How To Count Citations If You Must*

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Abstract

Citation indices are regularly used to inform critical decisions about promotion, tenure, and the allocation of billions of research dollars. Despite their importance, a systematic development is lacking. Instead, most indices (e.g., the h-index) are motivated by intuition and rules of thumb, often leading to undesirable conclusions. We take an axiomatic approach and are led to an essentially unique new index that has a simple functional form and avoids several critical shortcomings of the h-index and its successors. Our analysis includes a consistent methodology for adjusting for differences in field, ages of papers, scientific age, and the number of authors.

Keywords: citation indices, axiomatic, scale invariance

1. Introduction

Citation indices attempt to provide useful information about a researcher's publication record by summarizing it with a single numerical score. They provide government agencies, departmental committees, administrators, faculty, and students with a simple and potentially informative tool for comparing one researcher to another, and are regularly used to inform critical decisions about funding, promotion, and tenure. With decisions of this magnitude on the line, it is surprising that a systematic development of such indices is almost entirely lacking. We provide such a development here, one that, surprisingly perhaps, leads to an essentially unique new index.[1]

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[1] Clearly, reducing a research record to a single index number entails a loss of information. Consequently, no single index number is intended to be sufficient for making decisions about funding, promotion etc. It is but one tool among many for such purposes. But it is enough to hold the view that no tool should move one toward unsound or inconsistent decisions and it is this principle that forms the basis of our analysis.

Perhaps the best-known citation index beyond a total citation count is the h-index, see Hirsh (2005). A researcher's h-index is defined to be the largest number, h, of his/her papers, each of which has at least h citations. By design, the h-index limits the effect of a small number of highly cited papers, a feature which, though well intentioned, can produce intuitively implausible rankings. For example, consider two researchers, one with 10 papers, each with 10 citations (h-index $= 10$), and another with 8 papers, each with 100 (or even 1000) citations (h-index $= 8$).

Improvements to the h-index have been suggested. Consider, for example, the g-index (Egghe 2006a, 2006b), which is the largest number of papers, g, whose total sum of citations is at least g^2. The g-index is intended to correct for the insensitivity of the h-index to the number of citations received by the h papers with the most citations. Many other variations have been suggested since. Yet, like the h-index, they are ad hoc measures based almost entirely on intuition and rules of thumb with insufficient justification given for choosing them over the infinitely many other unchosen possibilities. But for a novel empirical approach to selecting among a new class of h-indices, see Ellison (2012, 2013).[2]

A formal way to find a good index is to take an axiomatic approach. The methodology of this approach is to *first* select, with care, a number of basic properties that any acceptable index must have. Restricting attention only to indices that satisfy all of these properties is a systematic way of narrowing down the set of potential indices. The advantage of this approach is that it focuses attention on the *properties* that an index should possess rather than the functional form that it should take. After all, it is the properties we desire an index to possess that should determine its functional form, not the other way around.

The approach adopted here is distinct from the literature's common exercise of axiomatizing a given *pre-existing* index function (e.g, Queseda, 2001; Woeginger, 2008a and 2008b; and Marchant, 2009a).[3] The aim of this latter exercise is to begin with a previously known index function, e.g., the h-index, and identify a set of properties that only that index can satisfy. The given index is then characterized by these properties. While this exercise can usefully identify an index's previously unknown properties, both good and bad, it is not designed to correct any inherent flaws. To find a good index one should instead first begin with well-chosen properties and then allow those properties to lead one to the index, whatever it may turn out to be.

Our goal is to provide a method for comparing one citation list to another in the most

[2] Ellison (2013) introduces the following class of generalized Hirsch indices. For any $a, b > 0$ and for any citation list, $h_{(a,b)}$ is defined to be the largest number h such that at least h papers have at least ah^b citations. Ellison estimates a and b so that $h_{(a,b)}$ gives the best fit in terms of ranking economists at the top 50 U.S. universities in a manner that is consistent with the observed labor market outcome.

[3] An exception is Marchant (2009b) where the consequences of a number of axioms are nicely worked out, even though most are not put forward as necessary for a good index.

basic sense of determining which list among any two is "better" than the other. Thus, the index theory developed here is ordinal. In particular, we do not attempt to say *how much* better one citation list is than another.

Also, our index is intended to judge an individual's record *as it stands*. It is not intended as means to predict future output. If a prediction of future output is important to the decision at hand (as in tenure decisions, for example) then separate methods must first be used to obtain a predicted citation list to which our index can then be applied (see also the discussion in Section 6.1).[4]

An important issue to consider for any index, is how it should be adjusted to adequately compare individuals with different characteristics, e.g. differences in field of study, scientific age, etc. A variety of adjustment methods have been proposed for the various indices within the literature (e.g., Hirsch, 2005; Van Noorden, 2013; and Kaur et. al., 2013). When an index provides only ordinal information, such adjustments, if not carried out very carefully, can lead to inconsistencies. Take, for example the online tool called "Scholarometer" (http://scholarometer.indiana.edu/). The purpose of this tool is to make it simple and convenient for practitioners to correct for disciplinary biases when conducting citation analyses whose objective is to rank scholars across fields. This tool relies on queries made through Google Scholar and normalizes a given scholar's h-index by dividing it by the average h-index within this scholar's field. This normalized h-index is then used to compare any two scholars, regardless of their field. In Section 4 we illustrate why such an approach is problematic and offer a more consistent methodology for correcting for differences in characteristics (such as field of study).

Related to the issue of ranking scholars is the issue of ranking scholarly journals (e.g., Palacios-Huerta and Volij, 2004 and 2014). But there is an important distinction. Because journal indices are typically used to translate citations and/or publications from distinct journals into a common currency, such indices must be cardinal in nature to be useful. In contrast, most indices that rank scholars are ordinal in nature because there is no fundamental distinction between them and any of their monotonic transformations. We expand upon these points in Section 4.

A matter of some technical interest is that the theory we develop here takes seriously the fact that citation lists are vectors of integers. Consequently, the space on which our index is defined is not connected and so otherwise standard results on when a binary relation has a representation with a particular functional form (e.g., results on additive separability as in Debreu (1960) are not directly applicable).

We next list four basic properties that, in our view, every good index should possess. It

[4] We thank Glenn Ellison for helpful discussions on this point.

is best to think of these basic properties as pertaining to citation lists whose cited papers differ only in their number of citations but otherwise have common characteristics, e.g., are published in the same year, in the same field of study, even in the same journal, and have the same number of authors, etc. One of the four properties, "scale invariance," is motivated with an eye toward the adjustments for differences in characteristics that will be considered in Section 4.

There is an essentially unique index that possesses the properties below. This new index has a simple functional form and is distinct from the h-index and all its successors.

2. Four Properties for Citation Indices

Citation indices work as follows. First, an individual's record is summarized by a finite list of citations, typically ordered from highest to lowest, so that the i-th number in the list is the number of citations received by the individual's i-th most highly cited paper. The list of citations is then operated upon by some function to produce the individual's index number. In Section 4.2, we argue that one must sometimes rescale one individual's citation list to adequately compare it to another individual's list (for example, when the two individuals are from different fields).[5] Consequently, one must be prepared to consider noninteger lists. Moreover, because the scaling factors will always be rational numbers, our domain of study is the set $\mathbb{L}$ consisting of all finite lists of nonnegative rational numbers ordered from highest to lowest. Any element of $\mathbb{L}$ is called a *citation list.*[6]

A *citation index* is any continuous function, $\iota : \mathbb{L} \to \mathbb{R}$, that assigns to each citation list a real number, where continuity means that for any citation list x, if a sequence of citation lists $x^1, x^2, ...$ each of whose length is the same as the finite length of x, converges to x – in the Euclidean sense – then $\iota(x^1), \iota(x^2), ...$ converges to $\iota(x)$.[7]

Consider now the following four properties. A short discussion of them follows.

1. Monotonicity (MON) For any citation list there is a positive integer m – which can depend on the list – such that the value of the index does not decrease if either, a new paper with m citations is added to the list, or any existing paper in the list receives any number of additional citations.

[5] And we also explain (in Section 4.1) why any adjustments should be made to the citation lists themselves and not to the index function.

[6] No non-singleton subset of $\mathbb{L}$ is connected. While this presents some difficulties, the end result is a theory that takes seriously the integer nature of citations.

[7] Note that the index is continuous *only* on the rationals – it is not even defined on the reals. In general, such functions cannot be extended from the rationals to the reals in a continuous manner without additional assumptions.

2. **Independence** (IND) The index's ranking of two citation lists must not change if a new paper is added to each list and each of the two new papers receives the same number of citations.

3. **Depth Relevance** (DR) There exist a citation list and a positive integer k such that, starting from that list, the index increases by more if one paper with k citations is added than if k papers with one citation each are added.

4. **Scale Invariance** (SI) The index's ranking of two citation lists must not change when each entry of each list is multiplied by any common positive rational scaling factor.

The first property, monotonicity, is self-explanatory and hardly seems to require further justification. Indeed, as far as we are aware, it is satisfied by all indices.

The second property, independence, is natural given that our index is intended to compare the records – as they currently stand – of any two individuals. It says, in particular, that a tie between two records cannot be broken by adding identical papers to each record.[8] The h-index fails to satisfy independence since its ranking of two citation lists can be reversed after adding identically cited papers to each of the two lists.

The third property, depth relevance, is satisfied by the vast majority of indices that we are aware of with the exception, of course, of the total citation count. When it is not satisfied, then for any total number of citations, one maximizes the value of one's index by spreading those citations as thinly as possible across as many publications as possible.

When comparing individuals in fields whose average numbers of citations differ, one must rescale the citation lists so that they can be compared (see Section 4.2). The fourth property, scale invariance, ensures that the final ranking of the population is independent of whether the lists are scaled down relative to the disadvantaged field or scaled up relative to the advantaged field. Because the h-index is not scale invariant it can, in particular, reverse the order of individuals in the same field when their lists are adjusted to account for differences across fields (see Example 4.2 in Section 4.3).

3. The main result

Call two citation indices *equivalent* if they always agree on the ranking of any two citation lists. For example, any index is equivalent to the index that assigns to each citation list the cube of the value assigned by the original index. We can now state our main result, whose proof is in the appendix.

[8] If one's aim were instead to use the current record to predict future output, then, as Glenn Ellison pointed out to us, adding identical papers can easily break a tie. See Section 6.1 for an example of how our index should be used when future output is of interest.

Theorem 3.1. *A citation index satisfies properties 1-4,* if and only if *it is equivalent to the index that assigns to each citation list* $(x_1, ..., x_n)$*, the number,*

$$\left(\sum_{i=1}^{n} x_i^{\sigma}\right)^{1/\sigma},$$

where σ *is some fixed number strictly greater than one.*[9]

According to the theorem, properties 1-4 determine the index (more precisely, its equivalence class) up to the single parameter $\sigma > 1$. For a rationale for restricting σ to values no greater than 2 see Section 5. Next, we discuss the interpretation of this important parameter.

The value of our index can increase in two different ways: when an existing paper receives more citations – in which case we will say that the list becomes deeper – and when a new paper is added to the list – in which case we will say that the list becomes longer. The parameter σ measures the extent to which depth is considered more important than length.

At the one extreme in which σ approaches its lower bound of 1, depth and length are equally important and citation lists are ranked according to their total citation count. At the other extreme in which σ becomes unboundedly large, citation lists are ranked according to which list is lexicographically higher and depth becomes infinitely more important than length.

In general, changes in both the depth and the length of a citation list affect its index value and the parameter σ determines the depth/length trade-off. Specifically, each $\sigma\%$ decrease in a citation list's length (i.e., for each k, a $\sigma\%$ decrease in the number of papers with k citations) requires a 1% increase in its depth (i.e., a 1% increase in each paper's number of citations) to restore the index to its original value. That is, σ is the elasticity of substitution of length for depth.[10] In particular, the higher is σ the more weight the index places on depth.

Additional intuition for σ is obtained by noting that it is characterized by the answer to the following question.

> "For what value of $\lambda \in (1, 2)$ would two papers, each with k citations, be considered equivalent to one paper with λk citations?"

[9] This index, being in units of citations and homogeneous of degree one, may be more convenient in practice than the equivalent index, $\sum_{i=1}^{n} x_i^{\sigma}$. Moreover, as defined, our index coincides with the l_σ norm from classical mathematical analysis, which is certainly an interesting happenstance.

[10] For any citation list x, define $v(d, l) := (l \sum_i (dx_i)^{\sigma})^{\frac{1}{\sigma}}$. Then, changes in d correspond to changes in the list's depth and changes in l correspond to changes in the list's length. The index's elasticity of substitution of length for depth is $\frac{d}{l} \frac{\partial v/\partial d}{\partial v/\partial l} = \sigma$.

The answer, of course can be different for different decision-makers. However, if λ^* is one's answer to this question, then the implied value of σ is $\ln 2/\ln\lambda^*$. So, for example if two papers each with 100 citations are considered equivalent to one paper with 140 citations, then the implied value of σ is approximately 2, while if they are instead equivalent to one paper with 160 citations, the implied value of σ is approximately 3/2. Thus, a value of σ between 3/2 and 2 may be suitable for most applications (again, see Section 5 for an argument leading to a value of σ no greater than 2).

3.1. An Implied Property

Properties 1 through 4 alone completely determine, up to the parameter $\sigma > 1$, the ordinal properties of the index. However, we wish now to point out a particularly interesting subsidiary property that holds as a consequence.

Consider two individuals, X and Y whose respective citations lists are $x_1 \geq ... \geq x_n$ and $y_1 \geq ... \geq y_n$, where we have added 0's if necessary to make the lists equal in length. Suppose that the two lists are related as follows.

$$\begin{gathered} x_1 \geq y_1 \\ x_1 + x_2 \geq y_1 + y_2 \\ \vdots \\ x_1 + ... + x_n \geq y_1 + ... + y_n \end{gathered} \tag{1}$$

That is, individual X's most cited paper has at least as many citations as Y's most cited paper, and X's two most cited papers have at least as many total citations as Y's two most cited papers, ..., and X has at least as many total citations as Y. It would seem rather difficult to argue that individual Y should be ranked above individual X. Yet, with the h-index, individual Y can easily be ranked above X (e.g., if X has one paper with 100 citations and Y has two papers each with 2 citations). In contrast, we have the following result.

If the citation lists of two individuals, X and Y are related as in (1), then any index satisfying properties 1-4 must rank individual X at least as high as individual Y, and must rank X strictly above Y if any of the inequalities in (1) is strict.

Note well that the result makes no mention of any particular value of σ. Indeed, in view of our Theorem, an equivalent way to state the result is to say that, for any value of $\sigma > 1$,

$$\left(\sum_{i=1}^{n} x_i^{\sigma}\right)^{1/\sigma} \geq \left(\sum_{i=1}^{n} y_i^{\sigma}\right)^{1/\sigma}, \tag{2}$$

whenever (1) holds, and the inequality is strict if any of the inequalities in (1) is strict (see

Hardy et al., 1934). Thus, whenever two candidate's lists are ranked as in (2), decision-makers who wish to satisfy properties 1-4 can safely rank X above Y without settling on a particular value of σ.

4. Adjusting for different characteristics

It is well understood that it can be inappropriate to compare the citation index values of two individuals who differ in important characteristics such as field of study, scientific age, ages of papers, the number of authors, etc. To make such comparisons meaningful, their indices should be adjusted. In this section, we provide a consistent method for making such adjustments and we apply this method to our index. But before doing so, we must discuss the problems that can arise, due to the ordinal nature of the index, if one is not sufficiently careful in how the adjustments are made. This point is important because many widely adopted adjustment procedures have not been created with sufficient care.

4.1. Problems with adjusting any ordinal index

If one's index is ordinal, then there is no distinction between that index and any of its monotone transformations. Consequently, if some procedure is used to adjust the index in order to rank individuals with different characteristics, then, absent an argument tying the procedure to that particular index, applying the procedure instead to a monotone transformation of the index should produce the same ranking of the population. But commonly used practical methods fail to satisfy this important principle.

For example, consider the online citation analysis tool "Scholarometer" (see http://scholarometer.indiana.edu), whose popularity for computing and comparing the h-indices of scholars across disciplines seems to be growing. Scholarometer implements the h-index adjustment in Kaur et. al. (2013), who suggest dividing a scholar's h-index by the average h-index in his/her field. This produces, for each scholar, an adjusted h-index, called the h_s-index, that is then used to compare any two scholars in the population, regardless of field. The following example shows, however, that the final ranking of the population depends on the particular representation of the h-index that is employed.

Example 4.1. *There are two mathematicians, M_1 and M_2, and two biologists, B_1 and B_2, with each individual representing half of his field. Their h-indices are 2, 8, 7 and 27, respectively so that their ranking according to the raw h-index is*

$$M_1 \prec B_1 \prec M_2 \prec B_2,$$

where $\prec$ means "is ranked worse than." To compare mathematicians to biologists using Scholarometer's method, we must divide each of the mathematician's indices by their field's average $((2+8)/2=5)$ and we must divide each of the biologist's h-indices by their field's average $((7+27)/2=17)$. The resulting adjusted indices, h_s, are respectively, 0.4, 1.6, 0.41, and 1.58, producing the final adjusted ranking,

$$M_1 \prec B_1 \prec B_2 \prec M_2. \tag{4.1}$$

But suppose that, instead of starting the exercise with the h-index, we start with the monotonic transformation $h^* = h + \sqrt{h}$. The h^*-indices of M_1, M_2, B_1, and B_2 are, respectively, $2+\sqrt{2}$, $8+\sqrt{8}$, $7+\sqrt{7}$, and $27+\sqrt{27}$, i.e., 3.4, 10.8, 9.6, and 32.2. Hence, their ranking according to the raw h^*-index is,

$$M_1 \prec B_1 \prec M_2 \prec B_2,$$

which is, of course, the same as the ranking from h because h^* is ordinally equivalent.

However, if for h^* we apply Scholarometer's adjustment procedure and divide each individual's h^*-index by the average h^* in his/her field (i.e., divide the mathematicians' h^*-indices by 7.1 and divide the biologists' by 20.9), then the final adjusted ranking becomes,

$$B_1 \prec M_1 \prec M_2 \prec B_2. \tag{4.2}$$

Comparing (4.1) with (4.2), we see that the ranking of M_1 and B_1 and of M_2 and B_2 depend on which of the ordinally equivalent indices h or h^* that one begins with. Absent a compelling argument for choosing one over the other, one must conclude that this adjustment procedure is not well-founded.

Pushing the example a little further shows in fact that the general procedure of rescaling indices to adjust for different fields – and not merely Scholarometer's particular rescaling method – is not well-founded. This is because rescaling the indices in one field relative to those in another can produce a ranking that is impossible to obtain with an ordinally equivalent index no matter how the fields' equivalent indices are rescaled relative to one another.[11]

The adjustment employed in the example is typical of those employed in practice and in the large literature on citation indices when comparisons between individuals with distinct characteristics are desired. Such adjustments tend to be applied to a particular index without

[11] Indeed, the ranking in (4.1) is impossible to obtain through any relative rescaling of the mathematicians' and the biologists' h^* indices because there is no $\lambda > 0$ such that $\lambda(2+\sqrt{2}) < (7+\sqrt{7}) < (27+\sqrt{27}) < \lambda(8+\sqrt{8})$.

regard to whether that adjustment would produce a different ranking were it applied to an ordinally equivalent index. What is required is a general methodology for making the necessary adjustments while respecting the ordinal nature of the index.

Fortunately, there is a simple way to ensure ordinally consistent adjustments, namely, by *adjusting the citation lists*, not the index. If one adjusts the citation lists to account for differences in characteristics, then the final ranking of any two individuals will be the same whether to those adjusted citation lists one applies a given index or any one of its monotonic transformations.

In the remaining subsections, we discuss how to adjust individual citation lists to take into account differences in a variety of characteristics.

4.2. Differences in fields and in the ages of papers

It is well-known that some fields are cited significantly more often than others (even within the same discipline), and that older papers typically have more citations than younger papers. It is therefore important to adjust the lists to take these differences into account.

In Radicchi et al. (2008), a convincing case is made that the appropriate way to adjust for these differences is to divide, for any given year t, each entry in an individual's citation list that was published in year t – call this a year-t paper – by the average number of citations of all the year-t papers in that individual's field.

In Figure 1 below, we reproduce Figure 1 in Radicchi et. al. (2008), showing normalized (relative to the total number of citations) histograms of unadjusted citations received by papers published in 1999 within each of 5 fields out of 20 considered. The main point of this figure is to show that the distributions of unadjusted citations differ widely across fields. For example, in Developmental Biology, a publication with 100 citations is 50 times more frequent than in Aerospace Engineering. However, after adjusting each citation received by a year-1999 paper by dividing it by the average number of citations of all the year-1999 papers in the same field, the distinct histograms in Figure 1, remarkably, align to form single histogram, shown in Figure 2 (in both Figures, the horizontal axis is on a logarithmic scale).[12]

We encourage the reader to consult the Radicchi et. al. paper for details, but let us point out that several statistical methods are used there to verify what seems perfectly clear

[12] A consequence of this alignment is that the suggested adjustment is unique in the following sense. Restricting to papers published in 1999, let $\tilde{c}_j$ denote the random variable describing the distribution of citations in field j, and let $\bar{c}_j$ denote its expectation. The suggested adjustment is to divide any field j paper's number of citations, c, by $\bar{c}_j$, giving the adjusted entry $c/\bar{c}_j$. If any increasing functions $\phi_1, \phi_2, ...$ align the distributions across fields – i.e., are such that the adjusted random variables $\phi_1(\tilde{c}_1), \phi_2(\tilde{c}_2), ...$ all have the same distribution – then there is a common increasing function f such that $\phi_j(c) = f\left(\frac{c}{\bar{c}_j}\right)$ for every j.

form the figures, namely, that dividing by the average number of citations per paper in one's field corrects for differences in citations across fields in the very strong sense that, after the adjustment, the distributions of citations are virtually identical across fields.[13]

In fact, Radicchi et. al. show that the adjustment yields again that same distribution when carried out for years 1990 and 2004. So, as noted in their paper, this adjustment corrects not only for differences in field, but also for differences in the ages of papers (see Figure 3 in Radicchi et. al. (2008).

Let us now apply this adjustment to our index.

[13] In Radicchi et al. (2008), the common histogram in Figure 2 is estimated as being log normal with variance 1.3 (the mean must be 1 because of the rescaling).

Consider a raw citation list $(x_1, ..., x_n)$. For each paper $i = 1, ..., n$, let a_i denote the average number of citations of all papers published in both the same year and field as paper i; and so each a_i is a rational number. Note that this permits an individual to have papers in more than one field.

To correct for field and paper-age differences, the requisite adjustment is to divide each x_i by a_i, producing the adjusted citation list $(x_1/a_1, ..., x_n/a_n)$; so each entry i is rational, being an integer x_i divided by a rational a_i. We then simply apply our index to this adjusted list. Consequently, any citation list $(x_1, ..., x_n)$ is given the adjusted index value,

$$\left(\sum_{i=1}^{n}\left(\frac{x_i}{a_i}\right)^{\sigma}\right)^{1/\sigma}.$$

4.3. Scale Invariance

Clearly, the above rescaling for field and paper-age that aligns the otherwise disparate distributions of citations across fields and paper-ages is arbitrary up to any positive scaling factor. In particular, a common distribution of citations across fields and paper-ages would also have obtained had Radicchi et. al. (2008) divided any citation list entry x_i, not by a_i, but instead by $2a_i$. Consequently, if after rescaling, one's index ranks the adjusted list $(x_1/a_1, ..., x_n/a_n)$ above the adjusted list $(x'_1/a'_1, ..., x'_n/a'_n)$, then it should also rank $(x_1/2a_1, ..., x_n/2a_n)$ above $(x'_1/2a'_1, ..., x'_n/2a'_n)$. Indeed, it should similarly rank $(x_1/\lambda a_1, ..., x_n/\lambda a_n)$ above $(x'_1/\lambda a'_1, ..., x'_n/\lambda a'_n)$ for any (rational) $\lambda > 0$.[14] But this is precisely what is ensured by our property 4, scale invariance.[15]

When scale invariance is violated, as it is with the h-index, serious difficulties can arise. For instance, within a biology department, biologist B_1 may be ranked above biologist B_2 according to the h-index, but when B_1 and B_2 are each compared to a mathematician, M, where rescaling is required to compare across the two fields, applying the h-index to the rescaled citation lists can result in B_2 being ranked first, M second, and B_1 third! Moreover, the final ranking can depend on whether one field's lists are scaled up or the other field's lists are scaled down. A typical example follows.

Example 4.2. *Suppose that B_1 has 15 papers, each with 15 citations; and B_2 has 10 papers, each with 30 citations. Hence, B_1 has an h-index of 15 which is above B_2's h-index of 10. Suppose that M has 8 papers, each with 7 citations and that to compare biologists to*

[14] This alignment argument holds of course also for irrational λ. However, there is never any need to consider irrational lists.

[15] Of course, once rescaling has aligned all the distributions, any subsequent common-across-fields increasing transformation applied to each citation list entry will maintain that alignment (and these are the only increasing transformations that can do so – see fn. 12). But the only such transformations compatible with properties 1-4 are linear.

mathematicians one must scale down the biologists' citations by a factor of 3*. Hence,* B_1*'s rescaled citation list contains* 15 *papers with* 5 *citations each and* B_2*'s rescaled citation list contains* 10 *papers with* 10 *citations each. Consequently, when comparing all three individuals,* B_2 *is ranked first* ($h = 10$)*,* M *second* ($h = 7$) *and* B_1 *third* ($h = 5$)*. On the other hand, had we instead scaled up the mathematician's list by a factor of 3, the final ranking would be* B_1 *first* ($h = 15$)*,* B_2 *second* ($h = 10$) *and* M *third* ($h = 8$)*.*

4.4. Other differences and group rankings

We now show how to adjust individual citation lists to also account for differences in the scientific ages of individuals and the number of authors, and we also show how our index can be used to rank groups of individuals such as departments, universities, or even countries.[16]

Suppose that an individual's raw citation list is $(x_1, ..., x_n)$ and that a_i denotes, as above, the average number of citations received by all of the papers published in the same field and in the same year as the i-th paper in the list. Let k_i denote the total number of authors associated with paper i and let t denote the scientific age of the individual.

To take into account all of these differences, the individual's citation list should be adjusted by dividing the i-th entry by $a_i(k_it)^{1/\sigma}$, where $\sigma > 1$ is the elasticity parameter of our index. Consequently, this individual's adjusted index value is,

$$\left(\sum_{i=1}^{n}\left(\frac{x_i}{a_i(k_it)^{1/\sigma}}\right)^{\sigma}\right)^{1/\sigma}. \tag{4.3}$$

We now explain why. Since we have already argued above that one must divide each x_i by a_i to account for differences in fields and the ages of papers, it remains only to adjust for scientific age and number of coauthors. We begin with scientific age.

Suppose that two individuals, X and X' have citation lists $x = (x_1, ..., x_n)$ and $x' = (x'_1, ..., x'_n)$ and that they differ in fields, ages of papers, and scientific age. Let us first eliminate the differences in fields and the ages of their papers by dividing each x_i by a_i, and dividing each x'_i by a'_i, as above. This first adjustment gives $\left(\frac{x_1}{a_1}, ..., \frac{x_n}{a_n}\right)$ as X's list and $\left(\frac{x'_1}{a'_1}, ..., \frac{x'_n}{a'_n}\right)$ as X''s list, and in particular renders all papers the same age.[17] It remains to account for the different scientific ages of X and X'.

Assume that for each individual and for any given number of citations, the number of papers written per year by that individual and with that number of citations is roughly constant. Hence, if X's (scientific) age is t, then under this assumption, over the next t years we would expect X to match his/her performance of the first t years and so we should *append*

[16]Ellison (2012) uses Hirsch-like indices to compare computer science departments.
[17]The entries as written here may no longer be in decreasing order.

to his/her (paper-age and field adjusted) citation list the n entries $\left(\frac{x_1}{a_1}, ..., \frac{x_n}{a_n}\right)$. Consequently, at age $2t$, X's (paper-age and field adjusted) citation list would be $\left(\frac{x_1}{a_1}, ..., \frac{x_n}{a_n}; \frac{x_1}{a_1}, ..., \frac{x_n}{a_n}\right)$, and at age $3t$ it would be $\left(\frac{x_1}{a_1}, ..., \frac{x_n}{a_n}; \frac{x_1}{a_1}, ..., \frac{x_n}{a_n}; \frac{x_1}{a_1}, ..., \frac{x_n}{a_n}\right)$, etc. Similarly, we can derive the (paper-age and field adjusted) citation lists of both X and X' when each of them reaches the common scientific age of $t \times t'$, at which point X is t' times as old as currently, and X' is t times as old as currently. This yields the following lists which adjust for both differences in the ages of papers and differences in the scientific ages of the individuals.

$$X\text{'s adjusted list} = \underbrace{\left(\frac{x}{a}, ..., \frac{x}{a}\right)}_{t' \text{ copies of } \frac{x}{a}}, \text{ where } \tfrac{x}{a} := \left(\tfrac{x_1}{a_1}, ..., \tfrac{x_n}{a_n}\right)$$

$$X'\text{'s adjusted list} = \underbrace{\left(\frac{x'}{a'}, ... \frac{x'}{a'}\right)}_{t \text{ copies of } \frac{x'}{a'}}, \text{ where } \tfrac{x'}{a'} = \left(\tfrac{x_1'}{a_1'}, ..., \tfrac{x_n'}{a_n'}\right)$$

Since these adjusted lists are such that fields have been adjusted for, all papers are the same age of one year, and both individuals have the same scientific age of $t \times t'$, we can apply our index to them. Doing so, we would rank X above X' if

$$\left(t' \sum_{i=1}^{n} \left(\frac{x_i}{a_i}\right)^{\sigma}\right)^{1/\sigma} > \left(t \sum_{i=1}^{n} \left(\frac{x_i'}{a_i'}\right)^{\sigma}\right)^{1/\sigma},$$

or equivalently if

$$\left(\frac{1}{t} \sum_{i=1}^{n} \left(\frac{x_i}{a_i}\right)^{\sigma}\right)^{1/\sigma} > \left(\frac{1}{t'} \sum_{i=1}^{n} \left(\frac{x_i'}{a_i'}\right)^{\sigma}\right)^{1/\sigma}.$$

Finally, let us consider adjusting the index for the number of authors. So, assume now that, in addition, the lists of X and X' differ in the number of authors on their papers. We make the simple assumption that more authors have the effect of increasing the number of papers that one can write in proportion to the number of authors. Hence, a list $x = (x_1, ..., x_n)$ of single-authored papers would have been $(x_1, ..., x_n;\ x_1, ..., x_n)$ had this author instead always coauthored his papers with one more author.

Let $k = k_1 k_2 ... k_n$ and let $k' = k_1' k_2' ... k_n'$. If X had instead always had kk' total authors on each of paper, then for each paper i he/she could have produced kk'/k_i papers like it. Hence, for each i, his list would have contained kk'/k_i papers with $x_i/(a_i t^{1/\sigma})$ citations (after adjusting for fields, paper-ages, and scientific age). And similarly for X'. Hence, if we adjust both X and X''s lists so that they each have kk' authors on all papers, we would rank X

above X' if

$$\left(\sum_{i=1}^{n}\frac{kk'}{k_i}\left(\frac{x_i}{a_i t^{1/\sigma}}\right)^{\sigma}\right)^{1/\sigma} > \left(\sum_{i=1}^{n}\frac{kk'}{k'_i}\left(\frac{x'_i}{a'_i (t')^{1/\sigma}}\right)^{\sigma}\right)^{1/\sigma},$$

or equivalently if

$$\left(\sum_{i=1}^{n}\left(\frac{x_i}{a_i (k_i t)^{1/\sigma}}\right)^{\sigma}\right)^{1/\sigma} > \left(\sum_{i=1}^{n}\left(\frac{x'_i}{a'_i (k'_i t')^{1/\sigma})}\right)^{\sigma}\right)^{1/\sigma}.$$

Taken altogether, we obtain the criterion in (4.3).

Consider now the problem of ranking *groups* of individuals. Because our approach is to first adjust individual lists to eliminate differences in important characteristics, it is not unreasonable after doing so to append all of the adjusted lists within a group to obtain a single citation list that represents that group. After adjusting for the size of the group (much like the adjustment for number of authors), our index is applied to the adjusted list.

For example, for each of a number of academic departments (which might be from different disciplines) a *department citation list* is obtained by stringing together all the adjusted citation lists of all of its members. Our index is then applied to this list.

Specifically, if the department has m members and member j is of scientific age t_j, has citation list $(x_{1j}, ...x_{nj})$, and a_{ij} is the average number of citations received by all of the papers published in the same field and in the same year as j's i-th paper, and k_{ij} is the number of authors associated with j's i-th paper, then our index assigns this department the index number,

$$\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{x_{ij}}{a_{ij}(mk_{ij}t_j)^{1/\sigma}}\right)^{\sigma}\right)^{1/\sigma}.$$

5. On the choice of σ

While any value of σ that is greater than 1 is consistent with properties 1-4, the following additional, though less basic, property provides a rationale for restricting the value of σ to be no greater than 2.[18] This property, called "balancedness," states in particular that when two given lists are equally ranked, the index's value of a new citation list that is a weighted average of the two given lists, is larger when the average is formed by placing the higher, rather than the lower, weight on the more balanced of the two given lists. The formal statement is as follows.

Balancedness For any two citation lists $x = (x_1, ..., x_n)$ and $y = (y_1, 0, ..., 0)$ of equal

[18]We are grateful to Faruk Gul and Debraj Ray for discussions that led us to this property.

length, if $\iota(x) \geq \iota(y)$, then $\iota(\beta x + \alpha y) \geq \iota(\alpha x + \beta y)$ for all rational α, β such that $\beta \geq \alpha \geq 0$ and $\alpha + \beta = 1$.

As the next proposition states, in order for our index to satisfy balancedness, it is necessary and sufficient that the value of σ be no greater than 2.

Proposition 5.1. *If $\sigma > 1$, the index $\left(\sum_{i=1}^{n} x_i^{\sigma}\right)^{1/\sigma}$ displayed in Theorem 3.1 satisfies balancedness if and only if $\sigma \leq 2$.*

6. Ranking Journals

Related to the topic of this paper is the issue of ranking scholarly journals. However, while the reason for ranking individuals is clear (limited resources must somehow be allocated), the reason for ranking journals is less clear. One possible view, and the view taken here, is that ranking journals is useful only in so far as it provides information on how to rank individuals.

6.1. Impact factors

Consider, for example the problem of assessing young scholars whose papers have not had sufficient time to collect citations. In such cases, one must form an estimate of the impact that their papers will have, i.e., one must estimate, for each paper, its expected number of future citations. This is, in fact, easily done by using a journal's so-called "impact factor," which is the average number of citations received per year by papers published in that journal. Once an estimate of the number of citations has been obtained for each paper, our citation index can be applied. This is perhaps one of the most useful roles that impact factors can play in practice.[19]

6.2. Endogenously-weighted citations

Palacios-Huerta and Volij (2004) provide an axiomatization of a cardinal index that assigns a numerical score to each journal. A journal's score is a weighted sum of all of the citations it receives from all journals, where the weights are endogenous in the sense that the weight placed on any given citation is proportional to the score of the journal from which it came.

[19] One might take this a step further if citations from higher-impact journals lead to more citations – due to the increased visibility they generate – than citations from lower impact journals. Thus, one might weight citations from different journals differently, and perhaps simply in proportion to journal impact factors, if this leads to a better prediction of future citations – something that is presumably testable. Being proportional to impact factors, the weights here are not "endogenously determined" in the sense of the next section. We thank Pablo Beker for suggesting this additional potential use of journal impact factors.

One might wonder whether a similarly endogenous approach to assigning weights to citations is possible/desirable when the objective is to rank individuals. While this is certainly an interesting question, we have only two small remarks. First, the endogenous approach allows the potential for over-weighting. For example, suppose that when individual A – with many citations – cites a paper, then that paper's expected number of additional citations is greater than when individual B – with fewer citations – cites that paper. Then, even with equally-weighted citations, a paper cited by A will receive a higher "score" than a paper cited by B because it will receive more citations. There then seems no need/reason to give citations received from A more weight than those from B. And if papers cited by A receive fewer citations than those cited by B, then it is perhaps even less reasonable to give higher weight to citations from A. Of course, what the endogenous-weight approach presumes is that being cited by a highly cited individual is better, *for its own sake*, than being cited by a less-cited individual. The rationale for this, not uncommon, presumption is not entirely clear to us in the context of evaluating a record as it currently stands.

Second, and more important, is the problem of the ordinality of the index for ranking individuals. Without a cardinal index there is no hope of producing a unique set of endogenous weights, since each new monotone transformation of the index can change the weights. Moreover, under ordinality, the trivial set of uniform weights is itself endogenously consistent at least as a limiting solution. And with uniform weights, we are back to a standard index.

A. Appendix

It is well-known that welfare (or utility) functions defined on various subsets of $\mathbb{R}^n$ that are continuous, increasing, additively separable and scale invariant must be equivalent to our index, or a closely related functional form (see Bergson, 1936; Roberts, 1980; and Blackorby and Donaldson, 1982). The difficulty in our case is that our domain $\mathbb{L}$ is quite fragmented in that no non-singleton subset is connected. Thus, the thrust of the proof is to show that our index (or, more precisely, one that is equivalent to it) can be extended to an index that (i) for each n, is continuous on $\mathbb{R}^n_+$, and (ii) satisfies all four of our properties. Once this is done we apply the standard results. For the reader's convenience, we restate our main result.

Theorem 3.1. A citation index satisfies properties 1-4 *if and only if* it is equivalent to the index that assigns to each citation list $(x_1, ..., x_n)$, the number,

$$\left(\sum_{i=1}^{n} x_i^\sigma\right)^{1/\sigma},$$

where σ is some fixed number strictly greater than one.

Proof. Sufficiency is clear so we focus only on necessity and suppose that the index function ι satisfies properties 1-4. For any natural number n, and to any n nonnegative rational numbers $x_1 \geq x_2 \geq ... \geq x_n$, ι assigns the real number, $\iota(x_1, ..., x_n)$. For every n, it will be convenient to extend ι to all of $\mathbb{Q}_+^n$ by defining $\iota(x_1, ..., x_n) := \iota(X_1, X_2, ..., X_n)$, where X_i is i-th order statistic of $x_1, ..., x_n$. Thus ι is now defined on the extended domain $\mathbb{L}^* := \cup_n \mathbb{Q}_+^n$ in such a way that it is symmetric; i.e., if $x \in \mathbb{L}^*$ is a permutation of $x' \in \mathbb{L}^*$, then $\iota(x) = \iota(x')$. Clearly, the extended function ι satisfies the associated extensions to $\mathbb{L}^*$ of properties 1-4. We work with the extended index ι from now on.

Let us first show that, for every $n > 0$, ι is *strongly increasing* on $\mathbb{Q}_+^n$ i.e.,

$$\iota(x_i', x_{-i}) > \iota(x_i, x_{-i}), \ \forall x \in \mathbb{Q}_+^n, \ \forall i, \text{ and } \forall \text{rational } x_i' > x_i. \tag{A.1}$$

By IND, it suffices to show that $\iota(x_i') > \iota(x_i)$, or by SI, that $\iota(1) > \iota\left(\frac{x_i}{x_i'}\right)$. Since $x_i' > x_i$, it suffices to show that $\iota(1) > \iota(\lambda)$ for all rational $\lambda \in [0, 1)$. We shall argue by contradiction.

If not, then there is a nonnegative rational $\lambda < 1$ such that $\iota(\lambda) \geq \iota(1)$. Applying scale invariance k times, each time using the scaling factor λ, gives $\iota(\lambda^k) \geq \iota(\lambda^{k-1})$. Since this holds for any positive integer k, we have $\iota(\lambda^k) \geq \iota(\lambda^{k-1}) \geq ... \geq \iota(\lambda) \geq \iota(1)$. For any positive rational $r < 1$, we may choose k such that $\lambda^k < r$ since $\lambda < 1$. Hence, $\iota(r) \leq \iota(1) \leq \iota(\lambda^k) \leq \iota(r)$, where the first and last inequalities are by MON. Hence, $\iota(1) = \iota(r)$ for every positive rational $r < 1$. But then, by SI, also $\iota(1) = \iota(\frac{1}{r})$ (use the scaling factor $1/r$). Hence, ι is constant on $\mathbb{Q}_{++}$ and, by continuity, on $\mathbb{Q}_+$. We next show inductively that, for each n, ι is constant on $\mathbb{Q}_+^n$, from which we will eventually derive a contradiction.

So, suppose that ι is constant on $\mathbb{Q}_+^n$. We must show that ι is constant on $\mathbb{Q}_+^{n+1}$. But this will follow if we can show that for all $(x_1, ..., x_{n+1}) \in \mathbb{Q}_+^{n+1}$

$$\iota(\mathbf{1}_{n+1}) = \iota(x_1, ..., x_n, 1) = \iota(x_1, ..., x_{n+1}),$$

where, for each m, $\mathbf{1}_m$ denotes the vector $(1, ..., 1) \in \mathbb{Q}_+^m$. But the first equality follows by IND because $\iota(\mathbf{1}_n) = \iota(x_1, ..., x_n)$ by the induction hypothesis, and the second equality follows by IND because $\iota(x_2, ..., x_n, 1) = \iota(x_2, ..., x_{n+1})$ by the induction hypothesis. Hence, ι is constant on $\mathbb{Q}_+^n$ for every n. We next derive the desired contradiction.

By DR there is a positive integer $\bar{k}$ and there is $\bar{x} \in \mathbb{L}^*$ such that $\iota(\bar{k}, \bar{x}) > \iota(\mathbf{1}_{\bar{k}}, \bar{x})$,where $\mathbf{1}_{\bar{k}} = (1, ..., 1)$ is of length $\bar{k}$. Hence, by IND, $\iota(\bar{k}) > \iota(\mathbf{1}_{\bar{k}})$. By MON, there exists m_2 large enough so that $\iota(\bar{k}, m_2) \geq \iota(\bar{k}) > \iota(\mathbf{1}_{\bar{k}})$. Applying MON once again, there exists m_3 large enough so that $\iota(\bar{k}, m_2, m_3) \geq \iota(\bar{k}, m_2) \geq \iota(\bar{k}) > \iota(\mathbf{1}_{\bar{k}})$. After $\bar{k}-1$ such successive applications of MON, we obtain $m_2, ..., m_{\bar{k}}$ such that $\iota(\bar{k}, m_2, ..., m_{\bar{k}}) \geq \iota(\bar{k}) > \iota(\mathbf{1}_{\bar{k}})$, implying that ι is not constant on $\mathbb{Q}_+^{\bar{k}}$. This contradiction establishes (A.1).

We can now prove the following claim.

Claim. There is a continuous function $\hat{\iota} : \mathbb{R}_+^n \to \mathbb{R}$ that is equivalent to ι on $\mathbb{Q}_+^n$ – i.e., $\forall x, y \in \mathbb{Q}_+^n$, $\hat{\iota}(x) \geq \hat{\iota}(y)$ iff $\iota(x) \geq \iota(y)$.

Proof of Claim. Since n is fixed, we write $\mathbf{1}$ instead of $\mathbf{1}_n$. Define a nondecreasing function $d : \mathbb{Q}_+^n \to \mathbb{R}_+$ as follows. For every $x \in \mathbb{Q}_+^n$,

$$d(x) := \inf\{r \in \mathbb{Q}_+ : \iota(r\mathbf{1}) \geq \iota(x)\}.$$

Then d is nondecreasing because ι is nondecreasing.

Let us show that,

$$d \text{ is equivalent to } \iota \text{ on } \mathbb{Q}_+^n. \tag{A.2}$$

Since it is obvious that $\iota(x) = \iota(y)$ implies $d(x) = d(y)$, it suffices to show that $\iota(x) > \iota(y)$ implies $d(x) > d(y)$.

First note that SI implies that d is homogeneous of degree 1 (since SI implies that for any positive rational λ, $\{r' \in \mathbb{Q}_+ : \iota(r'\mathbf{1}) \geq \iota(\lambda x)\} = \lambda\{r \in \mathbb{Q}_+ : \iota(r\mathbf{1}) \geq \iota(x)\}$). In particular, $d(r\mathbf{1}) = r$ for any nonnegative rational r ($d(0) = 0$, clearly).

Second, note that for $x \neq 0$, (A.1) implies that $\iota(x) > \iota(0)$. By continuity there is a rational $\varepsilon \in (0, 1)$ such that $\iota(x) > \iota(\varepsilon\mathbf{1}_n)$ and hence, $d(x) \geq \varepsilon > 0$.

Suppose now that $\iota(x) > \iota(y)$. Then, by continuity, there is a positive rational $\delta < 1$ such that $\iota(y) < \iota(\delta x)$, and so $d(y) \leq d(\delta x)$. But, by 1-homogeneity $d(\delta x) = \delta d(x) < d(x)$ since $d(x) > 0$. Hence, $d(y) \leq d(\delta x) < d(x)$, which proves (A.2).

Let $x_m, y_m \in \mathbb{Q}^n_+$ be sequences converging to the same point $x \in \mathbb{R}^n_+$. We will next establish that,

$$\lim_m d(x_m) = \lim_m d(y_m). \tag{A.3}$$

That is, the limits exist and are equal.

Let us first dispense with the case in which the common limit point is $x = 0$. Then, for any positive rational, r, $x_m << r\mathbf{1}$ for all m large enough. Since d is nondecreasing, $0 \leq d(x_m) \leq d(r\mathbf{1}) = r$ for all m large enough. Since r is arbitrary, we conclude that $\lim_m d(x_m) = 0$; and similarly for $\lim_m d(y_m)$. So (A.3) holds for $x = 0$ and we henceforth assume that $x \neq 0$.

Evidently, $d(x_m)$ is bounded because, for any $z \in \mathbb{Q}^n_+$ satisfying $z >> x$, we have $d(0) \leq d(x_m) \leq d(z)$ for m large enough since d is nondecreasing. Similarly, $d(y_m)$ is bounded. Hence, if either limit fails to exist or if the limits exist but are not equal, the sequences x_m and y_m would admit subsequences along which $d(x_m)$ converges, to a say, and $d(y_m)$ converges, to b say, and $a \neq b$. Restricting attention to these subsequences, reindexing, and supposing that $a < b$, this would mean that,

$$\lim_m d(x_m) = a < b = \lim_m d(y_m). \tag{A.4}$$

Thus, to establish (A.3) it suffices to assume (A.4) and derive a contradiction.

Choose rationals $q, q' \in (a, b)$ such that $q < q'$ and recall that $d(r\mathbf{1}) = r$ for any nonnegative rational r. Hence,

$$\lim_m d(x_m) = a < q = d(q\mathbf{1}) < d(q'\mathbf{1}) = q' < b = \lim_m d(y_m),$$

and so $d(x_m) < d(q\mathbf{1}) < d(q'\mathbf{1}) < d(y_m)$ for all sufficiently large m. Consequently, since d and ι are equivalent, letting $w := q\mathbf{1}$ and $w' := q'\mathbf{1}$, we have

$$\iota(x_m) < \iota(w) < \iota(w') < \iota(y_m), \tag{A.5}$$

for all sufficiently large m. Without loss, we may assume that (A.5) holds for all m.

Choose any rational $\delta \in (0, 1)$. Scale invariance applied to (A.5) implies in particular that,

$$\iota(\delta w') < \iota(\delta y_m), \text{ for all } m. \tag{A.6}$$

Choose $z', z \in \mathbb{Q}^n_+$ so that $\delta x_i < z'_i < z_i < x_i$ holds for every i such that $x_i > 0$, and such that $z'_i = z_i = 0$ for every i such that $x_i = 0$. Then both z' and z are nonzero (because x is nonzero) and so $\iota(z') < \iota(z)$ by (A.1). By continuity, there is a rational $\varepsilon \in (0, 1)$ such that $\iota(z' + \varepsilon\mathbf{1}) < \iota(z)$. For each i, define $z''_i = z'_i$ if $z_i > 0$ and $z''_i = \varepsilon$ if $z_i = 0$. The resulting vector $z'' \in \mathbb{Q}^n_{++}$ is such that all coordinates are positive and $z'' \leq z' + \varepsilon\mathbf{1}$. Consequently, by MON, $\iota(z'') \leq \iota(z' + \varepsilon\mathbf{1}) < \iota(z)$. Moreover, (i) $\delta x_i < z''_i = z'_i < z_i < x_i$ holds for every i such that $x_i > 0$, and (ii) $z''_i = \varepsilon$ and $z_i = 0$ holds for all i such that $x_i = 0$. Consequently, $\delta x << z''$ and $z \leq x$, with equality in the latter only for coordinates i such that $x_i = 0$.

Since x_m and y_m converge to x, there must be m' large enough so that $\delta y_{m'} << z''$ and $z \leq x_{m'}$. Applying MON to these two vector inequalities and using $\iota(z'') < \iota(z)$ gives,

$$\iota(\delta y_{m'}) \leq \iota(z'') < \iota(z) \leq \iota(x_{m'}).$$

Combined with (A.5) and (A.6), we obtain $\iota(\delta w') < \iota(\delta y_{m'}) < \iota(x_{m'}) < \iota(w) < \iota(w')$ which, in particular implies,

$$\iota(\delta w') < \iota(w) < \iota(w').$$

But since δ is an arbitrary rational in $(0, 1)$, this contradicts the continuity of ι, and we conclude that (A.3) holds.

A consequence of (A.3) is that we may extend d to all of $\mathbb{R}^n_+$ by defining its extension $\hat{\iota} : \mathbb{R}^n_+ \to \mathbb{R}_+$ as follows. For every $x \in \mathbb{R}^n_+$,

$$\hat{\iota}(x) := \lim_{x_m \to x} d(x_m),$$

where the limit is taken over any sequence $x_m \in \mathbb{Q}^n_+$ converging to x.

The proof of the claim will be complete if we can show that $\hat{\iota}$ is continuous on $\mathbb{R}^n_+$ and equivalent to ι on $\mathbb{Q}^n_+$. For the latter, suppose that $x \in \mathbb{Q}^n_+$. Since the limit defining $\hat{\iota}(x)$ is independent of the sequence $x_m \in \mathbb{Q}^n_+$ converging to x that is chosen, we can in particular choose the constant sequence $x_m = x$ (since $x \in \mathbb{Q}^n_+$). Therefore $\hat{\iota}(x) = d(x)$ for every $x \in \mathbb{Q}^n_+$ (and so $\hat{\iota}$ is indeed an extension of d). Since d is equivalent to ι on $\mathbb{Q}^n_+$ so too is $\hat{\iota}$.

For continuity on $\mathbb{R}^n_+$, suppose that $x_m \in \mathbb{R}^n_+$ converges to $x \in \mathbb{R}^n_+$. For every m, by the definition of $\hat{\iota}(x_m)$ we may choose $y_m \in \mathbb{Q}^n_+$ such that $\|x_m - y_m\| \leq \frac{1}{m}$ and $|\hat{\iota}(x_m) - d(y_m)| \leq \frac{1}{m}$. Hence, $y_m \to x$, and $\lim_m(\hat{\iota}(x_m) - d(y_m)) = 0$. But because $y_m \in \mathbb{Q}^n_+$ converges to x, $\hat{\iota}(x) = \lim_m d(y_m)$, by the definition of $\hat{\iota}(x)$. Hence, $\lim_m \hat{\iota}(x_m) = \lim_m(\hat{\iota}(x_m) - d(y_m) + d(y_m)) = \lim_m(\hat{\iota}(x_m) - d(y_m)) + \lim_m d(y_m) = 0 + \hat{\iota}(x) = \hat{\iota}(x)$, proving continuity, and completing the proof of the claim. ■

Let us now establish some additional properties of the continuous function $\hat{\iota} : \mathbb{R}^n_+ \to \mathbb{R}$. Since $\hat{\iota}$ is equivalent to ι on $\mathbb{Q}^n_+$, $\hat{\iota}$ is a symmetric and strongly increasing function on $\mathbb{Q}^n_+$ and satisfies all four of our properties, MON, IND, SI, and DR on $\mathbb{Q}^n_+$. We also note that $\hat{\iota}$ is nondecreasing on $\mathbb{R}^n_+$, a fact that is immediate from the definition because d is nondecreasing on $\mathbb{Q}^n_+$. We will use these facts, as well as the continuity of $\hat{\iota}$, freely.

Fact 1. (Symmetry on $\mathbb{R}^n_+$) If $x \in \mathbb{R}^n_+$ is a permutation of the coordinates of x', then $\hat{\iota}(x) = \hat{\iota}(x')$ follows from the observation that for any $x'_m \in \mathbb{Q}^n_+$ converging to x, we have also that x_m converges to x, where x_m is obtained from x'_m by applying the same permutation as performed on x' to obtain x. But since $\hat{\iota}$ is a symmetric function on $\mathbb{Q}^n_+$, we have $\hat{\iota}(x_m) = \hat{\iota}(x'_m)$ for every m. Hence, $\hat{\iota}(x) = \lim_m \hat{\iota}(x_m) = \lim_m \hat{\iota}(x'_m) = \hat{\iota}(x')$.

Fact 2. (Scale Monotonicity on $\mathbb{R}^n_+$). If $x \in \mathbb{R}^n_+ \backslash \{0\}$ and $\delta \in (0, 1)$ we wish to show that $\hat{\iota}(\delta x) < \hat{\iota}(x)$. Choose $z, z' \in \mathbb{Q}^n_+$ such that $\delta x_i < z_i < z'_i < x_i$ for every i such that $x_i > 0$ and set $z_i = z'_i = 0$ otherwise. Then z and z' are distinct (since $x \neq 0$) and $\delta x \leq z \leq z' \leq x$. Therefore, $\hat{\iota}(\delta x) \leq \hat{\iota}(z) < \hat{\iota}(z') \leq \hat{\iota}(x)$, where the two weak inequalities follow because $\hat{\iota}$ is nondecreasing and the strict inequality follows because $\hat{\iota}$, is strongly increasing on $\mathbb{Q}^n_+$.

Fact 3. (Scale Invariance on $\mathbb{R}^n_+$ with positive real scalars). Suppose that $x, y \in \mathbb{R}^n_+$, $\lambda \in \mathbb{R}_{++}$ and $\hat{\imath}(x) > \hat{\imath}(y)$. Therefore, in particular, $x \neq 0$. By the continuity of $\hat{\imath}$ there exist $x', y' \in \mathbb{Q}^n_+$ and a rational $\delta \in (0,1)$ such that $x' \leq x$, $y' \geq y$ and $\hat{\imath}(\delta x') > \hat{\imath}(y')$. Since $\hat{\imath}$ satisfies SI on $\mathbb{Q}^n_+$, $\hat{\imath}(\lambda' \delta x') > \hat{\imath}(\lambda' y')$ holds for any positive rational λ'. Taking the limit as $x' \to x$, $y' \to y$, and $\lambda' \to \lambda$ implies $\hat{\imath}(\lambda \delta x) \geq \hat{\imath}(\lambda y)$. By Fact 2, $\hat{\imath}(\lambda x) > \hat{\imath}(\lambda \delta x) \geq \hat{\imath}(\lambda y)$. Hence, we have shown that $\hat{\imath}(x) > \hat{\imath}(y)$ implies $\hat{\imath}(\lambda x) > \hat{\imath}(\lambda y)$ and so $\hat{\imath}$ is scale invariant.

Fact 4. (Independence on $\mathbb{R}^n_+$) We will show that for every $x^0, x^1 \in \mathbb{R}^n_+$ and every coordinate i, $\hat{\imath}(x^0_i, x^0_{-i}) > \hat{\imath}(x^0_i, x^1_{-i})$ implies $\hat{\imath}(x^1_i, x^0_{-i}) > \hat{\imath}(x^1_i, x^1_{-i})$. Without loss of generality, we may assume that x^1_i is rational. For if it is not rational, then it is strictly positive. In that case, by Fact 3, we may rescale all four of the vectors (x^0_i, x^0_{-i}), (x^0_i, x^1_{-i}), (x^1_i, x^0_{-i}) and (x^1_i, x^1_{-i}) by the factor $1/x^1_i$ (and so x^1_i is rescaled to be 1, a rational), without affecting any of their pairwise rankings according to $\hat{\imath}$. Thus, assume that $x^1_i \in \mathbb{Q}_+$. If the first inequality holds, then by the continuity of $\hat{\imath}$, there are rational vectors $z_{-i} \leq x^0_{-i}$, $w_{-i} \geq x^1_{-i}$, and a rational number r^0_i, such that $\hat{\imath}(r^0_i, z_{-i}) > \hat{\imath}(r^0_i, w^1_{-i})$. Since $\hat{\imath}$ satisfies IND on $\mathbb{Q}^n_+$, and because x^1_i is rational, we have $\hat{\imath}(x^1_i, z_{-i}) > \hat{\imath}(x^1_i, w_{-i})$. And since $\hat{\imath}$ is nondecreasing on $\mathbb{R}^n_+$, $\hat{\imath}(x^1_i, x^0_{-i}) \geq \hat{\imath}(x^1_i, z_{-i}) > \hat{\imath}(x^1_i, w_{-i}) \geq \hat{\imath}(x^1_i, x^1_{-i})$, as desired.

Fact 5. (Strongly Increasing on $\mathbb{R}^n_+$). Suppose that $x \in \mathbb{R}^n_+$ and that $x'_i > x_i$. We will show that $\hat{\imath}(x'_i, x_{-i}) > \hat{\imath}_n(x)$. Choose rationals r' and r such that $x'_i > r' > r > x_i$. Then, for every nonnegative rational vector z_{-i}, $\hat{\imath}(x'_i, z_{-i}) \geq \hat{\imath}(r', z_{-i}) > \hat{\imath}(r, z_{-i}) \geq \hat{\imath}(x_i, z_{-i})$, where the weak inequalities follow because $\hat{\imath}$ is nondecreasing, and the strict inequality follows because $\hat{\imath}$ is strongly increasing on $\mathbb{Q}^n_+$. Fact 4 (applied successively to each coordinate j of z_{-i} and x_{-i}) implies that $\hat{\imath}(x'_i, x_{-i}) > \hat{\imath}(x_i, x_{-i})$.

Because the continuous function $\hat{\imath} : \mathbb{R}^n_+ \to \mathbb{R}$ (which of course can depend on n) satisfies the properties given in Facts 1-5, we may apply Theorem 2, part (i), in Blackorby and Donaldson (1982) (see also Theorem 6 in Roberts, 1980) to conclude, using also that $\hat{\imath}$ is symmetric, that for all $n \geq 3$, there exists $a_n \in \mathbb{R}$ and there exists a strictly increasing $\phi_n : \mathbb{R}_{++} \to \mathbb{R}$ such that,

$$\hat{\imath}(x) = \phi_n \left(\left(\sum_i x_i^{a_n} \right)^{\frac{1}{a_n}} \right), \text{ for all } x \in \mathbb{R}^n_{++},$$

where we adopt the convention that $(\sum_i x_i^{a_n})^{\frac{1}{a_n}} := \Pi_i x_i$ if $a_n = 0$. From this we may conclude that $\hat{\imath}$ is equivalent to $(\sum_i x_i^{a_n})^{\frac{1}{a_n}}$ on the *strictly positive* orthant $\mathbb{R}^n_{++}$.

Let us first argue that $a_n > 0$. If not, then let $w(\alpha) = (\alpha, 1, ..., 1) \in \mathbb{R}^n_+$ and observe that, for $a_n \leq 0$, $\lim_{\alpha \to 0^+} (\sum_i (w_i(\alpha))^{a_n})^{\frac{1}{a_n}} = 0$. Consequently, because by MON $\hat{\imath}(1, 0, ..., 0) \leq \hat{\imath}(\alpha, 1, ..., 1)$, we have $\hat{\imath}(1, 0, ..., 0) \leq \lim_{\alpha \to 0^+} \hat{\imath}(w(\alpha)) = \lim_{\alpha \to 0^+} \phi_n \left((\sum_i (w_i(\alpha))^{a_n})^{\frac{1}{a_n}} \right) < \phi_n(\lambda n^{\frac{1}{a_n}}) = \hat{\imath}(\lambda \mathbf{1}_n) = \lambda$ holds for any $\lambda > 0$, where the strict inequality follows because ϕ_n is strictly increasing and $\lim_{\alpha \to 0^+} (\sum_i (w_i(\alpha))^{a_n})^{\frac{1}{a_n}} = 0 < \lambda n^{\frac{1}{a_n}}$. We conclude, since $\hat{\imath}$ is nonnegative, that $\hat{\imath}(1, 0, ..., 0) = 0$, which is a contradiction since $\hat{\imath}(0, ..., 0) \geq 0$ (since $\hat{\imath}$ is nonnegative) and $\hat{\imath}$ is strongly increasing on $\mathbb{R}^n_+$ (Fact 5).

Hence, for every $n \geq 3$, $a_n > 0$ and $\hat{\imath}$ is equivalent to $\sum_{i=1}^n x_i^{a_n}$ on $\mathbb{R}^n_{++}$. But since both $\hat{\imath}$ and $\sum_{i=1}^n x_i^{a_n}$ are continuous on $\mathbb{R}^n_+$ we may conclude that $\hat{\imath}$ is equivalent to $\sum_{i=1}^n x_i^{a_n}$ on $\mathbb{R}^n_+$. In particular, because ι is equivalent to $\hat{\imath}$ on $\mathbb{Q}^n_+$, we have that ι is equivalent to $\sum_{i=1}^n x_i^{a_n}$ on $\mathbb{Q}^n_+$.

Let us now show that $a_n = a_{n+1}$ for all $n \geq 3$. For any $x, y \in \mathbb{R}^n_+$, $\sum_{i=1}^n x_i^{a_n} \geq \sum_{i=1}^n y_i^{a_n}$ iff $\hat{\imath}(x) \geq \hat{\imath}(y)$ which by Fact 4 (independence on $\mathbb{R}^n_+$) is iff $\hat{\imath}(x, 0) \geq \hat{\imath}(y, 0)$, which holds iff $\sum_{i=1}^n x_i^{a_n+1} \geq \sum_{i=1}^n y_i^{a_{n+1}}$. But then $\sum_{i=1}^n x_i^{a_n}$ is equivalent to $\sum_{i=1}^n x_i^{a_{n+1}}$ on $\mathbb{R}^n_+$ from which we conclude (because for every i, j, the marginal rates of substitution $(x_i/x_j)^{a_{n+1}-1}$ and $(x_i/x_j)^{a_n-1}$ must then be identically equal on $\mathbb{R}^n_{++}$) that $a_n = a_{n+1}$ for every $n \geq 3$.

Letting $\sigma > 0$ denote the constant value of a_n, we have shown for all $n \geq 3$ that, on $\mathbb{Q}^n_+$, ι is equivalent to $\sum_{i=1}^n x_i^\sigma$. We next show that adding a paper with zero citations must leave the index unchanged.

By MON, for every $x_1 \in \mathbb{Q}_+$, there exists a positive integer m such that $\iota(x_1) \leq \iota(x_1, m)$. By SI, $\iota(\lambda x_1) \leq \iota(\lambda x_1, \lambda m)$ for every positive rational λ, and so by IND, $\iota(z, \lambda x_1) \leq \iota(z, \lambda x_1, \lambda m)$ for every $z \in \mathbb{L}^*$. Taking the limit as $\lambda \to 0^+$ gives, by continuity, that $\iota(z, 0) \leq \iota(z, 0, 0)$ from which IND implies that $\iota(z) \leq \iota(z, 0)$ for every $z \in \mathbb{L}^*$. It remains to establish the reverse inequality.

As we have seen above, DR implies the existence of a positive integer $\bar{k}$ such that $\iota(\bar{k}) > \iota(\mathbf{1}_{\bar{k}})$; observe then that $\bar{k} \geq 2$. By SI, $\iota(\lambda\bar{k}) > \iota(\lambda\mathbf{1}_{\bar{k}})$ for every positive rational λ, and so by IND, $\iota(z, \lambda\bar{k}) > \iota(z, \lambda\mathbf{1}_{\bar{k}})$ for every $z \in \mathbb{L}^*$. Taking the limit as $\lambda \to 0^+$ yields, by continuity, that $\iota(z, 0) \geq \iota(z, 0, ..., 0) \geq \iota(z, 0, 0)$, where the second inequality follows because $\bar{k} \geq 2$ and because, as shown in the previous paragraph, the index cannot fall when a paper with zero citations is added. But IND together with $\iota(z, 0) \geq \iota(z, 0, 0)$ implies that $\iota(z) \geq \iota(z, 0)$ for every $z \in \mathbb{L}^*$, as desired. Hence, adding a paper with zero citations leaves the value of the index unchanged.

We can now characterize the manner in which the index compares citation lists of different lengths.

Consider any $x, y \in \mathbb{L}^*$. Then $x \in \mathbb{Q}^m_+$ and $y \in \mathbb{Q}^k_+$ for some $m, k \geq 1$. Assume without loss that $m \geq k$. By IND, MON, and the fact that adding papers with zero citations leaves the index unchanged, $\iota(x) \geq \iota(y)$ iff $\iota(x, 0, 0) \geq \iota(y, 0, 0, \mathbf{0}_{m-k})$, which (since the common length n of the two lists $(x, 0, 0)$ and $(y, 0, 0, \mathbf{0}_{m-k})$ satisfies $n \geq m + 2 \geq 3$) is iff $\sum_i x_i^\sigma \geq \sum_i y_i^\sigma$. Hence, we may conclude that ι is equivalent to the index that assigns to any citation list $(x_1, ..., x_n) \in \mathbb{L}^*$ of any length n the number $\sum_{i=1}^n x_i^\sigma$.

It remains only to show that $\sigma > 1$. But this follows from DR because $\sigma > 1$ implies that the index is strictly convex, and so DR will hold (for *all* k and x, in fact), while $0 < \sigma \leq 1$ implies that the index is concave, and DR will fail. ■

Proposition 5.1: If $\sigma > 1$, the index $\left(\sum_{i=1}^n x_i^\sigma\right)^{1/\sigma}$ displayed in Theorem 3.1 satisfies balancedness if and only if $\sigma \leq 2$.

Proof. It suffices to work with the equivalent index $\sum_{i=1}^n x_i^\sigma$.

Suppose first that $\sum_{i=1}^n x_i^\sigma$ satisfies balancedness. Then, in particular, setting $n = 2$ and $y_1 = 1$, we have that $\forall \alpha, x_1, x_2 \in \mathbb{Q}_+$ s.t. $x_1 \geq x_2$ and $\alpha \in [0, 1/2]$,

$$\text{if } x_1^\sigma + x_2^\sigma \geq 1, \text{ then } ((1-\alpha)x_1 + \alpha)^\sigma + (\alpha x_2)^\sigma \geq (\alpha x_1 + 1 - \alpha)^\sigma + ((1-\alpha)x_2)^\sigma.$$

Since all of the functions in the above display are continuous, the implication must also hold when all of the variables are real. Hence, $\forall \alpha \in [0, 1/2]$ and $\forall x_1, x_2 \in \mathbb{R}_+$ such that $x_1 \geq x_2$ and $x_1^\sigma + x_2^\sigma \geq 1$,

$$((1-\alpha)x_1 + \alpha)^\sigma + ((1-\alpha)x_2)^\sigma \geq (\alpha x_1 + 1 - \alpha)^\sigma + (\alpha x_2)^\sigma.$$

In particular, fix any $x_1, x_2 \in (0,1)$ such that $x_1 \geq x_2$ and $x_1^\sigma + x_2^\sigma = 1$. Then $\forall \alpha \in [0, 1/2]$, we must have

$$((1-\alpha)x_1 + \alpha)^\sigma + ((1-\alpha)x_2)^\sigma - (\alpha x_1 + 1 - \alpha)^\sigma - (\alpha x_2)^\sigma \geq 0. \tag{A.7}$$

Because, $x_1^\sigma + x_2^\sigma = 1$, the left-hand side of (A.7) is zero when $\alpha = 0$. Hence, if the inequality is to hold for all $\alpha \in [0, 1/2]$, the derivative of the left-hand side of (A.7) with respect to α must be nonnegative at $\alpha = 0$. Since its derivative is

$$\sigma((1-\alpha)x_1 + \alpha)^{\sigma-1}(1-x_1) - \sigma((1-\alpha)x_2)^{\sigma-1}x_2 + \sigma(\alpha x_1 + 1 - \alpha)^{\sigma-1}(1-x_1) - \sigma(\alpha x_2)^{\sigma-1}x_2,$$

we must have, after evaluating at $\alpha = 0$ (recall that $\sigma > 1$),

$$\sigma x_1^{\sigma-1}(1-x_1) - \sigma x_2^\sigma + \sigma(1-x_1) \geq 0.$$

But, because $x_1^\sigma + x_2^\sigma = 1$, this is equivalent to

$$\sigma x_1(x_1^{\sigma-2} - 1) \geq 0,$$

which, because $x_1 \in [(1/2)^{\frac{1}{\sigma}}, 1)$ (recall $x_2 \leq x_1 < 1$), implies that $\sigma \leq 2$. We turn next to the converse.

For the converse, we must show that if $\sigma \in (1, 2]$, then $\forall \alpha, \beta, y_1, x_1, ..., x_n \in \mathbb{Q}_+$ s.t. $x_1 \geq ... \geq x_n$, $\alpha \leq \beta$ and $\alpha + \beta = 1$,

$$\text{if } x_1^\sigma + ... + x_n^\sigma \geq y_1^\sigma \text{ then } (\beta x_1 + \alpha y_1)^\sigma + (\beta x_2)^\sigma + ... + (\beta x_n)^\sigma \geq (\alpha x_1 + \beta y_1)^\sigma + (\alpha x_2)^\sigma + ... + (\alpha x_n)^\sigma. \tag{A.8}$$

We will in fact show that (A.8) holds $\forall \alpha, \beta, y_1, x_1, ..., x_n \in \mathbb{R}_+$ s.t. $\alpha \leq \beta$.

If $y_1 = 0$, then the conclusion follows trivially so we may assume that $y_1 > 0$. Dividing the expressions in (A.8) by $y_1^\sigma > 0$, we may assume without loss that $y_1 = 1$. So we have reduced the problem to the following.

We must show that $\forall \alpha, \beta \in \mathbb{R}_+$ s.t. $\alpha \leq \beta$, and $\forall x_1, ..., x_n \in \mathbb{R}_+$ s.t. $x_1^\sigma + ... + x_n^\sigma \geq 1$,

$$(\beta x_1 + \alpha)^\sigma + (\beta x_2)^\sigma + ... + (\beta x_n)^\sigma \geq (\alpha x_1 + \beta)^\sigma + (\alpha x_2)^\sigma + ... + (\alpha x_n)^\sigma. \tag{A.9}$$

If $x_1^\sigma + ... + x_n^\sigma > 1$, then we may reduce one or more of $x_1, x_2, ..., x_n$, without making any of them negative, until the strict inequality, $x_1^\sigma + ... + x_n^\sigma > 1$, becomes an equality. Since $\beta \geq \alpha$, this operation tightens the inequality in (A.9). Hence, if (A.9) holds after the operation it must have held initially. It therefore suffices to prove (A.9) when $x_1^\sigma + ... + x_n^\sigma = 1$.

But if $x_1^\sigma + ... + x_n^\sigma = 1$, we may rewrite (A.9) as

$$(\beta x_1 + \alpha)^\sigma + \beta^\sigma(1 - x_1^\sigma) \geq (\alpha x_1 + \beta)^\sigma + \alpha^\sigma(1 - x_1^\sigma),$$

or equivalently as,

$$(\beta x_1 + \alpha)^\sigma + \beta^\sigma(1 - x_1^\sigma) - (\alpha x_1 + \beta)^\sigma - \alpha^\sigma(1 - x_1^\sigma) \geq 0. \tag{A.10}$$

Fix any $\alpha, \beta \in \mathbb{R}_+$ s.t. $\alpha \leq \beta$. Since $x_1^\sigma + ... + x_n^\sigma = 1$ and all the x_k are nonnegative, we have $x_1 \in [0, 1]$. Hence, it suffices to show that for all $x_1 \in [0, 1]$, the function f defined by the left-hand side of (A.10), i.e.,

$$f(x_1) = (\beta x_1 + \alpha)^\sigma + \beta^\sigma(1 - x_1^\sigma) - (\alpha x_1 + \beta)^\sigma - \alpha^\sigma(1 - x_1^\sigma) \tag{A.11}$$

is nonnegative for all $x_1 \in [0, 1]$.

Since $f(0) = f(1) = 0$, it is enough to show that $f(x_1) \geq 0$ whenever $f'(x_1) = 0$ since, in this case, the minimum value attained by f on $[0, 1]$ cannot be negative (since if it is negative, this negative minimum value is attained at an interior point at which the derivative must be zero). The derivative of f is,

$$f'(x_1) = \sigma\beta(\beta x_1 + \alpha)^{\sigma-1} - \sigma\beta^\sigma x_1^{\sigma-1} - \sigma\alpha(\alpha x_1 + \beta)^{\sigma-1} + \sigma\alpha^\sigma x_1^{\sigma-1},$$

and setting this to zero gives (after division by σ and multiplication by x_1),

$$\alpha^\sigma x_1^\sigma - \beta^\sigma x_1^\sigma = \alpha x_1(\alpha x_1 + \beta)^{\sigma-1} - \beta x_1(\beta x_1 + \alpha)^{\sigma-1}. \tag{A.12}$$

It suffices therefore to show that $f(x_1) \geq 0$ whenever (A.12) holds. Substituting (A.12) into (A.11) yields,

$$\begin{aligned} f(x_1) &= (\beta x_1 + \alpha)^\sigma + \beta^\sigma - (\alpha x_1 + \beta)^\sigma - \alpha^\sigma + \alpha x_1(\alpha x_1 + \beta)^{\sigma-1} - \beta x_1(\beta x_1 + \alpha)^{\sigma-1} \\ &= (\beta x_1 + \alpha)^{\sigma-1}(\beta x_1 + \alpha - \beta x_1) - (\alpha x_1 + \beta)^{\sigma-1}(\alpha x_1 + \beta - \alpha x_1) + \beta^\sigma - \alpha^\sigma \\ &= (\beta x_1 + \alpha)^{\sigma-1}\alpha - (\alpha x_1 + \beta)^{\sigma-1}\beta + \beta^\sigma - \alpha^\sigma. \end{aligned}$$

To see that the last expression on the right-hand side is nonnegative, note first that its value is zero when $x_1 = 0$. It therefore suffices to show that it is nondecreasing in x_1. It's derivative with respect to x_1 is,

$$(\sigma - 1)\alpha\beta[(\beta x_1 + \alpha)^{\sigma-2} - (\alpha x_1 + \beta)^{\sigma-2}],$$

which is nonnegative because $0 \leq \alpha \leq \beta$, $x_1 \in [0, 1]$, and $\sigma \in (1, 2]$. Q.E.D.

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