4 Inverse function theorem

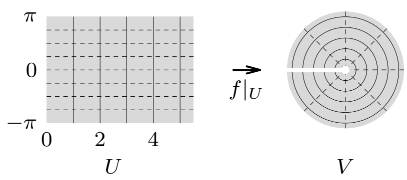
The solution $x = g(y)$ of an equation $f(x) = y$ near a given nondegenerate point is an easy matter in dimension one, but for $f : \mathbb{R}^n \to \mathbb{R}^n$ it means a system of n (nonlinear) equations in n unknowns. Still, under appropriate conditions, the inverse mapping to a continuously differentiable mapping is continuously differentiable. An iterative process converges to the solution.

4a What is the problem

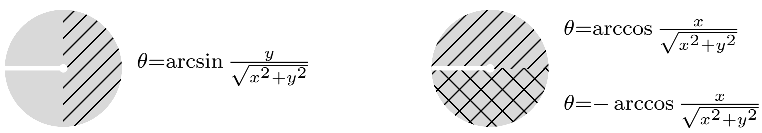
Recall the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(r, \theta) = (r \cos \theta, r \sin \theta) \,,$$

treated in 2e6. It is not one-to-one, since $f(r, \theta + 2\pi) = f(r, \theta)$ and $f(-r, \theta + \pi) = f(r, \theta)$. However, its restriction to the open set $U = (0, \infty) \times (-\pi, \pi)$ is one-to-one, and $f(U)$ is the open set $V = \mathbb{R}^2 \setminus (-\infty, 0] \times \{0\}$. Thus, $(f|_U)^{-1} : V \to U$. By 2e6, f is differentiable on U. We wonder, is $(f|_U)^{-1}$ differentiable on V?



The first coordinate $r = \sqrt{x^2 + y^2}$ of $(f|_U)^{-1}(x, y)$ evidently is differentiable on V. The second coordinate θ is differentiable on V by the argument used in 2b18(b):



However, this is just good luck. In general, the inverse mapping is not a combination of well-known functions. (Not even in dimension one; try for instance to find x from $x^5 + x = y$, or $x + \mathrm{e}^x = y$.) Can we deduce differentiability of f^{-1} from differentiability of f?

Of course, we need a multidimensional theory; $\mathbb{R}^2$ is only the simplest case.

4b Simple observations before the theorem

It is not a problem to differentiate the inverse mapping *assuming* that it is differentiable. By the chain rule,

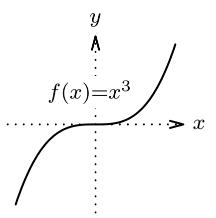
$$(Df^{-1})_{f(x_0)} \circ (Df)_{x_0} = \big(D(f^{-1} \circ f)\big)_{x_0} = I \,,$$

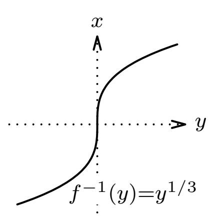
therefore

$$(Df^{-1})_{y_0} = \big((Df)_{f^{-1}(y_0)}\big)^{-1} \,.$$

The same argument shows that f^{-1} cannot be differentiable at $f(x_0)$ if the operator $(Df)_{x_0}$ is not invertible. (Recall also 2b13(b): x and y must be of the same dimension.)

It can happen that $(Df)_{x_0}$ is not invertible and nevertheless f is invertible. Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, $x_0 = 0$.

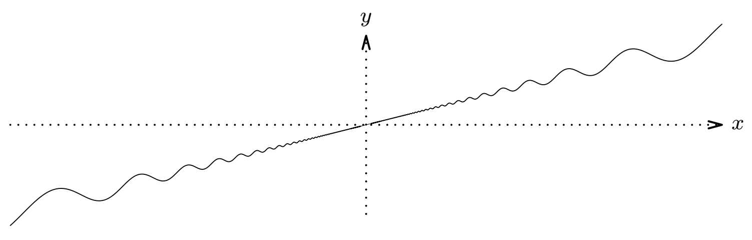




If $f : \mathbb{R} \to \mathbb{R}$ is such that the operator $(Df)_x$ is invertible (that is, $f'(x) \ne 0$) for all x then f is one-to-one (think, why). This is not the case for $f : \mathbb{R}^2 \to \mathbb{R}^2$. Example: $f(x, y) = (\mathrm{e}^x \cos y, \mathrm{e}^x \sin y)$.

Thus we turn to the local problem: the germ of f at x_0 is given, and we examine the germ of f^{-1} at $y_0 = f(x_0)$.

It can happen that f is differentiable near x_0 and $(Df)_{x_0}$ is invertible, but f is not one-to-one near x_0. Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 3x^2 \sin \frac{1}{x}$ for $x \ne 0$, $f(0) = 0$, $x_0 = 0$.[1]



[1]Hint: consider $f'(x)$ for $x \to 0$.

Thus, we assume that f is *continuously* differentiable near x_0. That is, $x \mapsto (Df)_x$ is continuous near x_0. It follows that $x \mapsto \big((Df)_x\big)^{-1}$ is continuous near x_0, see Exercise 4b1 below. Now, *assuming* again that the inverse mapping is differentiable (and therefore continuous) we see that it must be continuously differentiable, since $y \mapsto \big((Df)_{f^{-1}(y)}\big)^{-1}$ is continuous.

4b1 Exercise. If $A, A_n \in M_{n,n}(\mathbb{R})$, $A_n \to A$, and A is invertible then A_n is invertible for all n large enough, and $A_n^{-1} \to A^{-1}$.

Prove it.[1]

4c The theorem

4c1 Theorem. Assume that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable near x_0, and the operator $(Df)_{x_0}$ is invertible. Then there exists an open neighborhood U of x_0 and an open neighborhood V of $y_0 = f(x_0)$ such that $f|_U$ is a homeomorphism $U \to V$, continuously differentiable on U, and the inverse mapping $(f|_U)^{-1} : V \to U$ is continuously differentiable on V.

4c2 Remark. The equality

$$(Dg)_{y_0} = \big((Df)_{x_0}\big)^{-1}$$

for $g = (f|_U)^{-1}$ is often included into this theorem. However, it is just an immediate implication of the chain rule, as noted in Sect. 4b. Moreover, $(Dg)_y = \big((Df)_x\big)^{-1}$ whenever $x \in U$, $y \in V$, $y = f(x)$.

4c3 Remark. $\mathbb{R}^n$ may be replaced with an arbitrary n-dimensional vector or affine space. Likewise, the theorem applies to $f : S_1 \to S_2$ for two n-dimensional affine spaces S_1, S_2; in this case $x_0 \in S_1$, $y_0 \in S_2$, $(Df)_{x_0} : \vec{S}_1 \to \vec{S}_2$ and $(Dg)_{y_0} = \big((Df)_{x_0}\big)^{-1} : \vec{S}_2 \to \vec{S}_1$.

4c4 Remark. Only the germ of f at x_0 is relevant. Thus, the theorem may be applied to a function defined on a neighborhood of x_0 (rather than the whole $\mathbb{R}^n$). But never forget: U is generally smaller than the given neighborhood. In contrast, the next result applies to the whole given U.

4c5 Theorem. Assume that $U, V \subset \mathbb{R}^n$ are open, $f : U \to V$ is a homeomorphism, continuously differentiable, and the operator $(Df)_x$ is invertible for all $x \in U$. Then the inverse mapping $f^{-1} : V \to U$ is continuously differentiable.

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[1]Hint. One way: use determinants. Another way: first, reduce the general case to the special case $A = I$ (via $A^{-1}A_n \to I$); second, prove that $\|A^{-1} - I\| \le \frac{\|A-I\|}{1-\|A-I\|}$ whenever $\|A - I\| < 1$ (via the triangle inequality).

Proof of Theorem 4c1 given Theorem 4c5. Prop. 3b9 and Lemma 3b8 (or Prop. 3b7) provide open sets $U \ni x_0$ and $V \ni f(x_0)$ satisfying the conditions of Theorem 4c5. By Theorem 4c5 these U, V satisfy the conclusion of Theorem 4c1. $\square$

Proof of Theorem 4c5. Let $x_0 \in U$, $y_0 = f(x_0) \in V$; it is sufficient to prove that the mapping $g = f^{-1}$ is differentiable at y_0. (Continuity of Dg follows, see the end of Sect. 4b.)

Similarly to Sect. 3e we reduce the general case to a special case: $x_0 = 0$, $y_0 = 0$, and $(Df)_0 = \mathrm{id}$. Similarly to the proof of Prop. 3e2 we have U_ε such that

$$\begin{gathered} |(f(x) - f(y)) - (x - y)| \le \varepsilon |x - y| \,, \\ (1-\varepsilon)|x-y| \le |f(x) - f(y)| \le (1+\varepsilon)|x-y| \end{gathered} \qquad \text{for all } x, y \in U_\varepsilon \,,$$

and in particular (for $y = 0$),

$$\begin{gathered} |f(x) - x| \le \varepsilon |x| \,, \\ (1-\varepsilon)|x| \le |f(x)| \le (1+\varepsilon)|x| \end{gathered} \qquad \text{for all } x \in U_\varepsilon \,.$$

The set $V_\varepsilon = f(U_\varepsilon)$ is an open neighborhood of y_0 (recall 3b3), and

$$\begin{gathered} |y - g(y)| \le \varepsilon |g(y)| \,, \\ (1-\varepsilon)|g(y)| \le |y| \le (1+\varepsilon)|g(y)| \end{gathered} \qquad \text{for all } y \in V_\varepsilon \,.$$

Therefore

$$|g(y) - y| \le \frac{\varepsilon}{1-\varepsilon} |y| \quad \text{for all } y \in V_\varepsilon \,.$$

We see that $g(y) = y + o(|y|)$, that is, id is the derivative of g at y_0. $\square$

4c6 Remark. The equality $(Dg)_{y_0} = \big((Df)_{x_0}\big)^{-1}$ was known before, but also follows from the proof above.

4c7 Remark. Continuity of Dg was known before, but also follows readily from the arguments of the proof above, as follows. For all $x_1, x_2 \in U_\varepsilon$,

$$\begin{gathered} |(f(x_1) - f(x_2)) - (x_1 - x_2)| \le \varepsilon |x_1 - x_2| \,, \\ (1-\varepsilon)|x_1 - x_2| \le |f(x_1) - f(x_2)| \le (1+\varepsilon)|x_1 - x_2| \,; \\ |(f(x_1) - f(x_2)) - (x_1 - x_2)| \le \frac{\varepsilon}{1-\varepsilon} |f(x_1) - f(x_2)| \,; \end{gathered}$$

therefore for all $y_1, y_2 \in V_\varepsilon$

$$|(g(y_2) - g(y_1)) - (y_2 - y_1)| \le \frac{\varepsilon}{1-\varepsilon} |y_2 - y_1| \,.$$

On the other hand,

$$g(y_1 + h) - g(y_1) = (Dg)_{y_1}(h) + o(|h|)\,.$$

It follows that

$$|(Dg)_{y_1}(h) - h| \le \frac{\varepsilon}{1-\varepsilon}|h| + o(|h|)\,;$$

$|(Dg)_{y_1}(h) - h| \le \frac{2\varepsilon}{1-\varepsilon}|h|$ for all h near 0, therefore for all h;

$$\|(Dg)_{y_1} - \mathrm{id}\| \le \frac{2\varepsilon}{1-\varepsilon} \quad \text{for all } y_1 \in V_\varepsilon\,;$$

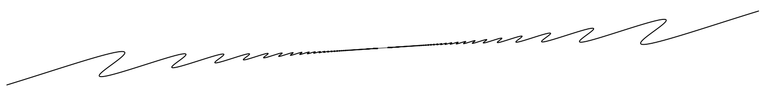
$(Dg)_{y_1} \to (Dg)_{y_0}$ as $y_1 \to y_0$.

4c8 Remark. We see that continuity of the map $A \mapsto A^{-1}$ is not necessarily used when proving the inverse function theorem. Curiously enough, the former can be deduced from the latter. To this end, consider the inverse to a mapping $(A, x) \mapsto (A, Ax)$ from $M_{n,n}(\mathbb{R}) \times \mathbb{R}^n$ to itself. It gives not only continuity of the map $A \mapsto A^{-1}$ (on the open set of all invertible matrices) but also its continuous differentiability. But this is not a revelation: elements of A^{-1} are just rational functions (that is, fractions of polynomials) of the elements of A. (See also 2e7.)

4c9 Exercise. (a) Let $f : U \to V$ be as in Theorem 4c5 and in addition $f \in C^2(U)$ (recall Sect. 2g). Prove that $f^{-1} \in C^2(V)$.[1]

(b) The same for $C^k(\dots)$ where $k = 3, 4, \dots$

4c10 Remark. Now we see that the paths γ used in Sect. 3f, 3g are not just continuous, they are continuously differentiable, which resolves the doubt of 3f3(a). In relation to 3f3(b) one may guess that a small ball must contain a connected portion of the path. This need not hold for a continuous path in general, not even for a differentiable path.



However, it holds for a continuously differentiable path (see 4c11).[2]

4c11 Exercise. Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ be continuously differentiable. Prove that the set $\{t \in (-\varepsilon, \varepsilon) : |\gamma(t) - \gamma(0)| < r\}$ is an interval provided that r is small enough.[3]

4c12 Exercise. Let $\psi : U \to V$ be as in 3e3(b). Prove that ψ is continuously differentiable.

[1] Hint: $(Dg)_y = ((Df)_{g(y)})^{-1}$ where $g = f^{-1}$.

[2] A similar fact for surfaces is beyond our course.

[3] Hint: $\frac{\mathrm{d}}{\mathrm{d}t}|\gamma(t) - \gamma(0)|^2 = 2\langle \gamma'(t), \gamma(t) - \gamma(0)\rangle \approx t|\gamma'(0)|^2$.

4d Iterations

We know that (under appropriate conditions) the solution x of the equation $f(x) = y$ exists and is unique. How to compute x numerically?

Taking into account that y is close to $y_0 = f(x_0)$, x must be close to x_0, and the operator $T = (Df)_{x_0}$ is invertible, we guess that

$$y = f(x) = f(x_0 + (x - x_0)) \approx y_0 + T(x - x_0)\,,$$

and hopefully,

$$x \approx x_0 + T^{-1}(y - y_0) = x_0 + T^{-1}(y - f(x_0))\,.$$

We iterate this operation,

$$x_{n+1} = x_n + T^{-1}\big(y - f(x_n)\big) \quad \text{for } n = 0, 1, 2, \dots$$

and hope that $x_n \to x$.

These iterations are well-defined for a mapping $f : S_1 \to S_2$ between affine spaces (as in 4c3). Choosing appropriate coordinates we return to the special case treated in the proof of 4c5: $f : \mathbb{R}^n \to \mathbb{R}^n$, $x_0 = 0$, $y_0 = 0$, and $T = \mathrm{id}$. Now we may use neighborhoods U_ε and the related inequalities. Also, the iterations become just $x_{n+1} = x_n + y - f(x_n)$, that is,

$$x_{n+1} - x_n = y - f(x_n)\,.$$

In particular, $x_1 - x_0 = y - y_0$. That is, $x_1 = y$.

We have

$$|(f(x_1) - f(x_2)) - (x_1 - x_2)| \le \varepsilon |x_1 - x_2|$$

for all $x_1, x_2 \in U_\varepsilon$. Assuming (for now) that $x_n \in U_\varepsilon$ for all n we get for all $n > 0$

$$\begin{aligned} |x_{n+1} - x_n| &= |(x_n + y - f(x_n)) - (x_{n-1} + y - f(x_{n-1}))| = \\ &= |(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1}))| \le \varepsilon |x_n - x_{n-1}|\,, \end{aligned}$$

therefore for all $n \ge 0$,

$$|x_{n+1} - x_n| \le \varepsilon^n |x_1| = \varepsilon^n |y|\,;$$
$$|x_{n+k} - x_n| \le (\varepsilon^n + \varepsilon^{n+1} + \dots)|x_1| = \frac{\varepsilon^n}{1 - \varepsilon}|y|\,;$$

we see that x_n are a Cauchy sequence, thus the limit $x = \lim_n x_n$ must exist, and

$$|x - x_n| \le \frac{\varepsilon^n}{1 - \varepsilon}|y|\,.$$

Also,
$$|y - f(x_n)| = |x_{n+1} - x_n| \le \varepsilon^n |y| \, ,$$
which implies $f(x) = y$, provided that $x \in U_\varepsilon$.

There exists $r > 0$ such that U_ε contains the open r-ball centered at 0. Assuming $|y| < (1-\varepsilon)r$ we get
$$|x_n| \le \frac{1}{1-\varepsilon}|y| < r$$
(since $|x_k| = |x_{0+k} - x_0| \le \frac{\varepsilon^0}{1-\varepsilon}|y|$), which ensures that U_ε contains $x_1, x_2, \dots$ and x.

We summarize.

4d1 Proposition. Assume that $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable near x_0, Df is continuous at x_0, and the operator $T = (Df)_{x_0}$ is invertible. Then for every y near $y_0 = f(x_0)$ the iterative process
$$x_{n+1} = x_n + T^{-1}\big(y - f(x_n)\big) \quad \text{for } n = 0, 1, 2, \dots$$
is well-defined and converges to a solution x of the equation $f(x) = y$. In addition, $|x - x_0| = O(|y - y_0|)$.

4d2 Remark. The proof of Prop. 4d1 does not use results of Sect. 3c, 3d. Thus, Prop. 4d1 is an alternative way[1] toward Lemma 3b8 and Prop. 3b7.

[1]The third way, if we count also the invariance of domain mentioned in Sect. 3b.