicians of Antiquity.

[2] The first Pappus's centroid theorem, about the surface area, has to wait for Analysis 4.

[3] Hint: use cylindrical coordinates: $\Psi(r, \varphi, z) = (r \cos \varphi, r \sin \varphi, z)$.

Equivalently,

$$\text{(8c1)} \qquad \frac{F([t-\varepsilon_1, t+\varepsilon_2))}{\varepsilon_1+\varepsilon_2} \to f(t) \qquad \text{as } \varepsilon_1, \varepsilon_2 \to 0+ \; .$$

And if f is integrable on $[s,t]$ then[1]

$$F([s,t)) = \int_{[s,t]} f \, .$$

In dimension 2 a similar construction exists, but is more cumbersome and less useful:

$$F([s_1,t_1) \times [s_2,t_2)) = \tilde F(t_1,t_2) - \tilde F(t_1,s_2) - \tilde F(s_1,t_2) + \tilde F(s_1,s_2) \, ;$$
$$\tilde F(s,t) = F([0,s) \times [0,t)) \, ;$$

this time $\tilde F$ is unique up to adding $\varphi(t_1) + \psi(t_2)$. In higher dimensions $\tilde F$ is even less useful; we do not need it. Instead, we generalize (8c1) as follows.

First, we define an additive box function.

8c2 Definition. An *additive box function* F (in dimension n) is a real-valued function on the set of all boxes (in $\mathbb{R}^n$) such that

$$F(B) = \sum_{C \in P} F(C)$$

whenever P is a partition of a box B.

Second, we define the *aspect ratio* $\alpha(B)$ of a box $B = [s_1,t_1] \times \dots \times [s_n,t_n] \subset \mathbb{R}^n$ by[2]

$$\alpha(B) = \frac{\max(t_1-s_1,\dots,t_n-s_n)}{\min(t_1-s_1,\dots,t_n-s_n)} \, .$$

Clearly, $\alpha(B) = 1$ if and only if B is a cube.

Third, we define the *derivative* of an additive box function F at a point x as the limit of the ratio $\frac{F(B)}{v(B)}$ as B tends to x in the following sense:

$$\text{(8c3)} \qquad B \ni x \, ; \quad v(B) \to 0 \, ; \quad \alpha(B) \to 1 \, .$$

[1]Can you prove it (a) for continuous f, (b) in general? Try 6b1 in concert with the mean value theorem. Anyway, it is the one-dimensional case of (8e4).

[2]It appears that "thin" boxes (of large aspect ratio) are dangerous to the main argument of the proof (see 8h1); this is why we need to control the aspect ratio.

Symbolically,

$$F'(x) = \lim_{B \to x} \frac{F(B)}{v(B)} \, .$$

It means: for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\left| \frac{F(B)}{v(B)} - F'(x) \right| \le \varepsilon$ for every box B satisfying $B \ni x$, $\operatorname{vol}(B) \le \delta$ and $\alpha(B) \le 1 + \delta$.

If this limit exists we say that F is differentiable at x (or on $\mathbb{R}^n$, if the limit exists for all x; or on a given box, etc).

In dimension one, F is differentiable if and only if $\tilde{F}$ is, and $F' = \tilde{F}'$.

In general the limit need not exist, and we introduce the lower and upper derivatives,

$$_*F'(x) = \liminf_{B \to x} \frac{F(B)}{v(B)} \, , \quad ^*F'(x) = \limsup_{B \to x} \frac{F(B)}{v(B)} \, .$$

8d Derivative of integral

Every locally integrable[1] function $f : \mathbb{R}^n \to \mathbb{R}$ leads to an additive box function $F : B \mapsto \int_B f$ (as was seen in Sect. 6j).

Can we restore f from F? Surely not, since F is insensitive to a change of f on a set of volume zero (by 6g1). However, the equivalence class of f can be restored, as we'll see soon.

We say that two functions f, g are *equivalent,* if $^*\!\int_B |f - g| = 0$ for every box B.

If two *continuous* functions are equivalent then they are equal (think, why).

8d1 Proposition. If $F : B \mapsto \int_B f$ for a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, then the three functions $_*F'$, f, $^*F'$ are (pairwise) equivalent.

Proof. Given a box B, we use Lipschitz functions $f_L^-, f_L^+ : B \to \mathbb{R}$ (introduced in Sect. 6i) and their limits $f_\infty^-, f_\infty^+ : B \to \mathbb{R}$;[2]

$$f_L^-(x) \uparrow f_\infty^-(x) \, , \quad f_L^+(x) \downarrow f_\infty^+(x) \quad \text{as } L \to \infty \, .$$

Clearly, $f_\infty^- \le f \le f_\infty^+$. We know that $\int_B f_L^- \uparrow \int_B f$ and $\int_B f_L^+ \downarrow \int_B f$ as $L \to \infty$. Thus,

$$\int_B^* |f - f_\infty^+| = \int_B^* (f_\infty^+ - f) \le \lim_L \int_B^* (f_L^+ - f) = 0 \, ,$$

[1] That is, integrable on every box.

[2] In fact, $f_\infty^-(x) = \liminf_{x_1 \to x} f(x_1)$ and $f_\infty^+(x) = \limsup_{x_1 \to x} f(x_1)$, but we do not need it.

therefore f and f^+_∞ are equivalent. Similarly, f and f^-_∞ are equivalent. On the other hand,

$$\frac{F(B)}{v(B)} = \frac{1}{v(B)} \int_B f \le \sup_B f \,,$$

therefore

$$^*F'(x) = \limsup_{B \to x} \frac{F(B)}{v(B)} \le \limsup_{B \to x} \sup_B f^+_L = f^+_L(x)$$

for all L, which shows that $^*F' \le f^+_\infty$. Similarly, $_*F' \ge f^-_\infty$. We see that $f^-_\infty \le {}_*F' \le {}^*F' \le f^+_\infty$ and $f^-_\infty, f, f^+_\infty$ are equivalent, therefore all these functions are equivalent. □

8e Integral of derivative

8e1 Proposition. (a) If an additive box function F is differentiable on a box B then

$$v(B) \inf_{x \in B} F'(x) \le F(B) \le v(B) \sup_{x \in B} F'(x) \,.$$

(b) For every additive box function F,

$$v(B) \inf_{x \in B} {}_*F'(x) \le F(B) \le v(B) \sup_{x \in B} {}^*F'(x) \,.$$

8e2 Lemma. For every partition P of a box B and every additive box function F,

$$\min_{C \in P} \frac{F(C)}{v(C)} \le \frac{F(B)}{v(B)} \le \max_{C \in P} \frac{F(C)}{v(C)} \,.$$

Proof. Denoting $a = \min_{C \in P} \frac{F(C)}{v(C)}$ and $b = \max_{C \in P} \frac{F(C)}{v(C)}$ we have $av(C) \le F(C) \le bv(C)$ for all $C \in P$; the sum over C gives $av(B) \le F(B) \le bv(B)$. □

8e3 Lemma. For every box B and every $\varepsilon > 0$ there exists a partition P of B such that $v(C) \le \varepsilon$ and $\alpha(C) \le 1 + \varepsilon$ for all $C \in P$.

Proof. Given $B = [s_1, t_1] \times \dots \times [s_n, t_n]$, for arbitrary natural number K we define natural numbers $k_1, \dots, k_n$ by

$$\frac{k_1 - 1}{K} \le t_1 - s_1 < \frac{k_1}{K} \,, \dots, \frac{k_n - 1}{K} \le t_n - s_n < \frac{k_n}{K} \,,$$

divide $[s_1, t_1]$ into k_1 equal intervals, ..., $[s_n, t_n]$ into k_n equal intervals, and accordingly, B into $k_1 \dots k_n$ equal boxes, each $C \in P$ being a shift of $[0, \frac{t_1 - s_1}{k_1}] \times \dots \times [0, \frac{t_n - s_n}{k_n}]$. For arbitrary $i, j \in \{1, \dots, n\}$ we have

$$\frac{\frac{t_i - s_i}{k_i}}{\frac{t_j - s_j}{k_j}} = \frac{(t_i - s_i) k_j}{k_i (t_j - s_j)} \le \frac{k_i k_j}{k_i (k_j - 1)} = \frac{k_j}{k_j - 1} = 1 + \frac{1}{k_j - 1} \le 1 + \frac{1}{K(t_j - s_j) - 1} \,,$$

thus,

$$\alpha(C) \le 1 + \frac{1}{K \min(t_1 - s_1, \dots, t_n - s_n) - 1} \to 0 \quad \text{as } K \to \infty \,.$$

Also,

$$v(C) = \frac{t_1 - s_1}{k_1} \dots \frac{t_n - s_n}{k_n} \le \frac{1}{K^n} \to 0 \quad \text{as } K \to \infty \,.$$

It remains to take K large enough. □

Proof of Prop. 8e1. Item (a) is a special case of (b); we'll prove (b).

Lemma 8e3 (with $\varepsilon = 1$) gives a partition P_1 of B such that $v(C) \le 1$ and $\alpha(C) \le 1+1$ for all $C \in P_1$. Lemma 8e2 gives $C_1 \in P_1$ such that $\frac{F(C_1)}{v(C_1)} \ge \frac{F(B)}{v(B)}$. We repeat the process for C_1 (in place of B) and $\varepsilon = 1/2$ and get $C_2 \subset C_1$ such that $v(C_2) \le 1/2$, $\alpha(C_2) \le 1 + 1/2$ and $\frac{F(C_2)}{v(C_2)} \ge \frac{F(C_1)}{v(C_1)} \ge \frac{F(B)}{v(B)}$. Continuing this way we get boxes $B \supset C_1 \supset C_2 \supset \dots$, $v(C_k) \to 0$, $\alpha(C_k) \to 1$, and $\frac{F(C_k)}{v(C_k)} \ge \frac{F(B)}{v(B)}$ for all k. The intersection of all C_k is $\{x\}$ for some $x \in B$, and $C_k \to x$ in the sense of (8c3). Thus, ${}^*F'(x) \ge \limsup_k \frac{F(C_k)}{v(C_k)} \ge \frac{F(B)}{v(B)}$, and therefore $F(B) \le v(B) \sup_{x \in B} {}^*F'(x)$. The other inequality is proved similarly (or alternatively, turn to $(-F)$). □

Combining 8e1(a) and 6b1 we get

$$F(B) = \int_B F' \qquad \text{(8e4)}$$

whenever F' exists and is integrable on B. Here is a more general result.

8e5 Exercise. Prove that

$$\underset{*}{\int}_B {}_*F' \le F(B) \le \overset{*}{\int}_B {}^*F'$$

for every box B and additive box function F such that ${}_*F'$ and ${}^*F'$ are bounded on B.

If $\underset{*}{\int}_B {}_*F' = \overset{*}{\int}_B {}^*F'$ then ${}_*F'$ and ${}^*F'$ are integrable and moreover, every function sandwiched between them is integrable (with the same integral).[1] In this case it is convenient to interpret F' as *any* such function and write

$$F(B) = \int_B F'$$

even though F may be non-differentiable at some points. (You surely know one-dimensional examples!) However, the equality $\underset{*}{\int}_B {}_*F' = \overset{*}{\int}_B {}^*F'$ may fail; here is a counterexample.

[1] A similar situation appeared in Sect. 7d.

8e6 Example. There exists a nonnegative box function F (in one dimension) such that ${}_*\int_{[0,1]} {}_*F' < {}^*\int_{[0,1]} {}^*F'$.

We choose disjoint intervals $[s_k, t_k] \subset [0,1]$, whose union is dense on $[0,1]$, such that $\sum_k (t_k - s_k) = a \in (0,1)$, define F by[1]

$$F([s,t]) = \sum_k \operatorname{length}\bigl([s_k, t_k] \cap [s,t]\bigr) \, ,$$

and observe that $F([0,1]) = a$, $0 \le {}_*F' \le {}^*F' \le 1$ and

$$F'(x) = 1 \quad \text{for all } x \in \bigcup_k (s_k, t_k)$$

(think, why). Thus, ${}^*\int_{[0,1]} {}^*F' = 1$ (since the integrand is 1 on a dense set). However, ${}_*\int_{[0,1]} {}_*F' \le F([0,1]) = a < 1$.[2]

8f Set function induced by mapping

Consider a mapping $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that the inverse image $\varphi^{-1}(B)$ of every box B is a bounded set. (An example: $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x,y) = x^2 + y^2$.) It leads to a pair of box functions $F_* \le F^*$ (in dimension n),

$$F_*(B) = v_*(\varphi^{-1}(B^\circ)) \, , \quad F^*(B) = v^*(\varphi^{-1}(B)) \, , \tag{8f1}$$

generally not additive but rather superadditive and subadditive: for every partition P of a box B,

$$F_*(B) \ge \sum_{C \in P} F_*(C) \, , \quad F^*(B) \le \sum_{C \in P} F^*(C) \, ,$$

which follows from (6f3), (6f4) and the fact that $\varphi^{-1}(C_1^\circ) \cap \varphi^{-1}(C_2^\circ) = \varphi^{-1}(C_1^\circ \cap C_2^\circ) = \emptyset$ when $C_1^\circ \cap C_2^\circ = \emptyset$.

If $F_*(B) = F^*(B)$ then $\varphi^{-1}(B)$ is Jordan measurable, and $\varphi^{-1}(\partial B)$ is of volume zero; if this happens for all B then the box function $F(B) = v(\varphi^{-1}(B))$ is additive. A useful sufficient condition is given below in terms of functions J^-, J^+ defined by

$$J^-(x) = \liminf_{B \to x} \frac{F_*(B)}{v(B)} \, , \quad J^+(x) = \limsup_{B \to x} \frac{F^*(B)}{v(B)} \, . \tag{8f2}$$

---

[1] Equivalently, $F([s,t]) = v_*(A \cap [s,t])$ where $A = \cup_k [s_k, t_k]$.

[2] In fact, F' is Lebesgue integrable, and its integral is equal to a.

8f3 Proposition. If J^-, J^+ are locally integrable and equivalent then

$$F_*(B) = F^*(B) = \int_B J^- = \int_B J^+$$

for every box B.

In this case[1]

$$v\big(\varphi^{-1}(B)\big) = \int_B J \tag{8f4}$$

where J is *any* function equivalent to J^-, J^+.

8f5 Exercise. Prove existence of J and calculate it for $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by (a) $\varphi(x,y) = x^2 + y^2$; (b) $\varphi(x,y) = \sqrt{x^2+y^2}$; (c) $\varphi(x,y) = |x| + |y|$, taking for granted that the area of a disk is πr^2.

8f6 Exercise. Prove existence of J and calculate it for $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\varphi(x,y,z) = \big(\sqrt{x^2+y^2}, z\big)$, taking for granted Prop. 8b8.

We generalize 8e2, 8e1, 8e4.

8f7 Exercise. For every partition P of a box B,

$$\min_{C \in P} \frac{F_*(C)}{v(C)} \le \frac{F_*(B)}{v(B)} \le \frac{F^*(B)}{v(B)} \le \max_{C \in P} \frac{F^*(C)}{v(C)} \,.$$

Prove it.

8f8 Exercise.

$$v(B) \inf_{x \in B} J^-(x) \le F_*(B) \le F^*(B) \le v(B) \sup_{x \in B} J^+(x) \,.$$

Prove it.

8f9 Exercise.

$${}_*\!\!\int_B J^- \le F_*(B) \le F^*(B) \le {}^*\!\!\int_B J^+ \,.$$

Prove it.[2]

Prop. 8f3 follows immediately.

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[1]Can this happen when $m < n$? If you are intrigued, try the inverse to the mapping of 6g11.

[2]Curiously, the left-hand and the right-hand sides differ thrice: ${}_*\!\int$, ${}^*\!\int$; lim inf, lim sup; v_*, v^*.

8f10 Remark. Similar statements hold for a mapping defined on a subset of $\mathbb{R}^m$ (rather than the whole $\mathbb{R}^m$). If $\varphi : A \to \mathbb{R}^n$ for a given $A \subset \mathbb{R}^m$ then $\varphi^{-1}(B) \subset A$ for every B, but nothing changes in (8f1), (8f2) and Prop. 8f3.

8f11 Remark. If J^-, J^+ are integrable and equivalent on a given box B (and not necessarily on every box) then $v(\varphi^{-1}(C)) = \int_C J$ for every box $C \subset B$.

8f12 Exercise. Calculate J for the projection mapping $\varphi(x, y) = x$ (a) from the disk $A = \{(x, y) : x^2 + y^2 \le 1\} \subset \mathbb{R}^2$ to $\mathbb{R}$; (b) from the annulus $A = \{(x, y) : 1 \le x^2 + y^2 \le 4\} \subset \mathbb{R}^2$ to $\mathbb{R}$. Is J (locally) integrable?

8f13 Exercise. Calculate J for the mapping $\varphi(x) = \sin x$ from the interval $[0, 10\pi] \subset \mathbb{R}$ to $\mathbb{R}$. Is J (locally) integrable?

8g Change of variable in general

8g1 Proposition. If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is such that[1] J^-, J^+ are locally integrable and equivalent then for every integrable $f : \mathbb{R}^n \to \mathbb{R}$ the function $f \circ \varphi : \mathbb{R}^m \to \mathbb{R}$ is integrable and

$$\int_{\mathbb{R}^m} f \circ \varphi = \int_{\mathbb{R}^n} f J \, .$$

Proof. First, the claim holds when $f = 1\!\!1_B$ is the indicator of a box, since

$$\int_{\mathbb{R}^n} f J = \int_B J \stackrel{(8f4)}{=} v(\varphi^{-1}(B)) = \int_{\mathbb{R}^m} 1\!\!1_{\varphi^{-1}(B)} = \int_{\mathbb{R}^m} f \circ \varphi \, .$$

Second, by linearity in f the claim holds whenever f is a step function (on some box, and 0 outside).

Third, given f integrable on a box B (and 0 outside), we consider arbitrary step functions g, h on B such that $g \le f \le h$. We have $g \circ \varphi \le f \circ \varphi \le h \circ \varphi$ and $\int_{\mathbb{R}^m} g \circ \varphi = \int_B gJ$, $\int_{\mathbb{R}^m} h \circ \varphi = \int_B hJ$, thus,

$$\int_B gJ \le {}_*\!\int_{\mathbb{R}^m} f \circ \varphi \le {}^*\!\!\int_{\mathbb{R}^m} f \circ \varphi \le \int_B hJ \, , \quad \int_B gJ \le \int_B fJ \le \int_B hJ \, .$$

We take M such that $|J(\cdot)| \le M$ on B and get

$$\int_B hJ - \int_B gJ = \int_B (h - g)J \le M \int_B (h - g) \, ;$$

thus, integrability of f implies integrability of $f \circ \varphi$ and the needed equality for the integrals. □

[1] We still assume that the inverse image of a box is bounded.

8g2 Corollary. If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is such that J^-, J^+ are locally integrable and equivalent then:

(a) for every Jordan measurable set $E \subset \mathbb{R}^n$ the set $\varphi^{-1}(E) \subset \mathbb{R}^m$ is Jordan measurable;

(b) for every integrable $f : E \to \mathbb{R}$ the function $f \circ \varphi$ is integrable on $\varphi^{-1}(E)$, and

$$\int_{\varphi^{-1}(E)} f \circ \varphi = \int_E f J \,.$$

Proof. (a) apply 8g1 to $f = 1\!\!1_E$; (b) apply 8g1 to $f 1\!\!1_E$. $\square$

8g3 Remark. If $\varphi : A \to \mathbb{R}^n$ is such that J^-, J^+ are integrable and equivalent on a given box B (and not necessarily on every box) then for every integrable $f : B \to \mathbb{R}$ the function $f \circ \varphi$ is integrable on $\varphi^{-1}(B)$, and

$$\int_{\varphi^{-1}(B)} f \circ \varphi = \int_B f J \,.$$

Also, 8g2 holds for $E \subset B$.

8g4 Exercise. (a) Prove that $\int_{x^2+y^2 \le 1} f\big(\sqrt{x^2+y^2}\big) \,\mathrm{d}x\mathrm{d}y = 2\pi \int_{[0,1]} f(r) r \,\mathrm{d}r$ for every integrable $f : [0,1] \to \mathbb{R}$;

(b) calculate $\int_{x^2+y^2 \le 1} \mathrm{e}^{-(x^2+y^2)/2} \,\mathrm{d}x\mathrm{d}y$. (Could you do it by iterated integrals?)

8h Change of variable for a diffeomorphism

8h1 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets and $\varphi : V \to U$ a diffeomorphism, then[1]

$$J^-(x) = J^+(x) = |\det(D\psi)_x|$$

for all $x \in U$; here $\psi = \varphi^{-1} : U \to V$.

Proof. Let $x_0 \in U$. Denote $T = (D\psi)_{x_0}$. By Theorem 6n1, $v(T(E)) = |\det T| v(E)$ for every Jordan measurable $E \subset \mathbb{R}^n$. Note that $\varphi^{-1}(E) = \psi(E)$. It is sufficient to prove that

$$\frac{v_*(\psi(B^\circ))}{v(T(B))} \to 1 \,, \quad \frac{v^*(\psi(B))}{v(T(B))} \to 1 \quad \text{as } B \to x \,.$$

Similarly to Sections 3e, 4c we may assume that $x_0 = 0$, $\psi(x_0) = 0$ and $T = \mathrm{id}$; also, for every $\varepsilon > 0$ we have a neighborhood U_ε of 0 such that

$$(1-\varepsilon)|x_1 - x_2| \le |y_1 - y_2| \le (1+\varepsilon)|x_1 - x_2|$$

[1] $\det D\psi$ is called the Jacobian of ψ and often denoted by J_ψ.

whenever $x_1, x_2 \in U_\varepsilon$ and $y_1 = \psi(x_1)$, $y_2 = \psi(x_2)$. Here $|\cdot|$ is the Euclidean norm; but we can get the same (taking a smaller neighborhood if needed) for an equivalent norm:

$$(1-\varepsilon)\|x_1 - x_2\| \le \|y_1 - y_2\| \le (1+\varepsilon)\|x_1 - x_2\|$$

where

$$\|x\| = \max(|x_1|, \dots, |x_n|) \quad \text{for } x = (x_1, \dots, x_n)\,.$$

That is, $\{x : \|x\| \le r\} = [-r, r]^n$ is a cube.

We may assume that $B \subset U_\varepsilon$ and $\alpha(B) \le 1 + \varepsilon$. Denoting the center of B by x_B we have

$$\|x - x_B\| \le r_B \quad \Longrightarrow \quad x \in B \quad \Longrightarrow \quad \|x - x_B\| \le (1+\varepsilon) r_B$$

for some $r_B > 0$. It is sufficient to prove that

$$(1-\varepsilon)^2 (B - x_B) \subset \psi(B) - y_B \subset (1+\varepsilon)^2 (B - x_B)$$

(where $y_B = \psi(x_B)$), since this implies $(1-\varepsilon)^{2n} v(B) \le v_*(\psi(B)) \le v^*(\psi(B)) \le (1+\varepsilon)^{2n} v(B)$.

On one hand, $\psi(B) - y_B \subset (1+\varepsilon)^2 (B - x_B)$ since

$$x \in B \quad \Longrightarrow \quad \|\psi(x) - y_B\| \le (1+\varepsilon)\|x - x_B\| \le (1+\varepsilon)^2 r_B \quad \Longrightarrow \\ \Longrightarrow \quad \psi(x) - y_B \in (1+\varepsilon)^2 (B - x_B)\,.$$

On the other hand, $(1-\varepsilon)^2 (B - x_B) \subset \psi(B) - y_B$ since

$$y - y_B \in (1-\varepsilon)^2 (B - x_B) \quad \Longrightarrow \\ \Longrightarrow \quad \|\varphi(y) - x_B\| \le \frac{1}{1-\varepsilon}\|y - y_B\| \le (1-\varepsilon)(1+\varepsilon) r_B \le r_B \quad \Longrightarrow \\ \Longrightarrow \quad \varphi(y) \in B \quad \Longrightarrow \quad y - y_B \in \psi(B) - y_B\,.$$

□

We see that J^-, J^+ are integrable and equivalent (moreover, equal and continuous) on every box $B \subset U$. According to 8g2 (and 8g3), for every Jordan measurable $E \subset B$ and integrable $f : E \to \mathbb{R}$,

(8h2) $\psi(E)$ is Jordan measurable,

(8h3) $f \circ \varphi$ is integrable on $\psi(E)$, and $\displaystyle\int_{\psi(E)} f \circ \varphi = \int_E f |\det D\psi|\,.$

Given a compact subset $K \subset U$, we generally cannot cover K by a single box $B \subset U$, but we can cover it by a finite collection of such boxes.

8h4 Lemma. If $U \subset \mathbb{R}^n$ is open and $K \subset U$ is compact then $K \subset B_1 \cup \dots \cup B_k \subset U$ for some boxes $B_1, \dots, B_k$ (and some k).

Proof. The number $\varepsilon = \inf_{x \in K} \operatorname{dist}(x, \mathbb{R}^n \setminus U)$ is not 0, since the function $x \mapsto \operatorname{dist}(x, \mathbb{R}^n \setminus U)$ is continuous (moreover, Lip(1)) on K. For $\delta = \frac{\varepsilon}{2\sqrt{n}}$ each δ-pixel (recall the end of Sect. 6k) intersecting K is contained in U. □

8h5 Corollary. $\psi(E)$ is Jordan measurable whenever $E \subset U$ is a Jordan measurable set contained in a compact subset of U.

Proof. $E \subset B_1 \cup \dots \cup B_k$; sets $\psi(E \cap B_i)$ are Jordan measurable by (8h2); their union $\psi(E)$ is thus Jordan measurable. □

Proposition 8a2 follows immediately. Theorem 8a5 needs a bit more effort.

Given $A = B_1 \cup \dots \cup B_k$ and $f : A \to \mathbb{R}$, can we represent it as $f = f_1 + \dots + f_k$ where each f_i vanishes outside B_i? Yes, we can; such technique is called "partition of unity" and will be used repeatedly in Analysis 4. This time its use is quite trivial, and could be avoided easily, but I do not want to miss a good opportunity to get acquainted with it.

We define functions $\rho_1, \dots, \rho_k : A \to [0, 1]$ by[1]

$$\rho_i(x) = \begin{cases} \frac{1}{\mathbf{1}_{B_1}(x) + \dots + \mathbf{1}_{B_k}(x)} & \text{if } x \in B_i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\rho_1 + \dots + \rho_k = 1$ on A, each ρ_i vanishes outside B_i and is integrable on B_i (just because it is a step function).

Given an integrable $f : A \to \mathbb{R}$, we introduce $f_1 = f\rho_1, \dots, f_k = f\rho_k$; by (8h3), $\int_{\psi(B_i)} f_i \circ \varphi = \int_{B_i} f_i |\det D\psi|$, that is, $\int_{\psi(A)} f_i \circ \varphi = \int_A f_i |\det D\psi|$; the sum over $i = 1, \dots, k$ gives $\int_{\psi(A)} f \circ \varphi = \int_A f |\det D\psi|$. Applying it to $f \mathbb{1}_E$ for a Jordan measurable $E \subset A$ we get

$$\int_{\psi(E)} f \circ \varphi = \int_E f |\det D\psi|$$

for integrable $f : E \to \mathbb{R}$.

In order to get Theorem 8a5 it remains to change notation. First, denote $g = f \circ \varphi$, then $f = g \circ \psi$, and $\int_{\psi(E)} g = \int_E (g \circ \psi) |\det D\psi|$. Second, rename g into f and ψ into φ.

[1] Do you want to propose a simpler construction of $\rho_1, \dots, \rho_k$? Well, you can; but let me exercise the construction that will be reused in less trivial situations in Analysis 4. I intentionally work with arbitrary (not just almost disjoint) boxes.

Proof of Corollary 8a6. Given $\delta > 0$, 6k11 gives us a compact Jordan measurable set $E_1 \subset U$ such that $v(U \setminus E_1) \le \delta$. Similarly, compact $F_1 \subset V$, $v(V \setminus F_1) \le \delta$. By 8a2, $\varphi(E_1)$ and $\varphi^{-1}(F_1)$ are Jordan measurable. Introducing $E = E_1 \cup \varphi^{-1}(F_1)$ and $F = F_1 \cup \varphi(E_1)$ we see that the sets $E \subset U$ and $F \subset V$ are compact, Jordan measurable, $v(U \setminus E) \le \delta$, $v(V \setminus F) \le \delta$ and $F = \varphi(E)$. By 8a5, $\int_F f = \int_E (f \circ \varphi) |\det D\varphi|$.

The inequality

$$\int_{U \setminus E} (f \circ \varphi) |\det D\varphi| \le (\sup_V |f|)(\sup_U |\det D\varphi|)\delta$$

shows that the function $(f \circ \varphi)|\det D\varphi|$ on U is approximated by integrable functions $(f \circ \varphi)|\det D\varphi| 1\!\!1_E$. By Prop. 6d15, the function $(f \circ \varphi)|\det D\varphi|$ is integrable on U, and $\int_U (f \circ \varphi)|\det D\varphi|$ is approximated by $\int_E (f \circ \varphi)|\det D\varphi| = \int_F f$. Also $\int_V f$ is approximated by $\int_F f$. In the limit we get $\int_V f = \int_U (f \circ \varphi)|\det D\varphi|$. □

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