On A Polygon Inequality Problem By Bernius And Blanchard

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Abstract

Bernius and Blanchard of Bielefeld University in Germany have conjectured the following *polygon inequality*: for any two set of vectors $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ in $\mathrm{I\!R}^{\mathrm{m}}$,

$$(*) \quad \sum_{i<j} \|\xi_i - \xi_j\| + \sum_{i<j} \|\eta_i - \eta_j\| \ \leq \ \sum_{i,j=1}^{n} \|\xi_i - \eta_j\|,$$

in the 2–norm and that, moreover, equality holds in $(*)$ if and only if there exists a permutation π on $\{1, 2, \ldots, n\}$ such that $\eta_i = \xi_{\pi(i)}$, $i = 1, \ldots, n$. We show that the fulfillment of $(*)$ for any two sets of vectors $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ in a Banach space X and in the norm of that space is equivalent to a condition on X due to Bretagnolle, Castelle, and Kirvine which, when it holds, is equivalent to the existence of an isometric embedding of X in a subspace of $L_1(\Omega, \mu)$ for some measure space (Ω, μ). Since it is known that for any $1 < p \leq 2$, $L_p(0, 1)$ is isometric to $L_1(\Omega, \mu)$, the Bernius and Blanchard polygon inequality follows. We continue by characterizing the case of equality in $(*)$ for $X = L_1(\Omega, \mu)$ and subsequently use this characterization to prove the remaining part of the conjecture about when equality holds in $(*)$ for the 2–norm.

1 Introduction

In the study of learning in artificial neural networks the following problem has arisen, see Bernius [**?**]:

Given a set of $2n$ *points in* $\{-1,1\}^m$*, divide it into two parts of equal sizes* $M_1 = \{\xi_1, \dots \xi_n\}$ *and* $M_2 = \{\eta_1, \dots, \eta_n\}$ *such that* M_1 *"represents"* M_2 *as good as possible. As an (also computationally convenient) measure of the degree of representation , Bernius introduced the function*

$$H_{1,2} \; = \; \sum_{i,j=1}^{n} \|\xi_i - \eta_j\| - \sum_{i<j} \|\xi_i - \xi_j\| - \sum_{i<j} \|\eta_i - \eta_j\|,$$

where $\|\cdot\|$ *is the Euclidean norm in the real* m*–dimensional space* $\mathbb{R}^{\mathrm{m}}$*.*

It was numerically observed that $H_{1,2} \geq 0$ and that $H_{1,2} = 0$ if and only if $M_1 = M_2$.

Starting from this observations, Bernius and Blanchard formulated in [**?**] the following conjecture:

<u>**CONJECTURE**</u> (**Polygon Inequality**): *Let* $\xi_1, \dots, \xi_n$ *and* $\eta_1, \dots, \eta_n$ *be vectors in* $\mathbb{R}^{\mathrm{m}}$*. Then:*

(i)

$$\sum_{i<j} \|\xi_i - \xi_j\|_2 + \sum_{i<j} \|\eta_i - \eta_j\|_2 \;\leq\; \sum_{i,j=1}^{n} \|\xi_i - \eta_j\|_2. \tag{1.1}$$

(ii) *Equality holds in* (**??**) *if and only if there exists a permutation* π *of* $\{1,2,\dots,n\}$ *such that* $\eta_i = \xi_{\pi(i)}$*,* $i = 1, \dots, n$*.*

Two special cases of the conjecture were proved in [**?**]: $m = 1$ and $n \in \mathbb{N}$ and $m \in \mathbb{N}$ and $n = 2$.

As usual, for any vector $x = (x_1 \; \dots \; x_m)^T \in \mathbb{R}^{\mathrm{m}}$ and for any positive number r, we shall let

$$\|x\|_r \; = \; \left(\sum_{i=1}^{m} |x_i|^r \right)^{1/r} .$$

and we shall let ℓ_r^m denote the space of all m–vectors $x = (x_1 \ \ldots \ x_m)^T$ of real numbers with the norm $\|x\|_r$.

It turns out that the natural setting for Blanchard's conjecture is the space $L_1(\Omega, \mu)$ instead of a Hilbert space. In this setting, inequality (**??**) is a special case of a theorem due to J. Bretagnolle, D. Dacuha Castelle, and J. Krivine, [**?**, Theorem 2] which states that:

A Banach space X is isometric to a subspace of $L_1(\Omega, \mu)$ if and only if the function

$$\phi(x, y) \;=\; \|x - y\|$$

defined on $X \times X$ is of negative type, that is,

$$\begin{cases} \text{for every } n > 2, \text{ every } \{z_i\}_{i=1}^n \subset X, \text{ and every real numbers} \\ \{\rho_i\}_{i=1}^n \text{ with } \sum_{i=1}^n \rho_i = 0, \\ \qquad \sum_{i,j=1}^n \|z_i - z_j\| \rho_i \rho_j \;\le\; 0. \end{cases} \tag{1.2}$$

The connection between (**??**) and (**??**) (in the L_1–norm) is established in the following:

PROPOSITION 1.1 *Let X be a Banach space. Then* (**??**) *is equivalent to the statement:*

$$\begin{cases} \text{for every } n \ge 2 \text{ and any collections } \{x_i\}_{i=1}^n \text{ and } \{y_i\}_{i=1}^n \text{ in } X, \\ \qquad \sum_{1 \le i < j \le n} (\|x_i - x_j\| + \|y_i - y_j\|) \;\le\; \sum_{i,j=1}^n \|x_i - y_j\|. \end{cases} \tag{1.3}$$

Proof: Assume (**??**) and let $\{x_i\}_{i=1}^n \cup \{y_i\}_{i=1}^n \subset X$. Put $z_i = x_i$ and $z_{n+i} = y_i$, $i = 1, \ldots, n$. Let $\rho_i = 1$ for $1 \le i \le n$ and let $\rho_i = -1$ for $n < i \le 2n$. It follows from (**??**) that

$$\sum_{i,j=1}^{2n} \|z_i - z_j\| \rho_i \rho_j \;\le\; 0.$$

Hence

$$\sum_{i=1}^n \sum_{j=1}^n \|z_i - z_j\| + \sum_{i=n+1}^{2n} \sum_{j=n+1}^{2n} \|z_i - z_j\|$$
$$-\; \sum_{i=1}^n \sum_{j=n+1}^{2n} \|z_i - z_j\| \;-\; \sum_{i=n+1}^{2n} \sum_{j=1}^n \|z_i - z_j\| \;\le\; 0.$$

Consequently,

$$\sum_{i,j=1}^{n} \|x_i - x_j\| + \sum_{i,j=1}^{n} \|y_i - y_j\| \;\leq\; 2 \sum_{i,j=1}^{n} \|x_i - y_j\|$$

which is just (**??**).

To prove the converse, note that we may assume without loss of generality that the numbers $\{\rho_i\}_{i=1}^n$ are rational. Put

$$\rho_i \;=\; \frac{m_i}{N}, \quad i = 1, \ldots, n,$$

where N and m_i are integers, and note that since $\sum_{i=1}^n \rho_i = 0$, we have that $\sum_{i=1}^n m_i = 0$.

Continuing, given $\{z_h\}_{h=1}^n \subset X$, for each

$$1 \;\leq\; i \;\leq\; \sum_{i=1}^{n} |m_i|,$$

define

$$w_i \;=\; z_h \;\text{ if }\; \sum_{j=1}^{h-1} |m_j| \;<\; i \;\leq\; \sum_{j=1}^{h} |m_j|$$

and let $M = \sum_{i=1}^n |m_i|$. Next define the sets

$$A \;=\; \left\{ 1 \leq i \leq M \mid \sum_{j=1}^{h-1} |m_j| < i \leq \sum_{j=1}^{h} |m_j| \text{ and } \rho_h > 0 \right\}.$$

and $B = \{1, 2, \ldots, M\} \setminus A$. Because $\sum_{i=1}^n m_i = 0$ we have that

$$|A| \;=\; |B| \;=\; \frac{M}{2}.$$

Put $K = M/2$. Using (**??**) with $n = K$, $\{x_i\}_{i=1}^K = \{w_j\}_{j \in A}$ and $\{y_i\}_{i=1}^K = \{w_j\}_{j \in B}$ we get that

$$\sum_{i,j \in A} \|w_i - w_j\| + \sum_{i,j \in B} \|w_i - w_j\| \;\leq\; 2 \sum_{i \in A, j \in B} \|w_i - w_j\|. \tag{1.4}$$

Fixing $1 \leq h, k \leq n$ with $\rho_h > 0$ and $\rho_k > 0$ we see that, in the first sum on the left hand side of (**??**), the equality

$$\|w_i - w_j\| \;=\; \|z_h - z_k\| \tag{1.5}$$

occurs $m_h m_k$ times. Similarly, fixing $1 \leq h, k \leq n$ with $\rho_h < 0$ and $\rho_k < 0$, we see that in the second sum on the left hand side of (**??**), (**??**) occurs $m_h m_k$ times. Finally, on the right hand side of (**??**), if we fix h with $\rho_h > 0$ and k with $\rho_k < 0$, then the equality (**??**) occurs $m_h|m_k|$ times. It thus follows from (**??**) that

$$\sum_{\rho_h>0,\rho_k>0} \|z_h - z_k\| |m_h| |m_k| + \sum_{\rho_h<0,\rho_k<0} \|z_h - z_k\| |m_h| |m_k|$$
$$\leq \; 2 \sum_{\rho_h>0,\rho_k<0} \|z_h - z_k\| |m_h| |m_k|.$$

Dividing both sides by N^2, we get the desired inequality that

$$\sum_{h,k=1}^{n} \|z_h - z_k\| \rho_h \rho_k \;\leq\; 0.$$

□

Theorem 2 of [**?**] and Proposition **??** yield:

COROLLARY 1.2 *A Banach space X is isometric to a subspace of $L_1(\Omega, \mu)$ for some measure space (Ω, μ) if and only if* (**??**) *holds.*

The following are two examples of families of Banach spaces which can be isometrically embedded in $L_1(\Omega, \mu)$ and hence satisfy (**??**):

EXAMPLE 1.3 It is well known that the space $L_p(0, 1)$ with $1 < p \leq 2$ is isometric to a subspace of $L_1(0, 1)$ (see Lindenstrauss and Tzafriri [**?**, pp.212–213] and the references cited therein). Hence the inequality (**??**) is satisfied in $X = L_p(0, 1)$, $1 \leq p \leq 2$.

EXAMPLE 1.4 Lindenstrauss proved in [**?**] that every two dimensional space X is isometric to a subspace of $L_1(0, 1)$. Hence (**??**) holds for every two dimensional space X.

The next two examples help build intuition concerning the Blanchard problem.

EXAMPLE 1.5 Let $\{u_i\}_{i=1}^n$ denote the unit vector basis of ℓ_∞^n, $n \geq 5$, let $e = (1, \ldots, 1)$, and put $y_i = e - 2u_i$ and $x_i = u_i$ for all $i = 1, \ldots, n$. Then $\|y_i - x_i\| = 2$ and for all $j \neq i$, $\|y_j - x_i\| = 1$. Hence $\sum_{i,j=1}^n \|x_i - y_j\| =$

$n(n-1+2) = n(n+1)$. On the other hand, $\|x_i - x_j\| = 1$ and $\|y_i - y_j\| = 2$ for all $1 \le i < j \le n$. Thus

$$\begin{aligned} \textstyle\sum_{1\le i<j\le n} (\|x_i - x_j\| + \|y_i - y_j\|) &= \tfrac{1}{2}n(n-1) + n(n-1) \;=\; \tfrac{3}{2}n(n-1) \\ &> n(n+1) \;=\; \textstyle\sum_{i,j=1}^{n} \|x_i - y_j\|. \end{aligned}$$

EXAMPLE 1.6 Let $X = \ell_1^2$ and let $\{x_i\}_{i=1}^2$ denote the unit basis of X. Put $y_1 = (1,1)$ and $y_2 = (0,0)$. Then $\|x_1 - x_2\| = 2 = \|y_1 - y_2\|$ and $\|x_i - y_j\| = 1$ for all $1 \le i, j \le 2$. Hence

$$\sum_{i,j=1}^{2} \|x_i - y_j\| \;=\; \|x_1 - x_2\| + \|y_1 - y_2\|,$$

but the sets$\{x_i\}_{i=1}^2$ and $\{y_i\}_{i=1}^2$ are not identical.

The main purpose of this note is to discuss the second part of the polygon inequality conjecture, namely, the case of equality in (**??**). First, in Section **??**, we will describe exactly when equality holds in the space $L_1(\Omega, \mu)$ and, in Section **??**, use this to prove this part of the conjecture in ℓ_2^m.

2 The case of equality in $L_1(\Omega, \mu)$

Our main tool in studying the equality case is the following:

LEMMA 2.1 *Let $s_1 \le \ldots \le s_n$ and $t_1 \le \ldots \le t_n$ be any two collections of real numbers. Then*

$$\sum_{i=1}^{n} |t_i - s_i| \;\le\; \sum_{i,j=1}^{n} |t_i - s_j| - \sum_{1\le i<j\le n} (|s_i - s_j| + |t_i - t_j|)\,. \tag{2.1}$$

Proof: We start our inductive argument with the case $n = 2$. We must show that:

$$|t_1 - s_1| + |t_2 - s_2| \;\le\; |t_1 - s_1| + |t_1 - s_2| + |t_2 - s_1| + |t_2 - s_2| - |s_1 - s_2| - |t_1 - t_2|,$$

which is the same as showing that the inequality

$$|s_2 - s_1| + |t_2 - t_1| \;\le\; |t_1 - s_2| + |t_2 - s_1|$$

holds. But this follows from $s_1 \leq s_2$ and $t_1 \leq t_2$ by checking. Let us pass to the induction step. Suppose that (**??**) holds for n and let $s_1 \leq \ldots \leq s_n \leq s_{n+1}$ and $t_1 \leq \ldots \leq t_n \leq t_{n+1}$ be given. Assume without loss of generality that $t_{n+1} \geq s_{n+1}$. Then

$$\sum_{i=1}^{n} |t_{n+1} - s_i| \;=\; n|t_{n+1} - s_{n+1}| + \sum_{i=1}^{n} |s_{n+1} - s_i|.$$

It now follows from the triangle inequality that

$$\begin{aligned} &\textstyle\sum_{i=1}^{n} |t_{n+1} - s_i| + \sum_{i=1}^{n} |s_{n+1} - t_i| \;= \\ &n|t_{n+1} - s_{n+1}| + \textstyle\sum_{i=1}^{n} |s_{n+1} - s_i| + \sum_{i=1}^{n} |s_{n+1} - t_i| \;\geq \\ &\textstyle\sum_{i=1}^{n} |s_{n+1} - s_i| + \sum_{i=1}^{n} |t_{n+1} - t_i|. \end{aligned} \tag{2.2}$$

But then (**??**) and the induction hypothesis give that

$$\begin{aligned} &\textstyle\sum_{i,j=1}^{n+1} |s_i - t_j| - \sum_{1 \leq i < j \leq n+1} (|s_i - s_j| + |t_i - t_j|) \\ =\; & |t_{n+1} - s_{n+1}| + \textstyle\sum_{i=1}^{n} |s_{n+1} - t_i| + \sum_{i=1}^{n} |t_{n+1} - s_i| + \sum_{i,j=1}^{n} |s_i - t_j| \\ -\; & \textstyle\sum_{i=1}^{n} (|s_{n+1} - s_i| + |t_{n+1} - t_i|) - \sum_{1 \leq i < j \leq n} (|s_i - s_j| + |t_i - t_j|) \\ \geq\; & |t_{n+1} - s_{n+1}| + \textstyle\sum_{i=1}^{n} |t_i - s_i| \;=\; \sum_{i=1}^{n+1} |t_i - s_i| \end{aligned}$$

and the proof is done. □

We are now ready to characterize the vectors $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ for which equality holds in (**??**).

THEOREM 2.2 *Let $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ be two collections of functions in $L_1(\Omega, \mu)$. Then equality holds in (**??**) if and only if for almost every $\omega \in \Omega$, the numerical sets $\{x_i(\omega)\}_{i=1}^{n}$ and $\{y_i(\omega)\}_{i=1}^{n}$ are identical.*

Proof: The "if" part is immediate because

$$\sum_{i,j=1}^{n} |x_i(\omega) - y_j(\omega)| - \sum_{1 \leq i < \leq j = 1}^{n} (|x_i(\omega) - x_j(\omega)| + |y_i(\omega) - y_j(\omega)|) \;=\; 0$$

for almost every ω (with respect to μ) and by integrating over Ω with respect to μ. Let us then consider the "only if" part. Suppose that equality holds in (**??**) and, for each $\omega \in \Omega$, let $\pi(\omega, i)$ and $\sigma(\omega, i)$ be permutations of $(1, 2, \ldots, n)$ for which

$$x_{\pi(\omega,1)}(\omega) \ \leq \ldots \leq \ x_{\pi(\omega,n)}(\omega)$$

and

$$y_{\sigma(\omega,1)}(\omega) \ \leq \ldots \leq \ y_{\sigma(\omega,n)}(\omega).$$

We will need the increasing order of these numbers in order to effectively use Lemma **??**. We must prove that

$$\text{for every } 1 \leq i \leq n, \ \ x_{\pi(\omega,i)}(\omega) \ = \ y_{\sigma(\omega,i)}(\omega) \text{ a.e.} \tag{2.3}$$

If (**??**) is false, then there exist $\delta > 0$, $\epsilon > 0$, and a subset $\Delta \subset \Omega$ with $\mu(\Delta) \geq \delta$ such that

$$\text{for every } \omega \in \Delta, \ \ \sum_{i=1}^{n} \left| x_{\pi(\omega,i)}(\omega) - y_{\sigma(\omega,i)}(\omega) \right| \ \geq \ \epsilon. \tag{2.4}$$

Note that, for every $\omega \in \Omega$, the expressions

$$\sum_{i,j=1}^{n} |x_i(\omega) - y_j(\omega)| \ \text{ and } \sum_{1 \leq i < j \leq n}^{n} (|x_i(\omega) - x_j(\omega)| + |y_i(\omega) - y_j(\omega)|)$$

are invariant under permutations of $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$. Therefore (**??**), Lemma **??**, and the equality

$$\sum_{i,j=1}^{n} \|x_i - y_j\| - \sum_{1 \leq i < j \leq n} (\|x_i - x_j\| + \|y_i - y_j\|) \ = \ 0$$

yield the inequality

$$\begin{aligned}
0 \ &= \ \textstyle\sum_{i,j=1}^{n} \|x_i - y_j\| - \sum_{1 \leq i < j \leq n} (\|x_i - x_j\| + \|y_i - y_j\|) \\
&= \ \textstyle\int_\Omega \left\{ \sum_{i,j=1}^{n} |x_i(\omega) - y_j(\omega)| - \sum_{1 \leq i < j \leq n} (|x_i(\omega) - x_j(\omega)| + |y_i(\omega) - y_j(\omega)|) \right\} d\mu \\
&= \ \textstyle\int_\Omega \Big\{ \sum_{i,j=1}^{n} |x_{\pi(\omega,i)}(\omega) - y_{\sigma(\omega,j)}(\omega)| \\
&\qquad - \ \textstyle\sum_{1 \leq i < j \leq n} \Big(|x_{\pi(\omega,i)}(\omega) - x_{\pi(\omega,j)}(\omega)| + |y_{\sigma(\omega,i)}(\omega) - y_{\sigma(\omega,j)}(\omega)| \Big) \Big\} \, d\mu \\
&\geq \ \textstyle\int_\Omega \sum_{i=1}^{n} |x_{\pi(\omega,i)}(\omega) - y_{\sigma(\omega,i)}(\omega)| d\mu \ \geq \ \epsilon\mu(\Delta) \ = \ \epsilon\delta,
\end{aligned}$$

a contradiction, and our proof is done. □

REMARK 2.3 Suppose that the measure space (Ω, μ) is a compact metric space and μ is a Borel measure. Assume also that $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are continuous functions. Then the proof of Theorem **??** shows that the equality

$$\sum_{i,j=1}^{n} \|x_i - y_j\| - \sum_{1 \le i < \le j=1}^{n} (\|x_i - x_j\| + \|y_i - y_j\|) \;=\; 0$$

implies that $\{x_i(\omega)\}_{i=1}^n$ and $\{y_i(\omega)\}_{i=1}^n$ are identical for every $\omega \in \Omega$.

Lemma **??** now yields the following result for monotonically increasing sequences of functions:

COROLLARY 2.4 *Let* $\{x_i\}_{i=1}^n \cup \{y_i\}_{i=1}^n \;\subset\; L_1(\Omega, \mu)$ *and assume that* $x_1(\omega) \le \ldots \le x_n(\omega)$ *and* $y_1(\omega) \le \ldots \le y_n(\omega)$ *a.e. Then*

$$\sum_{i=1}^{n} \|x_i - y_i\| \;\le\; \sum_{i,j=1}^{n} \|x_i - y_i\| - \sum_{1 \le i < j \le n}^{n} (\|x_i - x_j\| + \|y_i - y_j\|)\,.$$

3 Equality in (??) in the case ℓ_2^m

Let us now consider the second part of the polygon inequality conjecture. Suppose that $X = \ell_2^m$ and $\{x_i\}_{i=1}^n \cup \{y_i\}_{i=1}^n \subset X$ are such that (**??**) holds. Let

$$\Omega \;=\; S^m \;:=\; \{u \in X \mid \|u\| = 1\}.$$

Furthermore, let μ denote the normalized rotation–invariant measure on Ω (i.e., $\mu(\Omega) = 1$). Then the integral $\int_\Omega |\langle v, y\rangle| d\mu(y)$ depends only on $\|v\|$ and not on v itself. Put

$$c \;=\; \int_\Omega |\langle u, y\rangle| d\mu(y), \quad u \in \Omega.$$

Then the map

$$T:\; \ell_2^m \to L_1(\Omega, \mu),$$

defined by

$$(Tx)(\omega) \;=\; c^{-1}\langle x, \omega\rangle,$$

is an isometric embedding of ℓ_2^m into $L_1(\Omega, \mu)$ (see Lindenstrauss and Pelczynski [**?**, p.312, Prop. 7.5]). Moreover, Ω is a compact metric space and, for every $x \in \ell_2^m$, Tx is a continuous function. It follows by Theorem **??** and Remark **??** that, for every $\omega \in \Omega$, the numerical sets $\{Tx_i(\omega)\}_{i=1}^n$ and

$\{Ty_i(\omega)\}_{i=1}^n$ are identical for every $\omega \in S^m$.

Continuing, since the expression

$$\sum_{i,j=1} \|x_i - y_j\| - \sum_{1\le i<j\le n} (\|x_i - x_j\| + \|y_i - y_j\|)$$

does not depend on the ordering of the x_i's and y_i's we may assume that both sequences are arranged by decreasing order in the norms,viz.,

$$\|x_1\| \ge \ldots \|x_n\| \;\text{ and }\; \|y_1\| \ge \ldots \|y_n\|.$$

Suppose now that $\|y_1\| \ge \|x_1\|$ and let $\omega = \|y_1\|^{-1}y_1 \in S^m$. As mentioned earlier, by Theorem **??** and Remark **??**, the sets $\{Tx_i(\omega)\}_{i=1}^n$ and $\{Ty_i(\omega)\}_{i=1}^n$ are identical, and so for some $1 \le j \le n$, we must have that

$$c^{-1}\langle x_j, \omega\rangle \;=\; (Tx_j)(\omega) \;=\; (Ty_1)(\omega) \;=\; c^{-1}\|y_1\|^{-1}\langle y_1, y_1\rangle \;=\; c^{-1}\|y_1\|.$$

It follows that

$$\|y_1\| \;\ge\; \|x_j\| \;\ge\; \|y_1\|^{-1}\langle x_j, y_1\rangle \;=\; \|y_1\|^{-1}\langle y_1, y_1\rangle \;=\; \|y_1\|.$$

This implies that $x_j = y_1$. Clearly we can assume without loss of generality that $j = 1$. Since

$$\sum_{j=2}^n \|x_1 - y_j\| + \sum_{j=2}^n \|y_1 - x_j\| \;=\; \sum_{j=2}^n \|y_1 - y_j\| + \sum_{j=2}^n \|x_1 - x_j\|$$

we obtain that

$$\sum_{i,j=2}^n \|x_i - y_j\| \;=\; \sum_{2\le i<j\le n} (\|x_i - x_j\| + \|y_i - y_j\|)\,.$$

Repeating the above procedure we get that there is a j, $2 \le j \le n$, such that $x_j = y_2$. Continuing in this manner we arrive at the equality

$$\sum_{i,j=n-1}^n \|x_i - y_j\| \;=\; \|x_{n-1} - x_n\| + \|y_{n-1} - y_n\|$$

which implies, by the triangle inequality, that either $x_n = y_n$ and $x_{n-1} = y_{n-1}$ or $x_n = y_{n-1}$ and $x_{n-1} = y_n$. We have therefore proved part (ii) of the polygon inequality conjecture.

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