BOUNDARIES FOR MULTIPLIERS OF PERIODIC POINTS OF HOLOMORPHIC MAPPINGS

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1. We will describe the basic results of the article. Let W and W' be one-connected domains in $\overline{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$; T: $W \to W'$ is a polynomially similar mapping of degree $m \geq 2$ [1]. This means that $\overline{W} \subset W'$ and T: $W \to W'$ is an analytic m-sheeted ramified covering. We denote by F(T) the set of points that do not leave W under the action T: $F(T) = \{z \in W : T^n z \in W \ \forall n \in \mathbf{N}\}$ (f^n is the n-th iteration of arbitrary mapping f). The name of the mapping T is explained by the Douady–Hubbard theorem [1]: there exist polynomial mapping P: $\mathbf{C} \to \mathbf{C}$ of degree m and K-quasiconformal homeomorphism φ of neighborhood V of set $F(P) = \{z:\sup|P^n z| < \infty\}$ onto a neighborhood U of set F(T) such that φ joins $P|_V$ and $T|_U$, i.e.,

$$T \circ \varphi(z) = \varphi \circ P(z) \tag{1}$$

for all z for which $P(z) \in V$.

The boundary of set F(P) of polynomial P is called the Julia set J(P) of the given polynomial [2-5]. Repelling periodic points of P are dense on J(P) [point z is called periodic of period n for mapping f if $f^n z = z$, $f^i z \neq z$, $1 \leq i \leq n-1$; if its multiplier $\lambda = (f^n)'(z)$ is greater than 1 in modulus, point z is called repelling]. By analogy we denote $J(T) = \partial F(T)$. From (1) it follows that repelling periodic points of mapping T are dense on J(T).

In the present article we will show that if F(T) is connected, then for any repelling periodic point $a \in J(T)$ of period n, its multiplier $\lambda = (T^n)'(a)$ satisfies the inequality

$$|\lambda| \leqslant m^{2nK/r}, \tag{2}$$

where r is a natural number with simple geometric meaning: r is the number of ways in which point a is accessible from domain $W \backslash F(T)$ (see Sec. 5).

It is clear that if T is a polynomial with connected J(T), then we can take K = 1.

We define the number $K_0(T)$ as the lower bound of $x > 0$ such that the inequality $|\lambda| \leq m^{2nx}$ holds for multiplier λ of any periodic point of mapping T of period n (for all $n \in \mathbf{N}$). From (2) it immediately follows that $K_0(T) \leq K$.

The next result of the article can be stated this way: if the mapping T introduced above is hyperbolic on J(T) [4], then $K_0(T) < K$. In particular, for a polynomial T that is hyperbolic on its connected Julia set, $K_0(T) < 1$.

The proof of these and other assertions is based on application of the method of extremal lengths [6-8] and some facts of holomorphic dynamics [4, 5].

2. We will prove a lemma which we will call the fundamental lemma. Fix (to the end of this section) the mapping f: $U \to \mathbf{C}$, defined and conformal on a neighborhood U of point z = 0 and such that f(0), $f'(0) = \lambda$, $|\lambda| > 1$. Let $D(r) = \{\omega : |\omega| < r, \operatorname{Im}\omega > 0\}$. We will say that mapping h canonizes f on the domain Ω if

a) $\Omega \subset U$, $\Omega \subset f(\Omega)$, $0 \in \partial\Omega$,

b) there exist numbers $\rho > 1$, $\varepsilon > 1$, and $K \geq 1$ such that h: $D(\varepsilon\rho) \to f(\Omega)$ is a k-quasiconformal homeomorphism that joins $f|_\Omega$ with the expansion g: $\omega \to \rho\omega$, $\omega \in D(\varepsilon)$, i.e., $h \circ g = f \circ h$.

For mapping f let there exist no more than countably many pairwise nonintersecting domains Ω_i and mappings h_i that canonize f on Ω_i, $i \in I$. We will denote by K_i the coefficient of quasiconformality of mapping h_i and by ρ_i the coefficient of the corresponding expansion g_i: $\omega \mapsto \rho_i\omega$, $\omega \in D(\varepsilon_i)$, $i \in I$. In these conditions we have

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LEMMA 1.

$$\sum_{i\in I}\frac{1}{K_i\ln\rho_i}\leqslant\frac{2\alpha\ln|\lambda|}{|\ln\lambda|^2},\qquad(3)$$

where the number $\alpha, 0<\alpha\leq 1$, is the lower density of domain $\Omega=\bigcup_{i\in I}\Omega_i$ at point zero in logarithmic metric $|dz/z|$.

Proof. It is sufficient to prove (3) for one domain Ω_i. We will fix i and introduce the notation $\tilde{\Omega}=\Omega_i$, $h=h_i$, $g=g_i$, $K=K_i$, $\rho=\rho_i$, where h joins $g|_{D(\varepsilon)}$ with $f|_\Omega$. We will fix as well a small neighborhood of zero U_0, $U_0\subset U$, such that mapping f^{-1}: $f(U_0)\to U_0$ is conformally conjugate to mapping f_0^{-1}: $z\to\lambda^{-1}z$ (Schroeder's theorem [4]). For each $t\in(0, 1/2)$, the restriction of h to angle $\Pi_t=\{\omega: \arg\omega\in(\pi t, \pi-\pi t)\}$ is continuous at point $\omega=0$. Therefore we can find $\delta>0$, $\delta=\delta(t)$, such that image $h(V_t)$ of domain $V_t=D(\delta)\cap\Pi_t$ lies in neighborhood U_0. We introduce the domain

$$V=\bigcup_t V_t,\quad \Omega_0=\bigcup_t h(V_t).$$

By the construction we have $\Omega_0\subset U_0\cap\tilde{\Omega}$. Therefore it follows that it is sufficient to prove the inequality

$$\frac{1}{K\ln\rho}\leqslant\frac{2\alpha_0\ln|\lambda|}{|\ln\lambda|^2},\qquad(4)$$

where α_0 is the lower density of domain Ω_0 at point 0 in the logarithmic metric and mapping h is a K-quasiconformal homeomorphism such that

$$\lambda h(\omega)=h(\rho\omega),\quad \omega\in V.\qquad(5)$$

We denote by z_0A the set of products of number $z_0\in\mathbf{C}$ by elements of set $A\in\mathbf{C}$. We introduce domains

$$\Pi=\bigcup_{k=0}^{\infty}\rho^kU,\quad \Omega^*=\bigcup_{k=0}^{\infty}\lambda^k\Omega_0.$$

By the construction we have $V\subset\rho V$, $\Omega_0\subset\lambda\Omega_0$, Π is the upper halfplane and h can be continued to a K-quasiconformal homeomorphism Π on Ω^* with preservation of (5).

We will fix the boundary S_r of circle $B_r=\{z: |z|<r\}$ and ray $\alpha_\varphi=\{\omega: \arg\omega=\varphi\}$, $0<\varphi<\pi$. An arc $S\subset S_r\cap\Omega^*$ can be found with endpoints on boundary $\partial\Omega^*$ through which the curve $\beta_\varphi=h(\alpha_\varphi)$ exits from circle B_r. Then any curve $\beta_\psi=h(\alpha_\psi)$, $0<\psi<\pi$ intersects S. We denote for the rest of the article $\ell=h^{-1}(S)$. Each ray α_ψ, $0<\psi<\pi$ intersect ℓ. We introduce two families of curves $\tilde{\Gamma}$ and Γ. We will consider first the family of all segments that join points $\omega\in\ell$ and ω/ρ; then on each ray α_ψ, $0<\psi<\pi$ we leave exactly one such segment $\tilde{\gamma}=\tilde{\gamma}_\psi$ nearest to zero. We denote the resulting family of segments by $\tilde{\Gamma}$. It fills out the set of points $R\subset\Pi$. We define the second family Γ as the family of images $\gamma=h(\tilde{\gamma})$, $\tilde{\gamma}\in\tilde{\Gamma}$.

Now we introduce the logarithmic metric on $\mathbf{C}\setminus\{0\}$: $\sigma(z)=|z|^{-1}$ and the induced metric

$$\tilde{\sigma}(\omega)=\left.\frac{\sigma(z)}{|(h^{-1})'_z|-|(h^{-1})'_{\bar z}|}\right|_{z=h(\omega)},$$

existing almost everywhere on Π. We define the numbers

$$L=\inf_{\gamma\in\Gamma}\int_\gamma\sigma(z)|dz|,\quad A=\iint_{h(R)}\sigma^2(z)\,dx\,dy,$$

$$\tilde{L}=\inf_{\tilde{\gamma}\in\tilde{\Gamma}}\int_{\tilde{\gamma}}\tilde{\sigma}(\omega)|d\omega|,\quad \tilde{A}=\iint_R\tilde{\sigma}^2(\omega)\,du\,dv,$$

$$M=AL^{-2},\quad \tilde{M}=\tilde{A}\tilde{L}^{-2},$$

where $z=x+iy$, $\omega=u+iv$.

It is known [6] that

$$\tilde{M}\leqslant KM.\qquad(6)$$

We estimate $\tilde{M}$ from below. For this we use the Hölder inequality and obtain the relation

$$\pi\tilde{L}^2 \leqslant \int_0^\pi d\psi\left(\int_{\tilde{\gamma}_\psi} \tilde{\sigma}(\omega)|d\omega|\right)^2 \leqslant \int_0^\pi d\psi \int_{\tilde{\gamma}_\psi} \tilde{\sigma}^2(\omega)|\omega||d\omega| \int_{\tilde{\gamma}_\psi} \left|\frac{d\omega}{\omega}\right| = \tilde{A}\ln\rho, \tag{7}$$

which implies $\tilde{M} \geq \pi/\ln\rho$. Now we estimate M from above. Let $z \in H(R)$ and points $z_1 = \lambda z$, $z_2 = z/\lambda$ not be the endpoints of some curve $\gamma \in \Gamma$. From the definition of family Γ it follows that then z_1 and z_2 do not lie in h(R). Now if the point z from H(R) does not lie in the ring $C = \{z : r\backslash|\lambda| < |z| < r\}$ then we take z into C by mapping $z \to \lambda^k z$ with some $k = \pm 1, \pm 2, \ldots$. As a result

$$A = I(h(R)) \leqslant I(\Omega^* \cap C),$$

where I(X) is an area of set $X \subset \mathbf{C}\setminus\{0\}$ in metric $\sigma(z)$. Here we used equation $I(\lambda X) = I(X)$. On the other hand, from the definition of Ω^* it follows that the density α_0 of Ω_0 at point zero in the metric $\sigma(z)$ exists and equals

$$\alpha_0 = \frac{I(\Omega^* \cap C)}{2\pi\ln|\lambda|}.$$

Thus, we have shown that

$$M = \frac{A}{L^2} \leqslant \frac{A}{|\ln\lambda|^2} \leqslant \frac{2\pi\alpha_0\ln|\lambda|}{|\ln\lambda|^2} \tag{8}$$

and finally

$$\frac{1}{K}\frac{\pi}{\ln\rho} \leqslant \frac{\tilde{M}}{K} \leqslant M \leqslant \frac{2\pi\alpha_0\ln|\lambda|}{|\ln\lambda|^2},$$

i.e., inequality (4) and the lemma have been proved.

Remark 1. a) In the following we will use the equation

$$\alpha = \lim_{\delta\to 0}\frac{1}{\ln\frac{r}{\delta}}\int_\delta^r \frac{l_0(\rho)}{2\pi\rho}\,d\rho, \tag{9}$$

where $\ell_0(\rho)$ is the length of the part of the circle $|z| = \rho$ which lies in Ω.

b) Analysis of (6)-(8) allows the conclusion that if we put the equality sign in (3), then each mapping h_i extends to a continuous mapping of a closed semineighborhood $\overline{D(\varepsilon_i)}$ of point $\omega = 0$ and maps the boundary intervals to analytic arcs.

3. Now we apply Lemma 1 to evaluate the multipliers of periodic points of mappings that are defined "globally."

In [9, 10] the class of RB-domains and mappings is presented. This class contains, for example, immediate attraction domains of the attracting fixed points of a rational function.

Thus, let a one-connected hyperbolic domain $\Omega \subset \bar{\mathbf{C}}$ be given, along with a holomorphic mapping T: $U \to \mathbf{C}$ defined on a neighborhood U of boundary $\partial\Omega$. Following [9, 10], we impose on T and Ω the conditions

$$T(U\cap\Omega) \supset U\cap\Omega, \quad T(\partial\Omega) = \partial\Omega,$$
$$\bigcap_{n=1}^{\infty} T^{-n}(U\cap\overline{\Omega}) = \partial\Omega \tag{10}$$

[$T^{-n}(A)$ is the complete preimage of set A under mapping T^n]. We introduce the Riemann mapping h: $V = \{|\omega| < 1\} \to \Omega$ and the mapping $\tilde{g} = h^{-1} \circ T \circ \tilde{H} : D \to V$, $D = h^{-1}(U \cap \Omega)$ conjugate to T. Condition (10) is equivalent to the fact that $\tilde{g}$ holomorphically extends to a neighborhood of ∂V and hyperbolically to ∂V [10].

Consider this situation. Let $a \in \partial\Omega$ be a repelling fixed point of T. We will assume that point a is a radial limit of mapping h: $D \to \Omega$ at some point $\omega_0 \in \partial V$ and ω_0 is a periodic point of mapping $\tilde{g}$: $\partial V \to \partial V$.

Remark 2. This has been proved [1, 15, 3] in the following cases:

a) mapping T: $\partial\Omega \to \partial\Omega$ is hyperbolic;

b) point ω_0 is a repelling fixed point for mapping $\tilde{g}$; then there eixsts a radial limit $a = h(\omega_0)$ and $T(a)=a$, $|T'(a)|>1$;

c) T is a polynomial, $\Omega = D_\infty$ is its domain of attraction to ∞ and Julia set $J = \partial D_\infty$ is connected; in this case $\tilde{g}(\omega) = \omega^m$, where $m = \deg T$.

Assertions a) and b) were proved in [15] for rational functions, but the proof does not change in the case of RB-mappings.

Thus, let a be the radial limit over a finite number of directions, the endpoints of which form k cycles of $\tilde{g}$ with periods $\ell_1, \ell_2, \ldots, \ell_k$ and multipliers $\rho_1, \rho_2, \ldots, \rho_k$. We will call the total quantity $r = \ell_1 + \ldots + \ell_k$ the number of ways in which point a is accessible from Ω.

We will apply the fundamental lemma. Fix j and consider point b of a cycle of period $\ell = \ell_j$. By Schroeder's theorem, in a neighborhood V_b of point b the mapping $\tilde{g}^\ell$ is holomorphically conjugate to mapping g_j: $\omega \mapsto \rho_j\omega$ and here the part of the circle $\partial V \cap V_b$ passes to an interval. Further, semineighborhood $V \cap V_b$ determines in Ω the domain $h(V \cap V_b)$. Thus, we obtain r domains Ω_{jn}, $j = 1, \ldots, k$, $n = 1, \ldots, \ell_j$; to each cycle of period ℓ_j there correspond ℓ_j such domains. In this connection

$$T^L(\Omega_{jn}) \supset \Omega_{jn}, \quad j=1, \ldots, k, \; n=1, \ldots, l_j,$$

where we have set $L = \ell_1 \cdot \ldots \cdot \ell_k$. The fundamental lemma gives the inequality

$$\sum_{j=1}^{n} l_j \frac{\ln|\lambda|^L}{\ln \rho_j^{L/l_j}} \leqslant 2,$$

which implies

THEOREM 1.

$$\sum_{j=1}^{n} l_j \frac{l_j \ln|\lambda|}{\ln \rho_j} \leqslant 2. \tag{11}$$

Example 1. Let Ω be a Betkher domain [4, 5]. More precisely, T is defined on a neighborhood of the closure of Ω, which is a domain of attraction of some critical fixed point $q \in \Omega$: $T(q) = q$, $T'(q) = \ldots = T^{(m-1)}(q) = 0$, $T^{(m)}(q) \neq 0$ and in addition $T'(z) \neq 0$ $\forall z \in \Omega\backslash\{q\}$. We note that $m = \deg T|_\Omega$. We have

$$\tilde{g}(\omega) = \omega^m, \quad \rho_j = m^{l_j},$$

and inequality (11) acquires the form

$$\sum_{j=1}^{k} l_j \frac{l_j \ln|\lambda|}{l_j \ln m} = r\frac{\ln|\lambda|}{\ln m} \leqslant 2,$$

i.e., $|\lambda| \leq m^{2/r}$ for a multiplier of any fixed point a accessible from Ω. This example is interesting because it is realized for any nonlinear polynomial with connected Julia set (the role of Ω is played by the domain of attraction to ∞).

Example 2. Let $P_t(z) = z^2 + t$ and let $t \in \mathbf{C}$ be the center of the hyperbolic component of the Mandelbrot set [12-14, 4, 1] (for example, t = 0 or −1). This means that critical point z = 0 is a periodic point of some period n of mapping P_t. Let $f(z) = P_t^n(z)$, let Ω_1 be the domain of immediate access of fixed point 0 of mapping f(z), and let Ω_2 be the domain of attraction to ∞.

We evaluate multiplier λ of periodic point a of period m of mapping f(z) that lies on the boundary of Ω_1. Since f does not have critical points on $\partial\Omega_1$, then f: $\partial\Omega_1 \to \partial\Omega_1$ is hyperbolic. Therefore point a is accessible from Ω_1; from Ω_2 it is everywhere accessible. Now we apply the fundamental lemma. We have $f(\Omega_i) = \Omega_i$, i = 1, 2; the restriction $f|_{\Omega_2}$ is conformally equivalent to $\tilde{g}_2(\omega) = \omega^{\deg f}$, where $\deg f = 2^n$ and $f|_{\Omega_1}$ does not have critical points distinct from zero; since f(0) = 0, f'(0) = 0, $f''(0) = 2^n T_t'(0) \cdot \ldots \cdot T_t^{n-1}(0)$, then $\deg f|_{\Omega_1} = 2$ and $f|_{\Omega_1}$ is conformally equivalent to $\tilde{g}_1(\omega) = \omega^2$. If r_i is the number of ways of access of a from Ω_i, i = 1, 2, then by Lemma 1

$$r_1(r_1 \ln|\lambda|/r_1 m \ln 2) + r_2(r_2 \ln|\lambda|/r_2 nm \ln 2) \leqslant 2,$$

which implies $\ln|\lambda| \leq 2m \ln 1/(1 + 1/n)$.

4. We will strengthen the basic assertions in the hyperbolic case. Let compactum $F \subset \mathbf{C}$ be given with property (s): there exist numbers $\kappa \in (0, 1)$ and $\varepsilon > 0$ such that for any circle $S_r(z)$ with center at arbitrary point $z \in \partial F$ and radius $r < \varepsilon$

$$\mu(S_r(z) \cap F)/\mu(S_r(z)) \geqslant \varkappa, \tag{12}$$

where μ is the Lebesgue measure on $\mathbf{R}^2$.

We denote by $S_{u,\rho}$ the part of circle $|z - u| = \rho$, $u \in \partial F$ lying outside F. Let $\ell_u(\rho)$ be its length, $x = \ln(1/\rho)$, $\varphi_u(x) = \ell_u(\rho)/2\pi\rho$. It is obvious that $\varphi_u(\ln(1/\rho))$ is the angular measure of arc $S_{u,\rho}$.

LEMMA 2. When (s) holds, we can find a number α, $0 < \alpha < 1$, such that for any $u \in \partial F$, $R_0 > \pi + \ln(1/\varepsilon)$

$$\varliminf_{R\to\infty} \frac{1}{R - R_0} \int_{R_0}^{R} \varphi_u(x)\,dx \leqslant \alpha.$$

Proof. Let E(c) be a ring with center at point u and radii $r_1 = e^{-(c+\pi)}$, $r_2 = e^{-(c-\pi)}$. In (12) let $r = (r_1 + r_2)/2$ and choose a point $v \in \partial F$ at a distance $r + r_1$ from v. Since $\mu(S_r(v))/\mu(E(c)) = (e^{\pi} - e^{-\pi})/4(e^{\pi} + e^{-\pi}) = \beta$ and $S_r(v) \subset E(c)$, then

$$\mu(E(c)\setminus F)/\mu(E(c)) \leqslant \alpha, \tag{13}$$

where $\alpha = 1 - \varkappa\beta$. Now we recall the above-introduced notation and we rewrite (13):

$$\int_{r_1}^{r_2} l_u(\rho)\,d\rho \leqslant \alpha \int_{r_1}^{r_2} 2\pi\rho\,d\rho.$$

Here we make the change $c + t = \ln(1/\rho)$ and integrate the result with respect to c from R_0 to R, divide by $R - R_0$, and pass to the lower limit for $R \to \infty$, obtaining the desired result. The lemma is proved.

If Ω is an RB-domain and $T: \partial\Omega \to \partial\Omega$ is hyperbolic, then compactum $F = \mathbf{C} \setminus \Omega$ satisfies condition b). This was shown in [9]. Therefore the fundamental lemma, Eq. (9) and Lemma 2 imply this strengthening of Theorem 1:

THEOREM 2. In the conditions of Theorem 1, if T is hyperbolic, then we can find $\alpha \in (0, 1)$ such that for any periodic point

$$\sum_j l_j \frac{l_j \ln|\lambda|}{\ln \rho_j} \leqslant 2\alpha.$$

5. We will prove the results mentioned in Sec. 1 for polynomially-similar mappings.

Let $T: W \to W'$ be polynomially-similar mapping of degree m and let the set $F(T) = \{z \in W : T^n z \in W\ \forall n \in N\}$ be connected. There exist a polynomial P of degree m and a K-quasiconformal homeomorphism φ of neighborhoods V and U of sets F(P) and F(T), respectively, such that $T\circ\varphi = \varphi\circ P$. We will take the domain $D_\infty = \{z:\ P^n z \to \infty,\ n \to \infty\}$ and conformally transform the unit circle to D_∞ and $V \cap D_\infty$ by mapping φ to $U\setminus F(T)$. We obtain the mapping h, a K-quasiconformal homeomorphism joining $g(\omega) = \omega^m$ with T.

Now let $a \in \partial F(T)$ be a repelling periodic point of period n. A repelling periodic point $\varphi^{-1}(a) \in \partial F(P)$ of polynomial P accessible from D_∞ by some number r of ways corresponds to it (see Remark 2). Then from the fundamental lemma and Example 1 it follows that

$$2 \geqslant \sum_j \frac{l_j}{K} \frac{l_j \ln|\lambda|}{n l_j \ln m} = \frac{r}{K} \frac{\ln|\lambda|}{n \ln m},$$

i.e., we have proved

THEOREM 3. For multiplier λ of point a, we have

$$\ln|\lambda| \leqslant (2Kn/r)\ln m.$$

Now we note that $\mathbf{C} \setminus F(T)$ is an RB-domain for mapping T (since D_∞ is such a domain for P). Therefore from Lemma 2 we derive

THEOREM 4. If T: J(T) → J(T) is hyperbolic, then there exists α, 0 < α < 1, such that

$$\ln |\lambda| \leqslant (2Kn\alpha/r) \ln m.$$

COROLLARY 1. Let P be a polynomial of degree m ≥ 2, let its Julia set J(P) be connected, and let mapping P: J(P) → J(P) be hyperbolic. Then α ∈ (0, 1) can be found such that for a multiplier λ of any periodic point a of period n

$$\ln |\lambda| \leqslant (2\alpha n/r) \ln m,$$

where r is the number of ways of access of point a from D_∞. In particular, $|\lambda| \leq m^{2\alpha n}$.

Concluding Remarks. Thus, we have proved the inequality

$$\frac{1}{n} \ln |\lambda| \leqslant 2\chi, \tag{14}$$

where λ is the multiplier of any repelling periodic point of period n of polynomial T (of degree m ≥ 2) with connected Julia set J, and χ = χ(T) is the Lyapunov index of maximal measure of T, which in the case of a polynomial coincides with the harmonic measure of domain D_∞ [9, 10] (we note that χ = ln m, if J is connected). It turns out that for polynomial $t_c(z) = z^m + c$, m ≥ 2 inequality (14) remains true for any c ∈ ℂ. For the proof we can consider the harmonic function $\ln|\lambda|/n - 2\chi(T_c)$ on the complement of the Mandelbrot set [12] and apply the maximum principle. It is not known to the author whether (14) is preserved for an arbitrary polynomial with disconnected Julia set. In connection with this we state the lower limit for λ. Let

$$u = \min \{G(q): T'(q) = 0\},$$

where G is the Green's function of domain D_∞ with pole at ∞. Then $\ln|\lambda|/n \geq (m - 1)u$.

In conclusion the author would like to acknowledge the referee for pointing out [15], in which Theorem 1 is proved for rational functions. The fundamental lemma of the present article is a generalization of Theorem 3 of [15].

LITERATURE CITED

1. A. Douady, "Systems dynamiques holomorphes," in: Seminaire Bourbaki, 35th anniversary, No. 599 (1982).
2. G. Julia, "Memoire sur l'iteration des fonctions rationnelles," J. Math. Pure Appl., 8, 47-245 (1918).
3. P. Fatou, "Sur les equations fonctionnelles," Bull. Soc. Math. France, 47, 161-271 (1919); 48, 33-94, 208-314 (1920).
4. M. Yu. Lyubich, "Dynamics of rational transformations: the topological picture," Usp. Mat. Nauk, 41, No. 4, 35-95 (1986).
5. P. Montel, Normal Families of Analytic Functions, ONTI NKTP, Moscow—Leningrad (1936).
6. L. Ahlfors, Lectures on Quasiconformal Mappings, D. Van Nostrand Co., Toronto—New York—London (1966).
7. J. Jenkins, Univalent Functions and Conformal Mappings, Springer, Berlin (1958).
8. V. K. Dzyadyk, Introduction to the Theory of Uniform Approximation of Functions by Polynomials [in Russian], Nauka, Moscow (1977).
9. F. Przytycki, M. Urbanski, and A. Zdunik, "Harmonic, Gibbs, and Haussdorf measures on repellers for holomorphic maps," Preprint, Univ. Warwick (1986).
10. F. Przytychki, "Riemann map and holomorphic dynamics," Inv. Math., 85, 439-455 (1986).
11. M. V. Yakobson, "Markov partitions for rational endomorphisms of the Riemann sphere," in: Multicomponent Random Systems [in Russian], Nauka, Moscow (1978), pp. 309-319.
12. B. Mandelbrot, "Fractal aspects of the iteration of z → λz(1 − z) for complex λ and z," Ann. New York Acad. Sci., 357, 249-259 (1980).
13. A. Douady and J. H. Hubbard, "Iteration des polynomes quadratiques complexes," C. R. Acad. Sci. Paris, 249, No. 3, 123-126 (1982).
14. G. M. Levin, "A sequence of bifurcations of a one-parameter family of mappings," Usp. Mat. Nauk, 36, No. 3, 189-190 (1982).
15. C. Pommerenki, "On conformal mapping and iteration of rational functions," Complex Variables, 5, 117-126 (1986).