1 Euclidean space $\mathbb{R}^n$

1a	**Prerequisites: linear algebra**	**1**
1b	**Prerequisites: topology**	**2**
1c	**The joy of spaces**	**6**
1d	**Linearity and continuity**	**9**
1e	**Norms of vectors and operators**	**10**
1f	**Paths and connectedness**	**14**

The n-dimensional space $\mathbb{R}^n$ may be treated as a Euclidean space, or just a vector space, etc. Its topology is uniquely determined by its algebraic structure.

1a Prerequisites: linear algebra

You should know the notion of: *Forgot? Then see:*

Vector space (=linear space) [Sh:p.26 "Vector space axioms"]
Isomorphism of vector spaces: a linear bijection.
Basis of a vector ^{fd}space [Sh:Def.2.1.2 on p.28]
Dimension of a vector ^{fd}space: the number of vectors in every basis.
Two vector ^{fd}spaces are isomorphic if and only if their dimensions are equal.
Subspace of a vector space.
Linear operator (=mapping=function) between vector spaces [Sh:3.1]
Inner product on a vector space: $\langle x, y\rangle$ [Sh:p.31 "Inner product properties"]

A basis of a subspace, being a linearly independent system, can be extended to a basis of the whole vector ^{fd}space.

1a1 Definition. A *Euclidean vector space* consists of a vector space and an inner product.[1]

Isomorphism of Euclidean vector spaces: an isometric linear bijection.

On a Euclidean vector space:

Euclidean norm (=modulus=abs. value) of vector: $|x| = \sqrt{\langle x, x\rangle}$ [Sh:p.32]
The Cauchy-Schwartz inequality: $-|x||y| \le \langle x, y\rangle \le |x||y|$. [Sh:Th.2.2.5]
The triangle inequality: $|x + y| \le |x| + |y|$. [Sh:Th.2.2.6]

[1] I dislike this definition, since it does not exclude the pathology of incomplete infinite-dimensional Euclidean spaces. Fortunately, in this course we need only finite dimension.

Every Euclidean ^{fd}space has an orthonormal basis.
Two Euclidean ^{fd}spaces are isomorphic if and only if their dimensions are equal.
A subspace of a Euclidean space is another Euclidean space.

An orthonormal basis of a subspace can be extended to an orthonormal basis of the whole Euclidean ^{fd}space.

1a2 Proposition. Every linear operator from one Euclidean vector ^{fd}space to another sends some orthonormal basis of the first space into an orthogonal system in the second space.

This is called the Singular Value Decomposition.[1]

1a3 Exercise. Let a vector ^{fd}space be endowed with two Euclidean metrics. Then it contains a basis orthonormal in the first metric and orthogonal in the second metric.

Prove it.[2]

1a4 Exercise. Let V_1, V_2 be vector spaces and $T : V_1 \to V_2$ a linear bijection. Prove that $T^{-1} : V_2 \to V_1$ is linear.

1a5 Exercise. Prove that every isomorphism between two vector spaces preserves all the notions introduced for vector spaces: basis, dimension, subspace, inner product.

1b Prerequisites: topology

You should know the notion of:	*Forgot? Then see:*
A sequence of points of $\mathbb{R}^n$	[Sh:p.36][3]
Its convergence, limit	[Sh:p.42–43]

[1]See: Todd Will, "Introduction to the Singular Value Decomposition", `http://www.uwlax.edu/faculty/will/svd/index.html` *Quote:*

The Singular Value Decomposition (SVD) is a topic rarely reached in undergraduate linear algebra courses and often skipped over in graduate courses.

Consequently relatively few mathematicians are familiar with what M.I.T. Professor Gilbert Strang calls "absolutely a high point of linear algebra."

[2]Hint: $E_1 = (V, \langle\cdot,\cdot\rangle_1)$, $E_2 = (V, \langle\cdot,\cdot\rangle_2)$; apply 1a2 to the identity operator $E_1 \to E_2$.

[3]*Quote:* The only obstacle ... is notation ... n already denotes the dimension of the Euclidean space where we are working; and furthermore, the vectors can't be denoted with subscripts since a subscript denotes a component of an individual vector. ... As our work with vectors becomes more intrinsic, vector entries will demand less of our attention, and we will be able to denote vectors by subscripts.

More quote (p. 64–65): The author does not know any graceful way to avoid this notation collision, the systematic use of boldface or arrows to adorn vector names being

Mapping $\mathbb{R}^m \to \mathbb{R}^n$; continuity (at a point; on a set)	[Sh:p.41–48]
Subsequence; Bolzano-Weierstrass theorem	[Sh:p.52–53]
Subset of $\mathbb{R}^n$, its limit points; closed set; bounded set	[Sh:p.51]
Compact set	[Sh:p.54]
Open set	[Sh:p.191][1]
Closure, boundary, interior	[Sh:p.311,314]
Open cover; Heine-Borel theorem	[Sh:p.312]
Open ball, closed ball, sphere	[Sh:p.50,191–192]
Open box, closed box	[Sh:p.246]

[Sh:Exer. 2.3.8–2.3.11, 2.4.1–2.4.8]

1b1 Exercise. For a function $f : (0,\infty) \times (0,\infty) \to \mathbb{R}$ defined by $f(x,y) = y\sin(1/x)$ prove that the limits

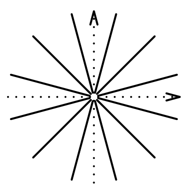
$$\lim_{(x,y)\to(0,0),x>0,y>0} f(x,y) \quad \text{and} \quad \lim_{x\to0+}\lim_{y\to0+} f(x,y)$$

exist and equal 0, but the second iterated limit

$$\lim_{y\to0+}\lim_{x\to0+} f(x,y)$$

does not exist.

1b2 Exercise.



Consider functions $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ constant on all rays from the origin; that is, $f(r\cos\varphi, r\sin\varphi) = h(\varphi)$ for some $h : \mathbb{R} \to \mathbb{R}$, $h(\varphi + 2\pi) = h(\varphi)$. Assume that h is continuous.

(a) Prove that the iterated limits

$$\lim_{x\to0+}\lim_{y\to0+} f(x,y) \quad \text{and} \quad \lim_{y\to0+}\lim_{x\to0+} f(x,y)$$

exist and are equal to $h(0)$ and $h(\pi/2)$ respectively.

(b) prove that the "full" limit

$$\lim_{(x,y)\to(0,0),x>0,y>0} f(x,y)$$

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heavyhanded, and the systematic use of the Greek letter ξ rather than its Roman counterpart x to denote scalars being alien. Since mathematics involves finitely many symbols and infinitely many ideas, the reader will in any case eventually need the skill of discerning meaning from context, a skill that may as well start receiving practice now.

[1] *Quote:* A set, however, is not a door: it can be neither open or closed, and it can be both open and closed. (Examples?)

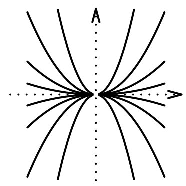
exists if and only if h is constant on $[0, \pi/2]$.

(c) It can happen that the two iterated limits exist and are equal, but the "full" limit does not exist. Give an example.

(d) The same as (c) and in addition, f is a rational function (that is, the ratio of two polynomials).[1]

(e) Generalize all that to arbitrary (not just positive) x, y.

1b3 Exercise.
Consider functions $g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ of the form $g(x,y) = f(x^2, y)$ where f is as in 1b2.



(a) Prove that the limit

$$\lim_{t \to 0+} g(ta, tb)$$

exists for every $(a,b) \neq (0,0)$; calculate the limit in terms of the function h of 1b2.

(b) It can happen that the "full" limit

$$\lim_{(x,y) \to (0,0)} g(x,y)$$

does not exist. Give an example.

1b4 Exercise. *"Componentwise nature of continuity"* Prove or disprove: a mapping $f : \mathbb{R} \to \mathbb{R}^n$ is continuous if and only if each coordinate function $f_k : \mathbb{R} \to \mathbb{R}$ is continuous; here $f(x) = \big(f_1(x), \dots, f_n(x)\big)$. [Sh:Th.2.3.9]

1b5 Exercise. Prove or disprove: a mapping $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous if and only if it is continuous in each coordinate separately; that is, $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for every x, and $f(\cdot, y) : \mathbb{R} \to \mathbb{R}$ is continuous for every y.

1b6 Exercise. Prove the Bolzano-Weierstrass theorem and the Heine-Borel theorem.

1b7 Exercise. (a) Prove that finite union of closed sets is closed, but union of countably many closed sets need not be closed; moreover, every open set in $\mathbb{R}^n$ is such union. However, intersection of closed sets is always closed.

(b) Formulate and prove the dual statement (take the complement).

1b8 Exercise. Prove that a set $K \subset \mathbb{R}^n$ is compact if and only if *every* continuous function $f : K \to \mathbb{R}$ is bounded.

[1]Hint: try $x^2 + y^2$ in the denominator.

1b9 Exercise. Prove that a continuous image of a compact set is compact, but a continuous image of a bounded set need not be bounded, and a continuous image of a closed set need not be closed; moreover, every open set in $\mathbb{R}^n$ is a continuous image of a closed set.[1]

1b10 Exercise. Prove that every decreasing sequence of nonempty compact sets has a nonempty intersection. Does it hold for closed sets? for open sets?

1b11 Exercise. Let $K \subset \mathbb{R}^n$ be compact, and $f : K \to \mathbb{R}^m$ continuous. Prove that f is uniformly continuous, that is,
$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, y \in K \; \big(|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon \big)$.

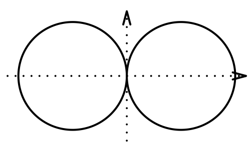
1b12 Exercise. Let $X \subset \mathbb{R}^n$ be a closed set, $f : X \to \mathbb{R}^m$ a continuous mapping. Prove that its *graph* $\Gamma_f = \{(x, f(x)) : x \in X\}$ is a closed subset of $\mathbb{R}^{n+m}$. Is the converse true?

1b13 Exercise. Prove existence of a bijection f from the open unit ball $B(0,1) \subset \mathbb{R}^n$ onto the whole $\mathbb{R}^n$ such that f and f^{-1} are continuous. (Such mappings are called *homeomorphisms*). What about the closed ball?

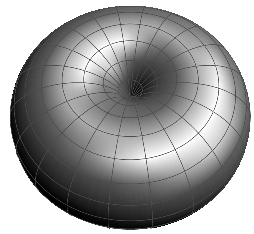
1b14 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bijection. Prove that $f^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous.

1b15 Exercise. Give an example of a continuous bijection $f : [0,1) \to S^1 = \{(x,y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ such that $f^{-1} : S^1 \to [0,1)$ fails to be continuous. The same for $f : [0,\infty) \to S^1$.

1b16 Exercise. Give an example of a continuous bijection $f : \mathbb{R} \to A = \{(x,y) : (|x| - 1)^2 + y^2 = 1\} \subset \mathbb{R}^2$ such that $f^{-1} : A \to \mathbb{R}$ fails to be continuous.



1b17 Exercise. Give an example of a continuous bijection $f : \mathbb{R}^2 \to B = \{(x,y,z) : (\sqrt{x^2 + y^2} - 1)^2 + z^2 = 1\} \subset \mathbb{R}^3$ such that $f^{-1} : B \to \mathbb{R}^2$ fails to be continuous.[2]



[1] Hint: the closed set need not be connected.

[2] What about a continuous bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$? In fact, f^{-1} is continuous, which can be proved using powerful means of topology (the Jordan curve theorem).

1c The joy of spaces [1]

To a mathematician, the word space *doesn't connote volume but instead refers to a set endowed with some structure.*[2] [Sh:p.24]

An affine space is nothing more than a vector space whose origin we try to forget about.
Marcel Berger, "Geometry I", p. 32.

Let V be a vector space.

1c1 Definition. An *affine space* with the difference space V consists of a set S and a map $V \times S \to S$ denoted $(v, a) \mapsto v + a$ such that for every $a \in S$

$$\begin{gathered} 0 + a = a\,; \\ v + (w + a) = (v + w) + a \quad \text{for all } v, w \in V\,; \\ \text{the map} \quad V \ni v \mapsto v + a \in S \quad \text{is bijective.} \end{gathered}$$

Here is an equivalent definition ("Weyl's axioms").[3]

1c2 Definition. An *affine space* with the difference space V consists of a set S and a map $S \times S \to V$ denoted $(a, b) \mapsto b - a$ such that

$$\begin{gathered} (c - b) + (b - a) = c - a \quad \text{for all } a, b, c \in S\,; \\ \forall a \in S \; \forall v \in V \; \exists!\, b \in S \quad b - a = v\,. \end{gathered}$$

The difference space of an affine space S is often denoted by $\vec{S}$.

1c3 Example. Given a continuous function $g : \mathbb{R} \to \mathbb{R}$, consider the set $S_1 = \{f : f' = g\}$ of all its antiderivatives (that is, indefinite integrals). This S_1 is an affine space with the difference space $P_0 = \{f : f' = 0\}$ of all constant functions.

More generally, $S_n = \{f : f^{(n)} = g\}$ is an affine space with the difference space $P_{n-1} = \{f : f^{(n)} = 0\}$ of all polynomials of degree (at most) $n - 1$.

1c4 Exercise. Fill in the details in 1c3. What about a more general linear differential equation?

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[1] Additional sources:
"Affine and Euclidean geometry", chapter II of a course in Madrid Politech. Univ.;
"Basics of Euclidean geometry", chapter 6 of the book: J. Gallier, "Geometric Methods and Applications" (pdf or djvu);
"Vector spaces, affine spaces, and metric spaces", chapter 2 of the book: Bærentzen, J. Gravesen, F. Anton, H. Aanæs, "Guide to Computational Geometry Processing";
chapters 1, 2 of the book: M. Audin, "Geometry".

[2] If you wonder why, see "Space (mathematics)" in Wikipedia.

[3] For several other equivalent definition see nLab.

1c5 Exercise. (a) Define isomorphism between affine spaces. Is it uniquely determined by the corresponding isomorphism between their difference spaces?

(b) Define the dimension of an affine ^{fd}space.

(c) Prove that two affine ^{fd}spaces are isomorphic if and only if their dimensions are equal.

(d) Define an affine subspace of an affine space. Is it uniquely determined by the corresponding subspace of the difference space? What about an affine subspace of a vector space?

(e) Recall S_n of 1c3. Is it finite-dimensional? What is its dimension? Check that it is a hyperplane in the vector space $B_n = \{f : \exists c \in \mathbb{R} \;\; f^{(n)} = cg\}$ (unless $g = 0$).

1c6 Exercise. Let S_1 be an affine plane (that is, 2-dimensional affine space), and $a_1, b_1, c_1 \in S_1$ not on a line (a line being a 1-dimensional affine subspace). Let the same hold for S_2 and $a_2, b_2, c_2 \in S_2$. Prove that one and only one isomorphism between S_1 and S_2 sends a_1 to a_2, b_1 to b_2 and c_1 to c_2.

Thus, up to isomorphism there is only one affine plane ("*the* affine plane") and only one triangle on it!

1c7 Definition. An *Euclidean affine space* is an affine space whose difference space is a Euclidean vector space (that is, endowed with a Euclidean metric).[1]

On *the* affine Euclidean plane triangles differ (up to isomorphism); you know, some are right-angled, acute-angled, obtuse-angled, isosceles, equilateral etc. Nothing like this can happen on *the* affine plane.

Recall a result from Euclidean geometry: the three bisectors of a triangle intersect [Sh:p.27]. Can we define bisector(s) on the affine plane? Yes, we can! The given Euclidean metric is irrelevant. We can simplify the task in two ways.

First way: work on the affine plane. No lengths, no angles. The smaller the labyrinth of possible arguments, the easier to find a proof.

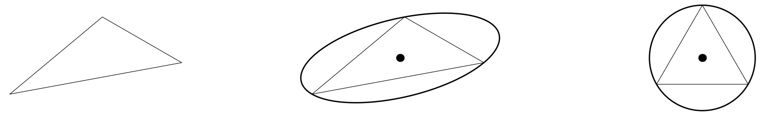
Second way: restrict yourself to equilateral triangles. That is, replace the given (irrelevant) Euclidean metric with another (relevant) Euclidean metric that makes the given triangle equilateral.

This is instructive.

Irrelevant structure is a nuisance. Downgrade the given structure as far as possible. A relevant structure may help. Try to upgrade the structure according to the given situation.

[1]Surely, Euclid himself did not treat some point as "the origin" of the space. Also, for him a length (or distance) was not a number; rather, the ratio of two lengths was a number.

1c8 Exercise. Given a triangle abc on an affine plane, upgrade the plane to a (vector, not affine) Euclidean plane such that $|a-b| = |b-c| = |c-a|$, $|a| = |b| = |c|$ and $a + b + c = 0$.

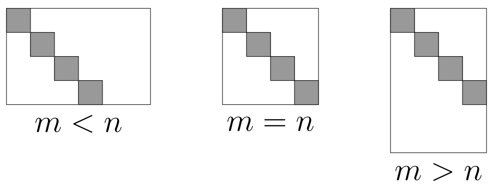


Also the three altitudes of a triangle intersect [Sh:p.37–38]. In this case Euclidean metric is relevant.

The space $\mathbb{R}^n$ is more than just a Euclidean vector space; it is a Euclidean vector space endowed with an orthonormal basis (or equivalently, Cartesian coordinates). And conversely, an n-dimensional Euclidean vector space endowed with an orthonormal basis is *canonically* isomorphic to $\mathbb{R}^n$.[1] Linear operators on such space (or from one such space to another) correspond bijectively *and canonically* to $n \times n$ matrices.

The Singular Value Decomposition 1a2 may be reformulated as follows.

1c9 Proposition. Every linear operator from an n-dimensional Euclidean vector space to an m-dimensional Euclidean vector space has a diagonal $m \times n$ matrix in some pair of orthonormal bases.[2]



In particular, this holds for every linear operator $\mathbb{R}^n \to \mathbb{R}^n$. It does not mean that every matrix is diagonalizable! Two bases give much more freedom than one basis.

This is instructive.

Whenever possible, downgrade a single space to a pair of spaces.

In other words: downgrade *canonically* isomorphic spaces to (just) isomorphic spaces.

We'll prove theorems much harder than the result about the three bisectors. The help of spaces will be relatively small. Still, even a relatively small simplification of a difficult proof should not be missed.

[1] Thus we feel comfortable saying that it *is* $\mathbb{R}^n$...

[2] Absolute values of the numbers on the diagonal of this matrix are well-known as *singular values* of the operator T; they are square roots of the eigenvalues of the operator T^*T, and do not depend (up to permutation) on the choice of the pair of bases.

1c10 Exercise. For every linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$ deduce the relation

$$\dim T(\mathbb{R}^n) + \dim T^{-1}(0) = n$$

from 1c9. Generalize to arbitrary pair of vector ^{fd}spaces.

1d Linearity and continuity

The general form of a linear function $f : \mathbb{R}^n \to \mathbb{R}$ [Sh:p.65]:

$$f(x) = \langle a, x \rangle \quad \text{for some } a \in \mathbb{R}^n \, .$$

Such f is continuous [Sh:p.65] (being a linear combination of coordinate functions).

The general form of a linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$ [Sh:p.68]:

$$T(x) = Ax \quad \text{for some } m \times n \text{ matrix } A \, .$$

Such T is continuous [Sh:Th.3.1.5] (since each coordinate of Tx is a linear function of x).

Affine function $f : \mathbb{R}^n \to \mathbb{R}$:

$$f(x) = \langle a, x \rangle + t \quad \text{for some } a \in \mathbb{R}^n, \, t \in \mathbb{R} \, .$$

Affine operator $T : \mathbb{R}^n \to \mathbb{R}^m$:

$$T(x) = Ax + b \quad \text{for some } m \times n \text{ matrix } A \text{ and some } b \in \mathbb{R}^m \, .$$

Such f and T are continuous.

Thus, every linear (as well as affine) bijection $T : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism (that is, T and T^{-1} are continuous).

1d1 Exercise. Prove that every homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ preserves all the topological notions introduced in Sect. 1b: convergence of sequence, limit point, closed set, bounded set,[1] compact set, open set, closure, boundary, interior.

Given an n-dimensional vector space V, we choose a linear bijection $T : V \to \mathbb{R}^n$ (equivalently, a basis of V, basically, coordinates on V) and transfer all topological notions from $\mathbb{R}^n$ to V via T. For instance, ($x_n \to x$ in V) $\iff$ ($T(x_n) \to T(x)$ in $\mathbb{R}^n$); (K is compact in V) $\iff$ ($T(K)$ is compact in $\mathbb{R}^n$). The choice of T does not matter. Here is why. Let

[1] "Bounded set" is generally not a topological notion; but in $\mathbb{R}^n$ it is equivalent to the notion "subset of a compact set".

$T_1, T_2 : V \to \mathbb{R}^n$ be linear bijections, then $T_2 T_1^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear bijection, therefore a homeomorphism. Thus, ($T_1(x_n) \to T_1(x)$) $\iff$ ($(T_2 T_1^{-1}) T_1(x_n) \to (T_2 T_1^{-1}) T_1(x)$) $\iff$ ($T_2(x_n) \to T_2(x)$); also, ($T_1(K)$ is compact) $\iff$ ($(T_2 T_1^{-1}) T_1(K)$ is compact) $\iff$ ($T_2(K)$ is compact); etc.

Topological notions are well-defined on every vector (as well as affine) *fd* space.

Every linear bijection between vector *fd* spaces is a homeomorphism. The same holds for an affine bijection between affine *fd* spaces.[1]

1d2 Exercise. Recall the n-dimensional vector space P_{n-1} of polynomials discussed in 1c3. Prove that the following conditions on $f, f_1, f_2, \dots \in P_{n-1}$ are equivalent:

(a) $f_k \to f$ in P_{n-1};

(b) $f_k(0) \to f(0)$, $f'_k(0) \to f'(0)$, ..., $f_k^{(n-1)}(0) \to f^{(n-1)}(0)$;

(c) $f_k(0) \to f(0)$, $f_k(1) \to f(1)$, ..., $f_k(n-1) \to f(n-1)$;

(d) $f_k(\cdot) \to f(\cdot)$ pointwise; that is, $f_k(x) \to f(x)$ for every $x \in \mathbb{R}$;

(e) $f_k(\cdot) \to f(\cdot)$ locally uniformly; that is, $\max_{|x| \le M} |f_k(x) - f(x)| \to 0$ for every M.

Hint: (c) consider the linear operator $P_{n-1} \ni g \mapsto \big(g(0), g(1), \dots, g(n-1)\big) \in \mathbb{R}^n$.

1d3 Exercise. The same as 1d2 for the n-dimensional affine space $S_n = \{f : f^{(n)} = g\}$ discussed in 1c3.

Hint: use 1d2.

1d4 Exercise. Let V be a vector *fd* space, and $V_1 \subset V$ its subspace.

(a) Upgrade V to $\mathbb{R}^n$ (by choosing a basis) getting $V_1 = \{(x_1, \dots, x_n) : x_{m+1} = \dots = x_n = 0\}$; here $n = \dim V$ and $m = \dim V_1$.

(b) Conclude that every subspace of a vector (as well as affine) *fd* space is closed (topologically).[2]

1e Norms of vectors and operators

1e1 Definition. The norm $\|T\|$ of a linear operator $T : E_1 \to E_2$ between Euclidean vector *fd* spaces E_1, E_2 is

$$\|T\| = \sup_{x \in E_1, x \ne 0} \frac{|T(x)|}{|x|} \, .$$

[1] In infinite dimension the situation is utterly different.

[2] In infinite dimension the situation is strikingly different.

Also,

$$\|T\| = \max_{|x|\le 1} |T(x)|$$

(think, why); this is the maximum of a continuous function on a compact set [Sh:p.73].

The *operator norm* $\|A\|$ of an $m \times n$ matrix A is, by definition, the norm of the corresponding operator $\mathbb{R}^n \to \mathbb{R}^m$.

1e2 Exercise. If a matrix $A = (a_{i,j})_{i,j}$ is diagonal then

$$\|A\| = \max_{i=1,\dots,\min(m,n)} |a_{i,i}|.$$

Prove it.

The set $M_{m,n}(\mathbb{R})$ of all $m \times n$ matrices (with real elements) evidently is an mn-dimensional vector space. Does the operator norm turn it to a Euclidean space? No, it does not. Even if we restrict ourselves to $M_{2,2}(\mathbb{R})$, and even to its 2-dimensional subspace of diagonal matrices, we get (by 1e2, up to isomorphism) $\mathbb{R}^2$ with the norm

$$\|(s,t)\| = \max(|s|,|t|)\,,$$

its unit ball $\{x : \|x\| \le 1\}$ being the square $[-1,1] \times [-1,1]$. This is not the Euclidean plane! For two non-collinear vectors $a = (1,1)$ and $b = (1,-1)$ we have $\|a\| = 1$, $\|b\| = 1$ and $\|a+b\| = 2$, which never happens on the Euclidean plane. Also, the "parallelogram equality" $|a-b|^2 + |a+b|^2 = 2|a|^2 + 2|b|^2$ holds for arbitrary vectors a, b of a Euclidean space, but fails for the operator norm.

1e3 Definition. (a) A *norm* on a vector space V is a function $V \ni x \mapsto \|x\| \in [0,\infty)$ such that

$$\begin{gathered}\|tx\| = |t| \cdot \|x\| \quad \text{for all } x \in V,\ t \in \mathbb{R}\,;\\ \|x+y\| \le \|x\| + \|y\| \quad \text{for all } x,y \in V\,;\\ \|x\| > 0 \quad \text{whenever } x \ne 0\,.\end{gathered}$$

(b) A *normed space* consists of a vector space and a norm on it.

Euclidean vector spaces are a special case of normed spaces.[1] Distances are well-defined in normed spaces, but angles — only in Euclidean spaces.

[1] In fact, a normed space is Euclidean iff the norm satisfies the parallelogram equality.

1e4 Exercise. Prove that

$$-\|x-y\| \le \|x\| - \|y\| \le \|x-y\|$$

for all $x, y \in V$.

1e5 Exercise. Prove that the operator norm is indeed a norm on $M_{m,n}(\mathbb{R})$.

1e6 Lemma. Every norm on $\mathbb{R}^n$ is continuous.

Proof. For arbitrary $t_1, \dots, t_n \in \mathbb{R}$,

$$\begin{aligned} \|(t_1, \dots, t_n)\| = \|t_1 e_1 + \dots + t_n e_n\| \le |t_1| \cdot \|e_1\| + \dots + |t_n| \cdot \|e_n\| \le \\ \le (\|e_1\| + \dots + \|e_n\|) \cdot \max(|t_1|, \dots, |t_n|) \le C \sqrt{t_1^2 + \dots + t_n^2} \end{aligned}$$

where[1] $C = \|e_1\| + \dots + \|e_n\|$ (and $e_1, \dots, e_n$ are the standard basis). Thus, $\|x\| \le C|x|$ for all $x \in \mathbb{R}^n$. Now, if $|x_n - x| \to 0$ then $\|x_n - x\| \to 0$, and by 1e4, $\|x_n\| \to \|x\|$. $\square$

The sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ being compact, a norm $\|\cdot\|$ reaches its minimum c and maximum C on S^{n-1}:

$$c = \min_{|x|=1} \|x\| \, , \quad C = \max_{|x|=1} \|x\| \, ;$$

$0 < c \le C < \infty$ (think, why $c > 0$). Thus,

$$\begin{gathered} \forall x \quad c|x| \le \|x\| \le C|x| \, ; \\ \|x_n\| \to 0 \quad \text{if and only if} \quad |x_n| \to 0 \, ; \end{gathered}$$

one says that $\|\cdot\|$ and $|\cdot|$ are equivalent norms. We conclude.

1e7 Proposition. For every vector ^{fd}space V,[2]

(a) for every norm $\|\cdot\|$ on V,

$$(x_n \to x) \quad \Longleftrightarrow \quad (\|x_n - x\| \to 0) \quad \text{for all } x, x_1, x_2, \dots \in V \, ;$$

(b) for every pair of norms $\|\cdot\|_1$, $\|\cdot\|_2$ on V,

$$\exists c, C \in (0, \infty) \;\; \forall x \in V \quad c\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \, .$$

All norms are equivalent on an arbitrary vector ^{fd}space.

[1] Even better, $C = \sqrt{\|e_1\|^2 + \dots + \|e_n\|^2}$ fits.

[2] In infinite dimension the situation is utterly different.

1e8 Exercise. Generalize 1e1 and 1e5 to the space $\mathcal{L}(X,Y)$ of all linear operators [Sh:p.71] between normed (not just Euclidean) *fd*spaces X,Y.[1]

1e9 Exercise. If $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(Y,Z)$ then $TS \in \mathcal{L}(X,Z)$ and $\|TS\| \le \|T\| \cdot \|S\|$. Prove it.

1e10 Exercise. (a) Prove equivalence of two definitions of the *Hilbert-Schmidt norm* $\|A\|_{\mathrm{HS}}$ of an $m \times n$ matrix $A = (a_{i,j})_{i,j}$:
$\|A\|_{\mathrm{HS}} = \big(\sum_{j,k} a_{j,k}^2\big)^{1/2}$;
$\|A\|_{\mathrm{HS}} = \sqrt{\operatorname{trace}(A^*A)}$.
(b) Is $(M_{m,n}(\mathbb{R}), \|\cdot\|_{\mathrm{HS}})$ a normed space? a Euclidean space?
(c) Prove that $\|A\| \le \|A\|_{\mathrm{HS}} \le \sqrt{n}\|A\|$.[2]

A bit about convexity.

1e11 Definition. (a) A set C in a vector space is *convex* if for all $x,y \in C$ the segment $[x,y] = \{\theta x + (1-\theta)y : 0 \le \theta \le 1\}$ is contained in C. (The same applies in an affine space.)
(b) A real-valued function f on a vector (or affine) space, or on a convex set therein, is called *convex* if

$$f\big(\theta x + (1-\theta)y\big) \le \theta f(x) + (1-\theta) f(y)$$

for all $\theta \in [0,1]$ and all x,y in the domain of f.

1e12 Exercise. Prove that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is convex if and only if the set $\{(x,y,z) : z \ge f(x,y)\} \subset \mathbb{R}^3$ is convex.

1e13 Exercise. Prove that convexity of the sets $\{x : f(x) \le t\}$ for all $t \in \mathbb{R}$ is necessary but not sufficient for convexity of a function f.[3]

1e14 Exercise. Prove that the second condition of 1e3 ($\|x+y\| \le \|x\|+\|y\|$) is equivalent (given the other two conditions) to (a) convexity of the norm, and also to (b) convexity of the ball $\{x \in V : \|x\| \le 1\}$.[4]

---

[1] Linear operators between spaces of operators are also well-defined, and sometimes called superoperators (mostly by physicists); see also “Superoperator” in Wikipedia.

[2] Hint to $\|A\| \le \|A\|_{\mathrm{HS}}$: using the Cauchy-Schwarz inequality, estimate first y_k^2 and then $\sum_{k=1}^m y_k^2$; here $y_k = \sum_j a_{k,j} x_j$.
Hint to $\|A\|_{\mathrm{HS}} \le \sqrt{n}\|A\|$: $|Ae_j| \le \|A\|$ for each $j = 1,\dots,n$.

[3] Hint: for “but not sufficient” try dimension one.

[4] Hint: (b) $\frac{x+y}{\|x\|+\|y\|} = \theta \frac{x}{\|x\|} + (1-\theta)\frac{y}{\|y\|}$.

1e15 Exercise. Let $p \in [1, \infty)$. Prove that the function

$$\mathbb{R}^n \ni (t_1, \dots, t_n) \mapsto \big(|t_1|^p + \dots + |t_n|^p \big)^{1/p} \in [0, \infty)$$

is a norm on $\mathbb{R}^n$.[1]

This norm is often denoted $\| \cdot \|_p$.[2]
In the limit $p \to \infty$ we get

$$\|(t_1, \dots, t_n)\|_\infty = \max(|t_1|, \dots, |t_n|) \,.$$

1f Paths and connectedness

1f1 Definition. A *path*[3] in $\mathbb{R}^n$ is a continuous map $\gamma : [0,1] \to \mathbb{R}^n$; the points $\gamma(0)$ and $\gamma(1)$ are called the endpoints of the path γ, and γ is called a path from $\gamma(0)$ to $\gamma(1)$. A path γ is *closed* if $\gamma(0) = \gamma(1)$. A path γ is *simple* if the restriction $\gamma\big|_{(0,1)}$ is one-to-one. The inverse path is $t \mapsto \gamma(1-t)$.

Two paths γ_1, γ_2 are called *equivalent*, if there exists an increasing bijection $\varphi : [0,1] \to [0,1]$ such that $\gamma_2(s) = \gamma_1(\varphi(s))$. Normally, we need not distinguish equivalent paths.

Sometimes it is convenient to use also $\gamma : [a,b] \to \mathbb{R}^n$ for given $a < b$. (Non-closed intervals, and even the whole $\mathbb{R}$, are also used sometimes, but then endpoints and closed paths are not defined.)

1f2 Example. (a) The segment with endpoints at x and y: $\gamma(t) = tx + (1-t)y$ for $t \in [0,1]$. The segment with the inverse orientation: $\gamma_{-1}(t) = ty + (1-t)x$.

(b) The unit circle with the natural orientation: $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. The circle with the opposite orientation: $\gamma_{-1}(t) = (\cos t, -\sin t)$ for $t \in [0, 2\pi]$. The path $\gamma_{10}(t) = (\cos 10t, \sin 10t)$ for $t \in [0, 2\pi]$ is also the circle, but run 10 times (not equivalent to γ).

(c) The Archimedean spiral: $\gamma(t) = (t \cos t, t \sin t)$ for $t \in [0, 2\pi]$.

1f3 Exercise. Draw the images (with orientation) of the following paths given in the polar coordinates:
(a) $r = 1 - \cos 2t$, $\varphi = t$ for $t \in [0, 2\pi]$;
(b) $r^2 = 4 \cos t$, $\varphi = t$ for $t \in [-\pi/2, \pi/2]$;

---

[1] Hint: the function $(t_1, \dots, t_n) \mapsto |t_1|^p + \dots + |t_n|^p$ is convex (being the sum of convex functions), therefore the set $\{(t_1, \dots, t_n) : |t_1|^p + \dots + |t_n|^p \le 1\}$ is convex.

[2] On the space of operators, the *Schatten norm* is $\|T\|_p = \big(|s_1|^p + \dots + |s_n|^p\big)^{1/p}$ where $s_1, \dots, s_n$ are the singular values of T.

[3] Note that (a) it is not a set of points, and (b) it can be space-filling.

(c) $r = 2\sin 3t$, $\varphi = t$ for $t \in [0, \pi]$.
Also, draw the image (with orientation) of the "path" in $\mathbb{R}^3$ defined as $\gamma(t) = (\cos t, \sin t, t)$ for $t \in \mathbb{R}$.

1f4 Definition. A set $X \subset \mathbb{R}^n$ is called *path-connected* if for each pair of points $x, y \in X$ there is a path in X[1] from x to y.

1f5 Example. $\mathbb{R}^1 \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{x_1 = 0\}$ are not path-connected; $\mathcal{S}^{n-1}$, $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{R}^3 \setminus \{x_1 = x_2 = 0\}$ are path-connected.

1f6 Example. Prove that a set $X \subset \mathbb{R}$ is path-connected if and only if it is an interval (of any kind: $[a, b]$, $[a, b)$, $[a, \infty)$, $(a, b]$, $[a, a]$ etc).

1f7 Exercise. Prove that a continuous image of a path-connected set is path-connected. That is, if $X \subset \mathbb{R}^n$ is a path-connected set and $f : X \to \mathbb{R}^m$ a continuous mapping then the image $f(X)$ is also path-connected.

1f8 Exercise. *"Mean-value property"* Suppose X is a path-connected set and $f : X \to \mathbb{R}$ a continuous function. If $\inf_X f < 0$ and $\sup_X f > 0$ then there exists a point $x \in X$ such that $f(x) = 0$.

1f9 Definition. An open path-connected subset of $\mathbb{R}^n$ is called a *domain* (or region).

1f10 Exercise. Let $G \subset \mathbb{R}^n$ be open, and $x \in G$. Introduce the set $U \subset G$ of all $y \in G$ such that there exists a path in G from x to y, and $V = G \setminus U$. Prove that U and V are open sets.

1f11 Exercise. Prove that every open set in $\mathbb{R}^n$ can be decomposed into at most countable union of disjoint domains.

1f12 Definition. A set $X \subset \mathbb{R}^n$ is *connected* if no pair of open sets $U, V \subset \mathbb{R}^n$ satisfies

$$X \subset U \cup V\,; \quad U \cap V = \emptyset\,; \quad X \cap U \neq \emptyset\,; \quad X \cap V \neq \emptyset\,.$$

1f13 Exercise. Prove that an open set in $\mathbb{R}^n$ is connected if and only if it is not the union of two disjoint nonempty open sets.

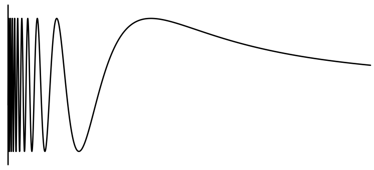
1f14 Exercise. Prove that every path-connected set in $\mathbb{R}^n$ is connected.[2]

1f15 Exercise. Prove that every connected open set in $\mathbb{R}^n$ is path-connected.[3]

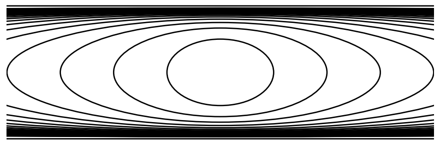
1f16 Exercise. The same for *polygonal connectedness*. (That is, a path is required to be a polygonal line.)

[1] That is, $\gamma : [0, 1] \to X$.
[2] Hint: think about $\inf\{t : \gamma(t) \notin U\}$.
[3] Hint: use 1f10.

Thus, all kinds of connectedness are equivalent for open sets in $\mathbb{R}^n$. For closed sets connectedness is more subtle.



"Topologist's sine curve"
This compact set is connected but not path-connected



This unbounded closed set contains two connected components (the two horizontal lines) that cannot be separated by U, V

A *connected component* is, by definition, a maximal connected subset.

According to Sect. 1d, all said above applies in every finite-dimensional vector (as well as affine) space.

Index

affine function, 9
affine operator, 9
affine space, 6

connected, 15
connected component, 16
convex function, 13
convex set, 13

difference space, 6
domain, 15

equivalent norms, 12
equivalent paths, 14
Euclidean affine space, 7
Euclidean vector space, 1

Hilbert-Schmidt, 13
homeomorphism, 9

inverse path, 14

linear function, 9
linear operator, 9

norm, 11
normed space, 11

operator norm, 11

path, 14
    closed, 14
    inverse, 14
    simple, 14
path-connected, 15
polygonally connected, 15

region, 15

Singular Value Decomposition, 2, 8

$\mathcal{L}(X, Y)$, 13
$M_{m,n}(\mathbb{R})$, 11
$\|T\|$, 10
$\|x\|$, 11
$|x|$, 1